Graphs Having the Local Decomposition Property

Yair Caro *   Raphael Yuster †

Abstract

Let H be a fixed graph without isolated vertices, and let G be a graph on n vertices. Let $2 \leq k \leq n-1$ be an integer. We prove that if $k \leq n-2$ and every k-vertex induced subgraph of G is H-decomposable then G or its complement is either a complete graph or a complete bipartite graph. This also holds for $k = n-1$ if all the degrees of the vertices of H have a common factor. On the other hand, we show that there are graphs H for which it is NP-Complete to decide if every $n-1$-vertex subgraph of G is H-decomposable. In particular, we show that $H = K_{1,h-1}$ where $h > 3$, are such graphs.

1 Introduction

All graphs considered here are finite, undirected and simple. Given two graphs, H and G, where H has no isolated vertices, the graph G is *H-decomposable*, denoted by $H \mid G$, if the edge-set of G is the union of edge-disjoint isomorphic copies of H. We refer to the recent book of Bosak [2] as a general reference for decomposition problems.

It has been proved by Dor and Tarsi [11] that for any fixed graph H having a connected component with at least three edges, the decision problem "does $H \mid G$" is NP-Complete. On the other hand, it is shown by Caro et al. in [7, 9] that the class of decomposition problems called "Random H-decompositions" is solvable in polynomial time, and several structural results were published by Beineke, Goddard and Hamburger, and many others [3, 13]. Aigner and Triesch [1] and Caro [5, 6] raised the problem of the possibility to determine the structure of a graph G in terms of the information given on its induced subgraphs. Inspired by this question Caro and Yuster [10] considered the following: Let F be a graph property (i.e. a family of graphs). For $n > k > 1$ a graph G on n vertices has the property $F(n,k)$ if every induced k-vertex subgraph of G has property F. In that paper, the computational complexity of deciding whether G has $F(n,k)$ is discussed

*Department of Mathematics, University of Haifa-ORANIM, Tivon 36006, Israel. e-mail: zeac603@uvm.haifa.ac.il

†Department of Mathematics, University of Haifa-ORANIM, Tivon 36006, Israel. e-mail: raphy@math.tau.ac.il

for a wide range of properties and values of k. Let H be a fixed graph and let F^H be the graph property of being H decomposable. The focus of this paper is to determine the computational complexity of $F^H(n,k)$, and provide a structure for $F^H(n,k)$ whenever this family of graphs is easily recognizable. For ease of notation we put $H(n,k) = F^H(n,k)$.

In order to present the results we need the following notations. For a graph $G = (V, E)$ denote by $e(G) = |E(G)|$ the cardinality of the edge-set of G, and denote by $e_m(G)$ the number of its edges modulo m where $m > 1$ is an integer. For a subset $A \subset V$ denote by $\langle A \rangle$ the induced graph of G with vertex-set A. For a graph H having h vertices with degrees $d_1, \ldots, d_h$ we put $gcd(H) = gcd(d_1, \ldots, d_h)$. Our main tool is the following theorem which is interesting in its own right.

Theorem 1.1 *Let G be a graph on n vertices and let $m \geq 2$ and $n-2 \geq k \geq 2$ be integers. Suppose that for any two subsets $A, B \subset V$ with $|A| = |B| = k$ we have $e_m(\langle A \rangle) = e_m(\langle B \rangle)$. Then, one of the following holds:*

1. $G \in \{K_n, \overline{K_n}\}$.
2. $G \in \{K_{1,n-1}, \overline{K_{1,n-1}}\}$ *where* $k \bmod m = 1$.
3. $G \in \{K_{a,n-a}, \overline{K_{a,n-a}}\}$ *where* $m = 2$ *and* $k \bmod 2 = 1$.

