Covering Graphs: The covering problem solved

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Abstract

For every fixed graph H, we determine the H-covering number of K_n, for all $n > n_0(H)$. We prove that if h is the number of edges of H, and $gcd(H) = d$ is the greatest common divisor of the degrees of H, then there exists $n_0 = n_0(H)$, such that for all $n > n_0$,

$$C(H, K_n) = \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil,$$

unless d is even, $n = 1 \bmod d$ and $n(n-1)/d + 1 = 0 \bmod (2h/d)$, in which case

$$C(H, K_n) = \lceil \frac{\binom{n}{2}}{h} \rceil + 1.$$

Our main tool in proving this result is the deep decomposition result of Gustavsson.

1 Introduction

All graphs considered here are finite, undirected and simple, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [4]. Let H be a graph without isolated vertices. An *H-covering* of a graph G is a set $L = \{G_1, \ldots, G_s\}$ of subgraphs of G, where each subgraph is isomorphic to H, and every edge of G appears in at least one member of L. The *H-covering number* of G, denoted by $C(H, G)$, is the minimum cardinality of an H-covering of G. An *H-packing* of a graph G is a set $L = \{G_1, \ldots, G_s\}$ of edge-disjoint subgraphs of G, where each subgraph is isomorphic to H. The *H-packing number* of G, denoted by $P(H, G)$, is the maximum cardinality of an H-packing of G. G has an *H-decomposition* if it has an H-packing which is also an H-covering. The H-covering and H-packing problems are, in general, NP-Complete as shown by Dor and Tarsi [8]. In case $G = K_n$, the H-covering and H-packing problems have attracted

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much attention in the last forty years, and numerous papers were written on these subjects (cf. [3, 7, 10, 14, 17, 18, 21] for various surveys). In a recent paper [6] the authors solved the H-packing problem, for K_n where $n \geq n(H)$. The purpose of this paper is to determine the H-covering number of K_n, for $n \geq n(H)$. In particular, our solution settles several special cases of the H-covering problem, which gained particular interest. Among them are:

1. $C(K_k, K_n)$ which has been linked to the Schonheim bound and the Túran numbers [3, 19]. Despite of much effort only the cases $k = 3$ [11] and $k = 4$ [15, 16] are solved. The case $k = 5$ is still open [1, 17].

2. $C(C_k, K_n)$ which is the cycle-system covering problem, solved completely only for $k = 3$ and $k = 4$ [20].

3. The *overlap* of an H-covering L of K_n is defined as the maximum number of appearances of an edge in members L. It is known [5] that if $n \geq n(H)$ then there exists an H-covering of K_n with overlap at most 2. Etzion [5] has conjectured that $CO(H, K_n) - C(H, K_n) \leq c(H)$ where $CO(H, K_n)$ is the minimum number of copies in an H-covering of K_n with overlap 2, and $c(H)$ is a constant depending only on H.

The H-decomposition problem of K_n is solved, for $n \geq n(H)$. This is due to the central theorem of Wilson [22], which states that for sufficiently large n, K_n has an H-decomposition if and only if $e(H) \mid \binom{n}{2}$ and $gcd(H) \mid n-1$ where $gcd(H)$ is the greatest common divisor of the degrees of H. In particular, whenever Wilson's conditions hold for K_n, the H-covering and H-packing numbers are known.

Our main result is the following:

Theorem 1.1 *Let H be a graph with h edges, and let gcd(H)=d. Then there exists $n_0 = n_0(H)$, such that for all $n > n_0$,*

$$C(H, K_n) = \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil,$$

unless d is even, $n = 1 \bmod d$ and $n(n-1)/d + 1 = 0 \bmod (2h/d)$, in which case

$$C(H, K_n) = \lceil \frac{\binom{n}{2}}{h} \rceil + 1.$$

2 Proof of the main result

As mentioned in the abstract, our main tool is the following result of Gustavsson [13]:

