# Decomposing oriented graphs into transitive tournaments

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**Abstract**

For an oriented graph $G$ with $n$ vertices, let $f(G)$ denote the minimum number of transitive subtournaments that decompose $G$. We prove several results on $f(G)$. In particular, if $G$ is a tournament then $f(G) < \frac{5}{21}n^2(1+o(1))$ and there are tournaments for which $f(G) > n^2/3000$. For general $G$ we prove that $f(G) \le \lfloor n^2/3 \rfloor$ and this is tight. Some related parameters are also considered.

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## 1 Introduction

All graphs and digraphs considered here are finite and have no loops or multiple edges. For the standard terminology used the reader is referred to [1]. An *oriented graph* is a digraph without directed cycles of length two (antiparallel edges). In other words, it is an orientation of a simple graph. A *tournament* on $n$ vertices is an orientation of $K_n$. An oriented graph is called *acyclic* if it has no directed cycles. An acyclic tournament is usually called a *transitive tournament*. We denote by $TT_k$ the unique (up to isomorphism) transitive tournament on $k$ vertices.

A *transitive decomposition* of a digraph $G$ is a set of edge-disjoint transitive subtournaments that occupy all the edges of the graph. Namely, each edge of $G$ belongs to precisely one transitive subtournament in the set. Let $f(G)$ denote the minimum size of a transitive decomposition of $G$. Since a digraph with $e(G) = m$ edges has a trivial transitive decomposition into $m$ copies of $TT_2$ we always have $f(G) \le e(G)$. The goal of this paper is to study transitive decompositions and to obtain nontrivial bounds for $f(G)$. We note that this problem is closely related to the problem of Erdős, Goodman and Pósa [3] who asked for the minimum number of cliques that decompose a graph $G$. They proved that if $G$ has $n$ vertices then $\lfloor n^2/4 \rfloor$ cliques always suffice, and this is tight.

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Let $f(n,m)$ denote the maximum possible value of $f(G)$ taken over all oriented graphs with $n$ vertices and $m$ edges. Particularly interesting is the value of $f(n,\binom{n}{2})$, namely, the minimal number of transitive tournaments that are needed in order to decompose an $n$-vertex tournament, in the worst case. For notational convenience we put $f(n) = f(n,\binom{n}{2})$.

In the next section we consider upper and lower bounds for $f(n)$. Notice that it is not obvious at first sight that $f(n) = \Theta(n^2)$. This is because every $n$-vertex tournament contains many copies of $TT_{\Theta(\log n)}$ which is easy to prove by induction. However, we prove that there are tournaments in which the large transitive subtournaments cannot be packed so as to cover all but $o(n^2)$ edges. In particular, we prove the following.

**Theorem 1.1** $\frac{1}{3000}n^2(1+o(1)) < f(n) < \frac{5}{21}n^2(1+o(1))$.

We note that both the upper and lower bounds can be slightly improved but they are still quite far. It seems very interesting to determine $f(n)$ even asymptotically.

In Section 3 we consider the more general parameter $f(n,m)$. Clearly, $f(n,m) = m$ if and only if there is an oriented graph with $n$ vertices and $m$ edges without a $TT_3$. It is not difficult to construct such graphs for all $m \le \lfloor n^2/3 \rfloor$ (as shown in the beginning of Section 3). We prove, however, that for larger $m$, $f(n,m)$ is still bounded by $\lfloor n^2/3 \rfloor$.

**Theorem 1.2** *$f(n,m) \le \lfloor n^2/3 \rfloor$. $f(n,m) = m$ for $m \le \lfloor n^2/3 \rfloor$.*

Section 4 contains some concluding remarks and some results on related parameters.

## 2 Decomposing tournaments into transitive subtournaments

Let $r(k)$ denote the minimum integer that guarantees that every tournament with $r(k)$ vertices has a $TT_k$. A trivial induction argument gives $r(k) \le 2^{k-1}$. Hence, it follows that in a tournament with $n$ vertices, every vertex appears in many copies of $TT_{\Theta(\log n)}$. Unfortunately, as wee shall see, in some cases it is impossible to pack these large transitive tournaments in order to obtain a transitive decomposition with $o(n^2)$ elements. In fact, Erdős and Moser [4] proved, using the probabilistic method, that $r(k) \ge 2^{0.5k(1+o(1))}$. This already shows that we *cannot* expect $f(n) = o(n^2/\log^2 n)$ even for random tournaments. It is easy to show $f(2) = 2$, $f(3) = 4$ an it is well known that $f(4) = 8$ and $f(5) = 14$ [7]. In fact, it is straightforward to construct the unique tournament $\mathcal{T}$ on seven vertices without a $TT_4$. We shall need the following lemmas in order to prove the upper and lower bounds of Theorem 1.1. Our first lemma is (a simple application of) the seminal result of Wilson [8] for undirected graphs.