Using Theorem 1.1 we prove:

Theorem 1.2 *Let H be a fixed graph on $h \geq 3$ vertices without isolated vertices.*

1. *If $gcd(H) \geq 2$ and $h \leq k \leq n-1$ then $H(n,k) \subset \{K_n, \overline{K_n}\}$.*
2. *If $gcd(H) = 1$ and $h \leq k \leq n-2$ and H has more than two edges then $H(n,k) \subset \{K_n, \overline{K_n}, K_{1,n-1}, \overline{K_{1,n-1}}\}$.*
3. *If H has two edges (i.e. $H = P_3$ or $H = 2K_2$) then $H(n,k) \subset \{K_n, \overline{K_n}, K_{a,n-a}, \overline{K_{a,n-a}}\}$.*

Furthermore, in all of the above cases we can decide if $G \in H(n,k)$ in polynomial time.

Theorem 1.2 shows that $H(n,k)$ has an easily recognizable structure whenever $k \leq n-2$. This is not the case for $H(n,n-1)$ (unless $gcd(H) > 1$) even for some very simple graphs H, as can be seen from the following theorem.

Theorem 1.3 *Let $H = K_{1,k}$ where $k \geq 3$. Given a graph G on n vertices, the decision problem "does $G \in H(n,n-1)$" is NP-Complete.*

We wish to emphasize that Theorem 1.1 essentially solves some problems mentioned in [4, 5] whose origin can be traced to an old open paper of Kelley and Merriell [12].

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.1 which provides us with the structure of graphs whose k-subgraphs have the same number of edges (modulo m). In section 3 we prove Theorem 1.2 thereby providing the structure for $H(n,k)$ for $k \leq n-2$ and, whenever $gcd(H) > 1$, also for $k = n-1$. In section 4 we turn to the case $k = n-1$ and $gcd(H) = 1$ and provide hardness results for some simple graphs H having this property. Section 5 contains concluding remarks and open problems.

2 k-subgraphs with the same number of edges

In this section we prove Theorem 1.1. It is convenient to resolve the case $k = n-2$ and deduce from it the result for smaller values of k.

Theorem 2.1 *Let G be a graph on n vertices and let $m \geq 2$ be an integer. suppose that for any two subsets $A, B \subset V$ with $|A| = |B| = n-2$ we have $e_m(\langle A \rangle) = e_m(\langle B \rangle)$. Then, one of the following holds:*

1. $G \in \{K_n, \overline{K_n}\}$.
2. $G \in \{K_{1,a}, \overline{K_{1,a}}\}$ *where* $a \bmod m = 2$.
3. $G \in \{K_{a,b}, \overline{K_{a,b}}\}$ *where* $m = 2$ *and* $a \neq b \bmod 2$.

Proof If $n < 3$ the claim is trivially true, so we assume $n \geq 3$. For $i = 0, \ldots, m-1$ define $D_i = \{v \in V \mid deg(v) \bmod m = i\}$. We need the following two lemmas.

Lemma 2.2 *Each $\langle D_i \rangle$ is either a complete graph or an empty graph.*

Proof Assume that some D_i is neither a complete nor an empty graph. Hence D_i has three vertices u, v, w such that $(u,v) \in E$ but $(v,w) \notin E$. But then deleting u and v from G changes the number of edges by $2i-1 \bmod m$ while deleting v and w from G changes the number of edges by $2i \bmod m$. Thus $e_m(\langle V \setminus \{u,v\} \rangle) \neq e_m(\langle V \setminus \{v,w\} \rangle)$, which contradicts our assumption. $\square$

Lemma 2.3 *There are at most two distinct indices i, j such that $|D_i| > 0$ and $|D_j| > 0$.*