Lemma 2.1 (Gustavsson's Theorem [13]) *Let H be a graph with h edges. There exists $N = N(H)$, and $\epsilon = \epsilon(H) > 0$, such that for all $n > N$, if G is a graph on n vertices and m edges, with $\delta(G) \geq n(1-\epsilon)$, $gcd(H) \mid gcd(G)$, and $h \mid m$, then G has an H-decomposition.* □

It is worth mentioning that $N(H)$ in Gustavsson's Theorem is a rather huge constant; in fact, it is a highly exponential function of h.
A sequence of n positive integers $d_1 \geq d_2 \geq \ldots \geq d_n$ is called *graphic* if there exists an n-vertex graph whose degree sequence is $\{d_1, \ldots, d_n\}$. We shall need the following theorem of Erdös and Gallai [9], which gives a necessary and sufficient condition for a sequence to be graphic.

Lemma 2.2 (Erdös and Gallai [9]) *The sequence $d_1 \geq d_2 \geq \ldots \geq d_n$ of positive integers is graphic if and only if its sum is even and for every $t = 1, \ldots, n$*

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}. \tag{1}$$

□

Recall that a *multigraph* is a graph in which multiple edges and loops are allowed. During the rest of this sequel, all multigraphs considered are assumed to have no loops. The degree of a vertex v in a multigraph is defined as the number of edges adjacent to v, taking multiplicity into account (i.e. an edge with multiplicity k contributes k to the degrees of its adjacent vertices). The next lemma, which is somewhat technical, is crucial to our proof of Theorem 1.1.

Lemma 2.3 *Let H be a graph with $h \geq 2$ edges and no isolated vertices, let $h \geq a \geq 1$, and let $n \geq 13h^3$. Then, if R is an n-vertex multigraph with $\Delta(R) \leq a$, then there exists an n-vertex multigraph G with the following properties:*

1. *R is a spanning sub-multigraph of G.*
2. *$G \setminus R$ is a graph (i.e. the edges of G not belonging to R have multiplicity one).*
3. *$\Delta(G) \leq 4h^2$.*
4. *G has an H-decomposition.*

Proof: We shall prove the lemma by induction on $e(R)$, the number of edges of R. In fact, we will show that if $e(R) = k$, then one may construct G, having the properties guaranteed by the lemma, and such that $e(G) \leq kh$, and $d_G(v) \leq h \cdot d_R(v) + 3h^2 \leq ha + 3h^2 \leq 4h^2$ for every vertex v. The basis of the induction, $k = 0$, holds since in this case $G = R$ is the empty graph, and all properties trivially hold. Now suppose $e(R) = k + 1$. Put $R' = R \setminus \{(a, b)\}$ where (a, b) is an arbitrary edge of

R. Since $e(R') = k$, we have, according to the induction hypothesis, that there exists a multigraph G', with all the above properties, with respect to R'. If $(a, b) \in G'$, we may take $G = G'$, and we are done. Assume, therefore, that $(a, b) \notin G'$. Since $e(G') \leq kh$, and since $k = e(R') \leq na/2$ we have $e(G') \leq nah/2$. Thus, there are at least $n/2$ vertices with degree at most $2ah$ in G'. Since $\Delta(G') \leq 4h^2$ we have, therefore, that there are is a set of vertices X, with $|X| \geq n/2 - 8h^2 - 2$, such that for every $v \in X$, $d_{G'}(v) \leq 2ah$, $v \neq a$, $v \neq b$, $(v, a) \notin G'$ and $(v, b) \notin G'$. We claim that there is an independent set in G' containing $2h - 2$ vertices of X. To see this, it suffices to show that $|X|/(2ah + 1) \geq 2h - 2$. Indeed,

$$|X| \geq \frac{n}{2} - 8h^2 - 2 \geq (2h - 2)(2ah + 1)$$

since $n \geq 13h^3$ and $a \leq h$. Thus, if t denotes the number of vertices of H, then since $t \leq 2h$, we have that there exists a set $Y \subset X$ with $t - 2$ vertices such that $Z = Y \cup \{a, b\}$ is an independent set of G', with t elements. Embed a copy of H on the vertex set Z, such that (a, b) is an edge of this copy. Let F denote the set of edges of this copy. Clearly, $|F| = h$ and $(a, b) \in F$. Put $G = G' \cup F$. Our construction shows that:

1. R is a spanning subgraph of G.

2. $G \setminus R = (G' \setminus R') \cup (F \setminus \{a, b\})$. This is a disjoint union of two graphs, and therefore $G \setminus R$ is a graph.