**Lemma 2.1** *Let $k$ be a positive integer. Then $K_n$ has $\frac{1}{k(k-1)}n^2(1-o(1))$ edge-disjoint copies of $K_k$.*

In fact, Wilson's theorem shows that if some obvious divisibility conditions hold then there is a $K_k$-decomposition of $K_n$, assuming $n$ is sufficiently large.

Our second lemma establishes $f(n)$ for some small values of $n$.

**Lemma 2.2** *$f(2) = 1$, $f(3) = 3$, $f(4) = 4$, $f(5) = 6$, $f(6) = 8$ and $f(7) = 10$.*

**Proof:** The values of $f(n)$ for $n \leq 5$ are easy exercises. We shall prove the case $f(7) = 10$. The case $f(6) = 8$ is easier. Let $S$ be a tournament with $t$ vertices. If $S = TT_7$ then $f(S) = 1$. If $S$ contains a $TT_6$ then $f(S) \leq 7$ since a $TT_6$ already contains 15 of the 21 edges. If $S$ contains a $TT_5$ and does not contain a $TT_6$ then let $(x, y)$ be an edge such that the other five vertices induce a $TT_5$ (the notation $(x, y)$ corresponds to an edge from $x$ to $y$). Since there is no $TT_6$, not all edges between $y$ and the other five vertices emanate from $y$. Hence there is a $TT_3$ containing $(x, y)$ which is edge-disjoint from the $TT_5$. Consequently, $f(S) \leq 10$. If $S$ has a $TT_4$ and does not have a $TT_5$ then we may assume, without loss of generality, that $a, b, c, d$ are the vertices of a $TT_4$ and $e, f, g$ are the other vertices. It is not difficult to verify that there are three edge-disjoint $TT_3$, each containing precisely two vertices from $e, f, g$. Thus, $S$ decomposes into a $TT_4$, three $TT_3$ and six $TT_2$. Consequently, $f(S) \leq 10$. If $S$ has no $TT_4$ then $S = \mathcal{T}$ (recall that $\mathcal{T}$ denotes the unique 7-vertex tournament without a $TT_4$). It is easy to verify that $\mathcal{T}$ has 6 edge-disjoint $TT_3$. Hence $f(\mathcal{T}) = 9$. We have shown that $f(7) \leq 10$. The following tournament $S$ has $f(S) = 10$. Let $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{5, 6, 7\}$. Orient all edges from $A$ to $B$, from $B$ to $C$ and from $C$ to $A$. The orientation of the edge $\{12\}$ and $\{34\}$ is arbitrary. Orient the edges inside $C$ in a cycle. It is easy to check that $S$ has no $TT_5$, and any $TT_3$ must contain two vertices from the same part. It follows that $f(S) = 10$. ■

The next two lemmas are needed for the lower bound in Theorem 1.1.

**Lemma 2.3** *There exist tournaments that do not have more than $n^2/14$ edge-disjoint $TT_4$.*

**Proof:** As before, let $\mathcal{T}$ denote the unique 7-vertex tournament without a $TT_4$. Assume the vertices of $\mathcal{T}$ are $1, \ldots, 7$. We blow up each vertex of $\mathcal{T}$ into either $\lceil n/7 \rceil$ or $\lfloor n/7 \rfloor$ vertices, so that the total number of vertices is $n$. Let $V_i$ denote the set of vertices blown up from $i$. For $x \in V_i$ and $y \in V_j$ the orientation of the edge $xy$ is the same as the orientation of $ij$ in $\mathcal{T}$. The orientation between two vertices in the same set is arbitrary. Since $\mathcal{T}$ has no $TT_4$, we have constructed a tournament with $n$ vertices in which every $TT_4$ must contain an edge connecting two vertices from the same set. As the total number of edges with both endpoints in the same set is at most $n^2/14$ the claim follows. ■

**Lemma 2.4** *For all $t \geq 2$, $K_t$ has a packing with edge-disjoint copies of $K_4$ so that the number of unpacked edges is at most $4t - 7$.*