Proof Assuming the contrary, let i, j, k be distinct integers such that none of D_i, D_j, D_k is an empty set. Since every graph with at least two vertices has two vertices with the same degree, we may assume $|D_i| > 1$. By Lemma 2.2 each $\langle D_i\rangle, \langle D_j\rangle, \langle D_k\rangle$ is a complete graph or an empty graph. Suppose first $\langle D_i\rangle$ is complete and that for some $v \in D_i$, $w \in D_j$, $(v, w) \in E$. Then with $A = V \setminus \{u, v\}$ for some $u \in D_i$ and with $B = V \setminus \{v, w\}$ we get $e_m(\langle A\rangle) = e(G) - (2i-1) \bmod m \neq e(G) - (i+j-1) \bmod m = e_m(\langle B\rangle)$, a contradiction. Suppose next that $\langle D_i\rangle$ is an empty graph and for some $v \in D_i$, $w \in D_j$, $(v, w) \notin E$. Defining A and B as above we again have $e_m(\langle A\rangle) \neq e_m(\langle B\rangle)$ which is a contradiction. By symmetry the same conclusions hold for D_i versus D_k. Hence if $\langle D_i\rangle$ is complete we may assume there exist $u \in D_i$, $v \in D_j$, $w \in D_k$ such that $(u, v) \notin E$ and $(u, w) \notin E$. Putting $A = V \setminus \{u, v\}$ and $B = V \setminus \{u, w\}$ we get $e_m(\langle A\rangle) = e(G) - (i + j) \bmod m \neq e(G) - (i + k) \bmod m = e_m(\langle B\rangle)$. If $\langle D_i\rangle$ is an empty graph we may assume there exist $u \in D_i$, $v \in D_j$, $w \in D_k$ such that $(u, v) \in E$ and $(u, w) \in E$. With $A = V \setminus \{u, v\}$ and $B = V \setminus \{u, w\}$ we get $e_m(\langle A\rangle) = e(G) - (i + j - 1) \bmod m \neq e(G) - (i + k - 1) \bmod m = e_m(\langle B\rangle)$. □

We now return to the proof of Theorem 2.1. Suppose first that we only have one index i with $|D_i| \geq 1$. Then by lemma 2.2 $G \in \{K_n, \overline{K_n}\}$, and we are done. Otherwise, by lemma 2.3, we have exactly two indices i, j with $|D_i| = a \geq 2$ and $|D_j| = b \geq 1$. Observe that the proof of Lemma 2.3 implies that if $\langle D_i\rangle$ is complete, then there are no edges between D_i and D_j, and if D_i is the empty graph, all possible edges between D_i and D_j exist. By reversing the roles of i and j in the proof we also get that if $\langle D_i\rangle$ is complete so is $\langle D_j\rangle$ and thus $G = K_a \cup K_b$, or else both $\langle D_i\rangle$ and $\langle D_j\rangle$ are empty graphs in which case $G = K_{a,b}$.

Assume first that $G = K_a \cup K_b$. If $b \geq 2$ then for $u, v \in D_i$, $w, z \in D_j$ we may choose $A = V \setminus \{u, v\}$, $B = V \setminus \{w, z\}$, $C = V \setminus \{u, w\}$ and since we must have $e_m(\langle A\rangle) = e_m(\langle B\rangle) = e_m(\langle C\rangle)$ we must have $2i-1 \bmod m = 2j-1 \bmod m = i+j \bmod m$. This is only possible if $m = 2$ and $a \neq b \bmod 2$. If $b = 1$ Then $G = K_a \cup K_1$ and by the above reasoning we infer that $2i - 1 \bmod m = i$ hence $i \bmod m = 1$ which implies $a \bmod m = 2$.

If $G = K_{a,b}$ we note that if G has the property that every two $n-2$-vertex subsets A and B have $e_m(\langle A\rangle) = e_m(\langle B\rangle)$ then $\overline{G}$ also has this property. Hence either $G = K_{a,1}$ with $a \bmod m = 2$ or $G = K_{a,b}$ with $m = 2$ and $a \neq b \bmod 2$. □