3. If $v \notin Z$ then $d_G(v) = d_{G'}(v) \leq h \cdot d_{R'}(v) + 3h^2 \leq h \cdot d_R(v) + 3h^2$. If $v \in Y$ then $d_G(v) \leq d_{G'}(v) + h \leq 2ah + h \leq 2h^2 + h \leq h \cdot d_R(v) + 3h^2$. Finally, if $v \in \{a, b\}$ then $d_G(v) \leq d_{G'}(v) + h \leq h \cdot d_{R'}(v) + 3h^2 + h = h \cdot d_R(v) + 3h^2$. In any case, we have shown that $d_G(v) \leq h \cdot d_R(v) + 3h^2$ for every vertex v.

4. G has an H-decomposition since G' has an H decomposition and since $G = G' \cup F$ where F is a copy of H, and no edge of F appears in G'.

5. $e(G) = e(G') + h \leq kh + h = (k + 1)h$.

This completes the induction step, and hence the proof. □

Proof of Theorem 1.1: Given H, we choose $n_0 = n_0(H) = \max\{N(H), \frac{1+4h^2}{\epsilon(H)}, 8h\}$, where $N(H)$ and $\epsilon(H)$ are as in Lemma 2.1. Now let $n > n_0$. Let $n - 1 = -a \bmod d$, where $0 \leq a \leq d - 1$. Let $n(n - 1 + a)/d = -b \bmod (2h/d)$, where $0 \leq b \leq 2h/d - 1$. Note that since $d = gcd(H)$ and $2h$ is the sum of the degrees of H, $2h/d$ must be an integer. Also note that $(n - 1 + a)/d$ is an integer, and so b is well-defined. We shall use the obvious fact that $h \geq d(d + 1)/2$, since $\delta(H) \geq d$. This means that

$$n > n_0 \geq 8h > 4d^2 > (a+d)^2.$$

Another useful fact is that $bd + na$ is even since if d is even then a and n have different parity, and if d is odd then $2h/d$ is even and so if b is odd then a and n are both odd, and if b is even then either n is even or a is even. In the first part of the proof we shall give an upper bound for $C(H, K_n)$, and in the second part we shall give a lower bound for $C(H, K_n)$, and notice that the upper and lower bounds coincide.

Proving an upper bound for $C(H, K_n)$: We shall first assume that $a \neq 0$ or $b > 1$ (or both). Our first goal is to show the existence of an n-vertex multigraph, R, which has b vertices with degree $d + a$, and $n - b$ vertices with degree a. In case $a = 0$ we can clearly construct R by taking $n - b$ isolated vertices, and b vertices which span a d-regular multigraph. This can be done since $bd + na = bd$ is even, as noted above, and since $b > 1$. Note that if $b \leq d$ R must contain multiple edges, but if $b > d$ we can insist that R be a *graph*. In case $a \neq 0$ we shall show the existence of R by using Lemma 2.2, with $d_i = a + d$ for $i = 1, \ldots, b$ and $d_i = a$ for $i = b + 1, \ldots, n$. (This will imply that the resulting R is, in fact, a graph, and not a proper multigraph). Notice first that the sum of the sequence is $bd + na$ and this number is even as mentioned above. Let $1 \leq t \leq a + d$. In this case, (1) holds since

$$\sum_{i=1}^{t} d_i \leq t(a+d) = t(t-1) + t(a+d-t+1) \leq t(t-1) + (a+d)(a+d-1) =$$

$$t(t-1) + (a+d)^2 - (a+d) < t(t-1) + n - (a+d) \leq t(t-1) + (n-t) \leq t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}.$$