**Proof:** It is well known that for $t \equiv 1, 4 \bmod 12$, $K_t$ has a $K_4$ decomposition (see, e.g., [2]). Suppose $t$ is not of this form. We may add or delete $s$ vertices where $1 \le s \le 4$ so as to obtain a graph whose number of vertices is either 1 or 4 modulo 12. In case $t \equiv 2, 5 \bmod 12$ we delete one vertex and $t - 1 \le 4t - 7$ edges. In case $t \equiv 0, 3 \bmod 12$ we add one vertex and the $t$ added edges are on $t/3$ copies of $K_4$ containing precisely $t \le 4t - 7$ original edges. In case $t \equiv 6 \bmod 12$ we delete two vertices and $2t - 3 \le 4t - 7$ edges. In case $t \equiv 7 \bmod 12$ we delete three vertices and $3t - 6 \le 4t - 7$ edges. In case $t \equiv 8 \bmod 12$ we delete four vertices and $4t - 10 \le 4t - 7$ edges. In case $t \equiv 10 \bmod 12$ we add three vertices $x, y, z$. We may assume that some $K_4$ of the decomposition contains $x, y, z, w$ where $w$ is an original vertex. This $K_4$ contains no original edges. The other $K_4$'s containing one of $x, y, z$ contain precisely $3t - 3 \le 4t - 7$ original edges. In case $t \equiv 11 \bmod 12$ we add two vertices. The unique $K_4$ containing the two new vertices contains only one original edge and the other $K_4$'s containing a new vertex contain precisely $2t - 4$ original edges. Again $(2t - 4) + 1 = 2t - 3 \le 4t - 7$. In case $t \equiv 9 \bmod 12$ we add one vertex and use the case of $10 \bmod 12$ to obtain a packing with $K_4$ which has $3(t + 1) - 3 = 3t$ unpacked edges. Since $6t/(t + 1) > 5$ there is some vertex which is incident with at least 6 unpacked edges. Deleting this vertex we obtain a graph with $t$ vertices and the number of unpacked edges is now at most $(3t - 6) + (t - 6) = 4t - 12 \le 4t - 7$. ■

**Proof of the upper bound in Theorem 1.1** Let $T$ be a tournament with $n$ vertices. By Lemma 2.1 we can pack $K_n$ with $\frac{1}{k(k-1)} n^2 (1 - o(1))$ edge-disjoint $K_k$. Particularly, we can pack $T$ with $\frac{1}{42} n^2 (1 - o(1))$ edge-disjoint subtournaments each having 7 vertices. By Lemma 2.2, $f(7) = 10$. Thus, each of these subtournaments can be decomposed into at most 10 transitive tournaments. It follows that $f(T) \le \frac{10}{42} n^2 (1 + o(1))$.

We note that in [9] it is proved that a tournament on $n$ vertices has at least $0.13n^2(1 + o(1))$ edge-disjoint $TT_3$. These $TT_3$ cover $0.39n^2(1 + o(1))$ edges which implies an upper bound of 0.24 in Theorem 1.1. This is only slightly inferior to our 5/21 upper bound. By computing specific values of $f(k)$ for larger $k$ one may be able to obtain an improve upper bound, but this approach must converge as suggested by the lower bound.

**Proof of the lower bound in Theorem 1.1** Let $T$ be the $n$-vertex tournament constructed in Lemma 2.3, and recall that $T$ has at most $n^2/14$ edge-disjoint $TT_4$. Consider a transitive decomposition of $T$ with $k = f(T)$ elements whose vertex sizes are $p_1, \ldots, p_k$. Thus, $\binom{n}{2} = \sum_{i=1}^{k} \binom{p_i}{2}$. By Lemma 2.4, the element whose size is $p_i$ contains a set of edge-disjoint $TT_4$ covering all but at most $4p_i - 7$ edges. It follows that $T$ has at least

$$\frac{\binom{n}{2} - \sum_{i=1}^{k} (4p_i - 7)}{6}$$

edge-disjoint copies of $TT_4$. Clearly the last sum is minimized when $\sum_{i=1}^{k} p_i$ is maximized. This happens when all the $p_i$ are equal and their common value $p$ satisfies $p(p - 1) = n(n - 1)/k$. For convenience, put $k = \alpha n(n-1)$. Thus, $p = 1/2 + \sqrt{1/4 + 1/\alpha}$ and the number of edge-disjoint copies of $TT_4$ is at least

$$n(n-1)\left(\frac{1}{12} + \frac{7}{6}\alpha - \frac{2}{3}\alpha(1/2 + \sqrt{1/4 + 1/\alpha})\right).$$

Taking $\alpha = 1/3000$ gives, for $n$ sufficiently large, more than $0.071438n^2 > \frac{1}{14}n^2$ edge-disjoint $TT_4$ in $T$, a contradiction. ■

## 3 Proof of Theorem 1.2

Consider the Turán graph $T(n,3)$. Recall that this graph is a complete 3-partite graph whose vertex classes are as equal as possible. Hence, it has $\lfloor n^2/3 \rfloor$ edges. Let the vertex classes be $V_0, V_1, V_2$. We orient all edges from $V_i$ to $V_{(i+1) \bmod 3}$ for $i = 0, 1, 2$. Notice that this orientation does not contain a $TT_3$. Hence, we have that $f(n,m) = m$ for all $m \le \lfloor n^2/3 \rfloor$.