Proof of Theorem 1.1. We apply induction on n, fixing k and m. Clearly, for $n = k + 2$ the claim reduces to Theorem 2.1. Also, for $k = 2$ the claim becomes trivial, so we assume $k \geq 3$ and $n \geq k + 3$. We first show that, subject to the conditions of Theorem 1.1, $G \in \{K_n, \overline{K_n}, K_{1,n-1}, \overline{K_{1,n-1}}, K_{a,n-a}, \overline{K_{a,n-a}}\}$. Since $n - 1 \geq k + 2$, we have that for every $n-1$-subset $A \subset V$, all its k-subsets have the same number of edges modulo m. Hence by the induction hypothesis, $\langle A\rangle \in \{K_{n-1}, \overline{K_{n-1}}, K_{1,n-2}, \overline{K_{1,n-2}}, K_{a',n-1-a'}, \overline{K_{a',n-1-a'}}\}$. An easy check shows that G itself must belong to the family $\{K_n, \overline{K_n}, K_{1,n-1}, \overline{K_{1,n-1}}, K_{a,n-a}, \overline{K_{a,n-a}}\}$. But now case 1 follows trivially, and for case 2 observe that if a k-subset A does not contain the center of the star

$K_{1,n-1}$ then $e_m(\langle A\rangle)=0$, while a k-subset B containing the center has $e_m(\langle B\rangle)=k-1$. Hence, $k \bmod m=1$. By taking complements (as in the last part of the proof of Theorem 2.1), the second possibility in case 2, namely $\overline{K_{1,n-1}}$, holds only if $k \bmod m=1$.

For case 3, if $G=K_{a,n-a}$, we may assume $2\le a\le n-a$. Write $k=k_1+k_2$ where $0<k_1<a$, $0<k_2<n-a$ which is possible as $n\ge k+3$, $a\ge 2$ and $n-a\ge 2$. Now, consider the k-subsets A,B,C having bipartitions $A=A_1\cup A_2$, $|A_1|=k_1$, $|A_2|=k_2$, $B=B_1\cup B_2$, $|B_1|=k_1-1$, $|B_2|=k_2+1$, $C=C_1\cup C_2$, $|C_1|=k_1+1$, $|C_2|=k_2-1$. By equating $e_m(\langle B\rangle)$ and $e_m(\langle C\rangle)$ we obtain the condition $2(k_1-k_2) \bmod m=0$. By equating $e_m(\langle A\rangle)$ and $e_m(\langle B\rangle)$ we obtain the condition $k_1-k_2 \bmod m=1$. This implies that $m=2$ and $k \bmod 2=k_1+k_2 \bmod 2=k_1-k_2 \bmod 2$, hence $k \bmod 2=1$. The second possibility in case 3 is solved, as before, by taking complements. This completes the proof of Theorem 1.1. $\square$

3 The local decomposition property

Proof of Theorem 1.2 We begin with the case $gcd(H)>1$. We apply induction on n, while k is fixed. The basis of the induction is $n=k+1$. Suppose that G is neither the complete nor the empty graph. Then there exist vertices u,v,w such that $(u,v)\in E$ but $(u,w)\notin E$. The degree of u in $\langle G\setminus v\rangle$ differs by one from the degree of u in $\langle G\setminus w\rangle$. Thus in one of these graphs $gcd(H)$ does not divide the degree of u, and hence it is not H-decomposable. Assuming we have proved our claim for $n-1$, we prove it for n. The induction hypothesis implies that every $n-1$-subset induces K_{n-1} or $\overline{K_{n-1}}$. Thus it immediately follows that $G\in\{K_n,\overline{K_n}\}$.

Suppose now that $gcd(H)=1$. Since every induced k-subgraph of G has an H-decomposition it follows that for every two k-subsets $A,B\subset V$, $e_{e(H)}(\langle A\rangle)=e_{e(H)}(\langle B\rangle)$. Hence by Theorem 1.1 we infer that if $e(H)=2$ then $G\subset\{K_n,\overline{K_n},K_{a,n-a},\overline{K_{a,n-a}}\}$, otherwise $G\subset\{K_n,\overline{K_n},K_{1,n-1},\overline{K_{1,n-1}}\}$. We now need to show that, given a graph G, we can tell in polynomial time if $G\in H(n,k)$. We show this according to the structure of G.