For $a + d \leq t \leq n$ we shall prove that (1) holds by induction on t, where the base case $t = a + d$ was proved above. If $t > a + d$ we use the induction hypothesis to derive that

$$\sum_{i=1}^{t} d_i = d_t + \sum_{i=1}^{t-1} d_i \leq d_t + (t-1)(t-2) + \sum_{i=t}^{n} \min\{t, d_i\} =$$

$$d_t + \min\{t, d_t\} - 2(t-1) + t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}$$

$$\leq (a+d) + (a+d) - 2(a+d) + t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\} = t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\}.$$

Thus, in any case, the desired multigraph R exists. Note that $\Delta(R) \leq d + a \leq 2d - 1 \leq d(d+1)/2 \leq h$. According to Lemma 2.3, there exists a multigraph G on n vertices, which contains R as a spanning submultigraph, $\delta(G) \leq 4h^2$, and G has an H-decomposition. Furthermore, the multigraph

F obtained from G by deleting the edges of R is, in fact, a graph. Let G^* be the graph obtained from K_n by deleting the edges of F. We claim that $d \mid gcd(G^*)$. To see this, note that the fact that G has an H-decomposition implies that $d \mid gcd(G)$. Since the degree of every vertex of R is a mod d, it follows that the degree of every vertex of F is $(-a)$ mod d. Since the degree of every vertex of K_n is $n-1 = (-a)$ mod d, it follows that the degree of every vertex of G^* is 0 mod d. Now we claim that $e(G^*)$, the number of edges of G^*, is 0 mod h. This is because $e(G) = 0$ mod h, and

$$e(G^*) = \binom{n}{2} - e(G) + e(R) = \binom{n}{2} - e(G) + \frac{bd+na}{2} = \frac{d}{2}(\frac{n(n-1+a)}{d} + b)) - e(G) = 0 \bmod h.$$

Also note that $\delta(G^*) \ge n - 1 - 4h^2 = n(1 - \frac{1+4h^2}{n}) \ge n(1 - \epsilon(H))$, since $n > n_0 \ge \frac{1+4h^2}{\epsilon(H)}$. Thus, G^* satisfies the conditions of Lemma 2.1, and therefore G^* has an H-decomposition. The union of the H-decomposition of G^* and the H-decomposition of G yields a covering of K_n in which all the edges of K_n, but the edges of R, are covered once. Furthermore, if an edge of R has multiplicity t, then this edge is covered $t+1$ times in the resulting H-covering of K_n. The overall number of copies of H in both decompositions is, therefore, exactly $(\binom{n}{2} + e(R))/h$. Thus,

$$C(H, K_n) \le \frac{\binom{n}{2} + e(R)}{h} = \frac{\binom{n}{2} + (bd+na)/2}{h} = \frac{d}{2h}(\frac{n(n-1+a)}{d} + b)) = \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil.$$

We now deal with the case $a = 0$ and $b = 0$. Note that in this case K_n satisfies the condition in Wilson's Theorem [22], (or according to Lemma 2.1), so, trivially,

$$C(H, K_n) = \frac{\binom{n}{2}}{h} = \frac{dn}{2h}\frac{n-1}{d} = \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil.$$

The only remaining case is $a = 0$ and $b = 1$. This can only happen if d is even, since, recall, $bd + na$ is always even. In this case we create a graph R on $1 + \frac{2h}{d}$ vertices which is d regular (we then add to R a set of $n - 1 - \frac{2h}{d}$ isolated vertices to obtain an n-vertex graph). This can be done since $h \ge d(d+1)/2$ which implies $d < \frac{2h}{d} < \frac{2h}{d} + 1$. Once again, since $\Delta(R) = d \le h$, using Lemma 2.3 we obtain an n-vertex *graph* G, containing R as a subgraph, $\Delta(G) \le 4h^2$, and G has an H-decomposition. As in the case where $a \ne 0$, the graph G^* obtained from K_n be first deleting the edges of G and then returning the edges of R, satisfies the conditions of Lemma 2.1, and thus G^* has an H-decomposition, and the union of the H-decomposition of G and the H-decomposition of G^* forms a covering of K_n where every edge is covered once, but the edges of R which are covered twice. The overall number of copies of H in both decompositions is, therefore, exactly $(\binom{n}{2} + e(R))/h$. Thus,

$$C(H, K_n) \le \frac{\binom{n}{2} + e(R)}{h} = \frac{\binom{n}{2} + h + d/2}{h} = \frac{\binom{n}{2} + d/2}{h} + 1 = \lceil \frac{\binom{n}{2}}{h} \rceil + 1.$$