It remains to show that every oriented graph $G$ with $n$ vertices has $f(G) \le n^2/3$. We prove this by induction on $n$. The theorem clearly holds for $n = 1$. Assume it holds for oriented graphs with $n-1$ vertices. Let $G = (V,E)$ be a graph with $n$ vertices. For a vertex $u$, let $d^+(u)$ and $d^-(u)$ be the out and in degrees of $u$, respectively, and let $d(u) = d^+(u) + d^-(u)$ be the total degree. Let $v \in V$ have minimal total degree. Let $G'$ be the induced subgraph of $G$ on $V - v$. By the induction hypothesis, $f(G') \le \lfloor (n-1)^2/3 \rfloor$. Clearly, $f(G) \le d(v) + f(G')$ since we may trivially decompose the edges incident with $v$ into $d(v)$ copies of $TT_2$. Thus, if $d(v) \le 2n/3$ we have

$$f(G) \le \lfloor \frac{2}{3}n \rfloor + \lfloor \frac{(n-1)^2}{3} \rfloor = \lfloor \frac{n^2}{3} \rfloor.$$

We may now assume that $d(v) = \lfloor 2n/3 \rfloor + a$ where $a > 0$. It suffices to prove that there are $a$ edge-disjoint copies of $TT_3$ containing $v$ since this would give $f(G) \le a + (d(v) - 2a) + f(G')$ and we can again use the induction hypothesis to obtain $f(G) \le \lfloor \frac{n^2}{3} \rfloor$.

Without loss of generality, assume $d^+(v) \ge d^-(v)$. Let $N^+(v)$ and $N^-(v)$ be the set of out-neighbors and in-neighbors of $v$, respectively. Hence, $|N^+(v)| = d^+(v) = \lfloor n/3 \rfloor + b$ where $b > 0$ and $|N^-(v)| = d^-(v) = \lfloor n/3 \rfloor + c$ and note that it may be that $c < 0$. Let $H_1$ ($H_2$) be the undirected subgraph of $G$ induced by $N^+(v)$ ($N^-(v)$). The minimum degree of $H_1$ satisfies

$$\delta(H_1) \ge d(v) - (n - d^+(v)) = \lfloor 2n/3 \rfloor + \lfloor n/3 \rfloor - n + a + b \ge a + b - 1.$$

Similarly,

$$\delta(H_2) \ge d(v) - (n - d^-(v)) = \lfloor 2n/3 \rfloor + \lfloor n/3 \rfloor - n + a + c \ge a + c - 1.$$

We shall use the well known fact that a graph with minimum degree $\delta$ has a path of length $\delta$ and hence a matching of size $\lceil \delta/2 \rceil$ (see, e.g., [1]). Consider first the case $c < 0$. In this case we must have $b \geq a$ and hence there is a matching of size at least $\lceil (2a-1)/2 \rceil = a$ in $H_1$. If $c \geq 0$ then there is a matching of size $\lceil (a+b-1)/2 \rceil$ in $H_1$ and a matching of size $\lceil (a+c-1)/2 \rceil$ in $H_2$ which together is a matching of size at least $a$ in $N^+(v) \cup N^-(v)$. Now, each element of this matching, together with $v$, yields a $TT_3$. We have shown that there are $a$ edge-disjoint copies of $TT_3$ containing $v$, as required. ■

## 4 Concluding remarks

- As mentioned in Section 2, if $T$ is an $n$-vertex random tournament (the orientation of each edge is chosen uniformly at random, and independently) then $f(T) = \Omega(n^2/\log^2 n)$ almost surely (that is, with probability tending to 1 as $n$ tends to infinity). On the other hand, unlike the general case where Theorem 1.1 shows that $f(n) = \Theta(n^2)$, it is not difficult to show that for every $\epsilon > 0$, if $T$ is an $n$-vertex random tournament then almost surely $f(T) < \epsilon n^2$ for $n$ sufficiently large. Hence $f(T) = o(n^2)$ almost surely for random tournaments. This follows from the result of [5] which, when applied to our setting, gives that for every fixed positive integer $k$, there is, almost surely, a packing of $T$ with copies of $TT_k$ so that the number of unpacked edges is only $o(n^2)$. In particular this implies that $f(T) \leq \binom{n}{2}/\binom{k}{2} + o(n^2)$.