- If G is the empty graph $\overline{K_n}$, every k-subgraph of it is trivially H-decomposable.
- If $G=K_n$ then every k-subgraph is K_k, and we need to determine whether K_k is H-decomposable. A necessary condition (which is easily checked) is $e(H)\,|\,\binom{k}{2}$. This condition is also sufficient if $k>k_0=k_0(H)$, by Wilson's Theorem [14]. For $k\le k_0$ the problem is solved in constant time, as H is fixed.
- If $G=\overline{K_{1,n-1}}=K_{n-1}\cup K_1$ we need both K_k and K_{k-1} to be H-decomposable. Each is determined as in the previous case.

- If $G = K_{1,n-1}$ we must have $H = K_{1,h-1}$ with $h-1 \mid n-k-1$. This is clearly a necessary and sufficient condition which can be easily verified.

- If $G = K_{a,n-a}$ and $H = P_3 = K_{1,2}$, we must have, by Theorem 1.1 that $k \bmod 2 = 1$. Thus every k-subgraph of G is either the empty graph or it is complete bipartite with an even number of edges. In both cases it is H-decomposable according to a theorem of Caro and Schönheim [8] which states that a graph is P_3 decomposable if every connected component has an even number of edges.

- If $G = K_a \cup K_{n-a}$, $a \le n/2$ and $H = P_3$ we again must have k odd. Every k-subgraph of G is a union of an even and an odd clique where, according to [8], each must have an even number of edges in order to ensure P_3 decomposition. Thus each clique must have $0, 1 \bmod 4$ edges. This is only possible for $a = 1$.

- If $G = K_{a,n-a}$, $a \le n/2$ and $H = 2K_2$ we have, as before, that k must be odd. By Caro's Theorem [4] a graph G is has a $2K_2$ decomposition iff $e(G)$ is even, $\Delta(G) \le e(G)/2$ and $G \ne K_3 \cup K_2$. Thus, we must have $n - a < k - 1$, and since $k \le n-2$, we must also have $4 \le a \le n/2$. These conditions are also sufficient, by applying Caro's Theorem.

- If $G = K_a \cup K_{n-a}$, $a \le n/2$ and $H = 2K_2$ then by a parity argument $k \bmod 4 = 1$ since only in this case it is true that for every choice of $0 \le k_1 \le a$, $0 \le k_2 \le n-a$, $k_1 + k_2 = k$ we get the necessary condition $\binom{k_1}{2} + \binom{k_2}{2} \bmod 2 = 0$. In view of the forbidden $K_3 \cup K_2$ either $k \ge 9$, $k \bmod 4 = 1$ and $a \le n/2$ is unrestricted, or $k = 5$ and $a = 1$. $\square$

As an immediate corollary of Theorem 1.2 we have:

Corollary 3.1 *Let H be a fixed graph without isolated vertices. Deciding membership in $H(n,k)$ can be done in polynomial time for $1 \le k \le n-2$. If $gcd(H) > 1$, deciding membership in $H(n,n-1)$ can also be done in polynomial time.*

4 Hardness of $n-1$ decomposition of stars

Corollary 3.1 leaves open the complexity of deciding membership in $H(n,n-1)$ for graphs having $gcd(H) = 1$. The purpose of this section is to show that this problem is probably much harder, as it is NP-Complete even for a simple family of graphs, namely the stars with three or more edges. Note that for the star with two edges, P_3, we have the Theorem of Caro and Schönheim [8], mentioned in the previous section.