- As we define $f(G)$ on general digraphs, we may also define the analog of $f(n,m)$ on this wider class where antiparallel edges are allowed. Notice that in this more general case, there is no interesting analog for $f(n)$ since the complete $n$-vertex digraph can be trivially decomposed into two edge-disjoint copies of $TT_n$ and hence $f(n, n(n-1)) = 2$ in this case. However, using a similar inductive approach as in the proof of Theorem 1.2 it can be shown that $f(n,m) \leq \lfloor n^2/2 \rfloor$ and $f(n,m) = m$ for all $m \leq \lfloor n^2/2 \rfloor$. The construction here is obtained by the existence of an $n$-vertex bipartite digraph with $m$ edges for $m \leq \lfloor n^2/2 \rfloor$.

- As mentioned in Section 2, it is possible to slightly improve the upper bound in Theorem 1.1 by computing higher explicit values of $f(k)$. This seems to be a difficult task already for relatively small values of $k$. However, there is another approach which yields a minor improvement of the upper bound and which requires no additional explicit computations. A *fractional* transitive decomposition of a digraph $G$ is an assignment of nonnegative weights to all the transitive subtournaments of $G$ so that for any edge, the sum of the weights of all the transitive subtournaments that contain the edge is precisely one. The *value* of the fractional transitive decomposition is the sum of all assigned weights. Let $f^*(G)$ be the smallest possible value of a fractional transitive decomposition of $G$. Trivially, $f^*(G) \leq f(G)$. Let $f^*(n)$ be the fractional analog of $f(n)$. Thus, $f^*(n) \leq f(n)$. Let us first show how $f(k)$ can be used to obtain a nontrivial upper bound for $f^*(k')$, where $k' > k$. Let $T$ be a tournament with $k'$ vertices. There are $\binom{k'}{k}$ subtournaments on $k$ vertices. In each of them we may take a transitive decomposition with at most $f(k)$ elements. Since each edge of $T$ appears in $\binom{k'-2}{k-2}$ subtournament with $k$ vertices, we may assign the value $1/\binom{k'-2}{k-2}$ to each element of each transitive decomposition and obtain a fractional transitive decomposition of $T$ whose value is at most

$$\binom{k'}{k}\frac{1}{\binom{k'-2}{k-2}}f(k) = \frac{k'(k'-1)}{k(k-1)}f(k) \geq f^*(k').$$

However in some cases we can do better. Consider, for example, the case $k = 7$ and $k' = 64$. By the last inequality we have $f^*(64) \leq 960$. Using the same notation as in Section 2, we have that $r(7) \leq 64$ and, in fact, every vertex of a 64-vertex tournament $T$ is a source or a sink of some $TT_7$. Hence, $T$ has at least 32 distinct $TT_7$. Since $f(TT_7) = 1$ and $f(7) = 10$ we have

$$f^*(64) \leq \frac{1}{\binom{62}{5}}\left(10\left(\binom{64}{7} - 32\right) + 32\right) = 960 - \frac{288}{\binom{62}{5}}.$$

Now, using Lemma 2.1 applied to $k = 64$ in the proof of Theorem 1.1 would give $f^*(n) \leq (\frac{5}{21} - \frac{1}{14\binom{62}{5}})n^2(1+o(1))$. By Theorem 2 of [6], applied to the family of transitive tournaments the values of $f^*(n)$ and $f(n)$ differ only in $o(n^2)$. Thus,

$$f(n) \leq \left(\frac{5}{21} - \frac{1}{14\binom{62}{5}}\right)n^2(1 + o(1)).$$

Although this is only a negligible improvement over the upper bound in Theorem 1.1, the approach presented here may be useful in other settings as well.

- By the same argument as in the last paragraph we have that $f^*(n)/(n(n-1)) \leq f^*(n-1)/((n-1)(n-2))$. In particular, this shows that $f^*(n)/n^2$ converges to some limit $c$. Since $f(n) - f^*(n) = o(n^2)$ we also have that $f(n)/n^2$ converges to $c$. Theorem 1.1 shows that $\frac{5}{21} > c > \frac{1}{3000}$. We leave as an open problem determining $c$ exactly.

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