Proof of Theorem 1.3. Our first ingredient is the construction of a (fixed) graph H_k with the following properties:

1. H_k has $3k+2$ vertices, one vertex has degree 1 and the rest have have degree $k-1 \bmod k$.

2. H_k has a $K_{1,k}$ decomposition.

H_k is constructed as follows. The vertex set of H_k is $\{a_1,\dots,a_k,b_1,\dots,b_k,c_1,\dots,c_k,u,v\}$. The vertices $\{a_1,\dots,a_k,b_1,\dots,b_k\}$ induce a clique K_{2k}. It is well known (e.g. Wilson's Theorem) that K_{2k} is $K_{1,k}$-decomposable. We now add to H_k k copies of $K_{1,k}$ whose roots are the a_i's as follows. a_1 is connected to all $c_1,\dots,c_k$. a_i, for $i=2,\dots,k$ is connected to u and v and to all $c_2,\dots,c_k$ but not to c_i. Our construction shows that H_k is $K_{1,k}$-decomposable. The vertex c_1 has degree 1. The vertices $a_1,\dots,a_k$ have degree $3k-1$, the vertices $b_1,\dots,b_k$ have degree $2k-1$, and the vertices $c_2,\dots,c_k,u,v$ have degree $k-1$.

Denote by $H_{k,t}$ for $1 \le t \le k-1$ the union of t copies of H_k that intersect only in the unique degree 1 vertex of H_k. Thus, $H_{k,t}$ has $(3k+1)t+1$ vertices, all vertices but one having degree $k-1 \bmod k$, and one vertex (the "unifier") has degree t. Clearly, $H_{k,t}$ is $K_{1,k}$-decomposable.

We recall that by the theorem of Dor and Tarsi, deciding if a graph G is $K_{1,k}$-decomposable ($k \ge 3$ fixed) is NP-Complete. We perform a polynomial transformation from this problem to our problem by constructing a graph G' having the property that G has a $K_{1,k}$ decomposition iff the deletion of every vertex from G' induces a subgraph which has a $K_{1,k}$ decomposition. Given the input graph G, we first test if $k \mid e(G)$. If this is not the case then G is not $K_{1,k}$ decomposable and we are done. So we assume $k \mid e(G)$. We construct G' as follows:

For each vertex v of G with degree $t \bmod k$ we add to G a copy of $H_{k,k-1-t}$ by identifying v with the unifier vertex of a copy of $H_{k,k-1-t}$. (Note that if v already has degree $k-1 \bmod k$ we do not attach anything to it). Note that after this modification v has degree $k-1 \bmod k$, and the newly added $(3k+1)(k-1-t)$ vertices also have degree $k-1 \bmod k$. We do this for every vertex v and obtain the graph G'', which we shall later use to define G'. Note that G'' is constructed in polynomial time, and has $n'' \le n(3k+1)(k-1)$ vertices, where n is the number of vertices of G. Every vertex of G'' has degree $k-1 \bmod k$, and since G'' is the edge-disjoint union of G and copies of H_k, it is $K_{1,k}$-decomposable if G is. We claim that the converse is also true. Consider a $K_{1,k}$-decomposition of G'', and a copy of $K_{1,k}$ in such a decomposition. The edge that is adjacent to the degree 1 vertex of H_k is a bridge in G'' in every occurrence of H_k in G''. Since H_k is $K_{1,k}$-decomposable it follows that each copy of $K_{1,k}$ in the decomposition of G'' is either entirely within G or entirely within one of the added copies of H_k. Hence, G is also $K_{1,k}$-decomposable. Note also that $n'' \bmod k = 0$. To see this, note that the sum of the degrees of the vertices of G'' must divide $2k$ and is also $n''(k-1) \bmod k$. The graph G' is defined by adding to G'' a new vertex x, and connecting it to all vertices of G''. Thus, x has degree 0 mod k. Put $n' = n'' + 1$.

Suppose first that G is not $K_{1,k}$-decomposable. Then, G'' is also not $K_{1,k}$-decomposable, and $G'' = G' \backslash x$ is an $n'-1$-vertex induced subgraph of G'. Now, suppose G is $K_{1,k}$-decomposable. Thus, G'' is also $K_{1,k}$-decomposable. We claim that for each vertex $v \in G'$, $G' \setminus v$ is $K_{1,k}$-decomposable. This is clearly true if $v = x$. Otherwise, $v \in G''$. We construct a $K_{1,k}$-decomposition of $G' \setminus v$ from a given decomposition of G'' as follows. We replace each occurrence of v in the decomposition for G'' by x. We have used $deg(v)$ edges of x in this way. We still remain with $n'' - 1 - deg(v)$ unused edges of x. But $n'' \bmod k = 0$ and $deg(v) \bmod k = k - 1$ hence $k \mid n'' - 1 - deg(v)$, and we can decompose these edges into copies of $K_{1,k}$.

Finally, we note that the $H(n, n-1)$ recognition problem is in NP for every graph H by providing n distinct decompositions, one for each $n-1$ induced subgraph. $\square$

Note that the proof of Theorem 1.3 also shows that G' is $K_{1,k}$-decomposable if G'' is and hence if G is. This means that the following "intersection" problem is also NP-Complete: Given a graph G, is it, and all its $n-1$-vertex induced subgraphs, $K_{1,k}$-decomposable ($k \geq 3$).

5 Concluding remarks and open problems

We note that for some simple graphs H, deciding whether G is H-decomposable can be done in polynomial time. This holds, for example, whenever every connected component of H is an edge or when every connected component of H is a path of length 2. Although the Theorem of Dor and Tarsi shows that H-decomposition is NP-Complete whenever H has a connected component consisting of more than two edges, (for example if H is a triangle), it can be seen from Theorem 1.2 that $H(n, n-2)$ is easily recognizable for all graphs, and even $H(n, n-1)$ is, assuming $gcd(H) > 1$. A triangle provides a good example where decomposition is difficult, but local decomposition is easy, for all values of k.

It is interesting to find the complexity of deciding membership in $H(n, n-1)$ for graphs other than stars (for which it is NP-Complete) and for graphs other than the ones where H-decomposition is polynomial, or that have $gcd(H) > 1$ (for which it is polynomial).

References

[1] M. Aigner and E. Triesch, *Realizability and uniqueness in graphs*, Discrete Mathematics 136 (1994), 3-20.

[2] J. Bosak, *Decomposition of Graphs*, Kluner Academic, Dordrecht 1990.

[3] L. W. Beineke, W. Goddard and P. Hamburger, *Random packings of graphs*, Discrete Mathematics, in press.

[4] Y. Caro, *Chromatic decomposition of a graph*, Ars Combinatoria 21 (1986), 29-32.

[5] Y. Caro, *On graphs with prescribed subgraphs of order k and a theorem of Kelly and Merriell*, Czechoslovak Mathematical Journal 44 (1994), 623-629.

[6] Y. Caro, *On graphs determined by their k subgraphs*, Ars Combinatoria, to appear.

[7] Y. Caro, J. Rojas and S. Ruiz, *A forbidden subgraphs characterization and a polynomial time algorithm for randomly decomposable graphs*, Czechoslovak Mathematical Journal 46 (1996), 413-419.

[8] Y. Caro and J. Schönheim, *Decomposition of trees into isomorphic subtrees*, Ars Combinatoria 9 (1980), 119-130.

[9] Y. Caro, A. Sebo and M. Tarsi, *Recognizing greedy structures*, Journal of Algorithms 20 (1996), 137-156.

[10] Y. Caro and R. Yuster, *Recognizing global occurrence of local properties*, submitted.

[11] D. Dor and M. Tarsi, *Graph decomposition is NPC, a complete proof of Holyer's conjecture*, Proceedings of the 20^{th} ACM Symposium on the Theory of Computing, ACM Press (1992), 252-263.

[12] P. Kelley and D. Merriell, *A class of graphs*, Transactions of the American Mathematical Society 96 (1960), 488-492.

[13] D. P. Sumner, *Randomly Matchable Graphs*, Journal of Graph Theory 3 (1979), 183-186.

[14] R. M. Wilson, *Decomposition of complete graphs into subgraphs isomorphic to a given graph*, Congressus Numerantium XV (1975), 647-659.