Proving a lower bound for $C(H, K_n)$: Let L be an arbitrary H-covering of K_n. Let s denote the cardinality of L. Let G be the n-vertex multigraph obtained by the edge-union of all the members of L. That is, an edge of G has multiplicity k if it appears in k members of L. Clearly, G contains sh edges. Since K_n is a spanning *subgraph* of G, we may define the multigraph $G^* = G \setminus K_n$. G^* contains $sh - \binom{n}{2}$ edges. The degree of every vertex in G is $0 \bmod d$ and so the degree of every vertex in G^* is $a \bmod d$. Therefore, the number of edges in G^* satisfies

$$sh - \binom{n}{2} = \frac{an + cd}{2}$$

for some non-negative integer c. In particular, $\binom{n}{2} = (-\frac{an+cd}{2}) \bmod h$. This, in turn, implies that $n(n-1+a)/d = (-c) \bmod (2h/d)$. Thus, we must have $c \geq b$. Therefore,

$$s = \frac{\binom{n}{2} + \frac{an+cd}{2}}{h} \geq \frac{\binom{n}{2} + \frac{an+bd}{2}}{h} = \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil.$$

Since L was an arbitrary H-covering, we have

$$C(H, K_n) \geq \lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil.$$

We must now show that in case $a = 0$ and $b = 1$, the last bound can be improved by 1. To see this, note that in this case we cannot have $c = 1$. This is because every non-isolated vertex of G^* has degree at least d, and therefore there are at least $d(d+1)/2$ edges in G^*, and since the number of edges in G^* is $cd/2$, we cannot have $c = 1$. We must, therefore have $c \geq b + 2h/d$. Therefore,

$$s = \frac{\binom{n}{2} + \frac{an+cd}{2}}{h} \geq \frac{\binom{n}{2} + \frac{an+(b+2h/d)d}{2}}{h} = \frac{\binom{n}{2}}{h} + \frac{d}{2h} + 1 = \lceil \frac{\binom{n}{2}}{h} \rceil + 1.$$

□

3 Concluding remarks

1. Theorem 1.1, applied to $H = K_k$ yields, for $n \geq n_0(k)$, that

$$C(K_k, K_n) = \lceil \frac{n}{k} \lceil \frac{n-1}{k-1} \rceil \rceil,$$

unless $k-1$ is even and $k-1 \mid n-1$ and $n(n-1)/(k-1) + 1 = 0 \bmod k$, in which case the above formula should be increased by 1.

2. Theorem 1.1, applied to $H = C_k$ yields, for $n \geq n_0(k)$, that

$$C(C_k, K_n) = \lceil \frac{n}{k} \lceil \frac{n-1}{2} \rceil \rceil,$$

unless n is odd $\binom{n}{2} + 1 = 0 \bmod k$.

3. If $n \geq n_0(H)$ and $gcd(H) = 1$, then $C(H, K_n) = \lceil \frac{\binom{n}{2}}{e(H)} \rceil$. This bound can also be obtained from the packing bound, as shown in [6] where it is proved that in this case, $P(H, K_n) = \lfloor \frac{\binom{n}{2}}{e(H)} \rfloor$.

4. The proof of the upper bound in Theorem 1.1 shows that whenever $n - 1 \neq 0 \bmod d$, or whenever $n - 1 = 0 \bmod d$ and $b \in \{0, 1, d + 1, d + 2, \ldots\}$ the multigraph R is, in fact, a graph. Thus the obtained optimum covering has overlap 2. This shows that whenever $n \geq n_0(H)$, and n and b satisfy the above, $CO(H, K_n) = C(H, K_n)$. In case $n - 1 = 0 \bmod d$ and $2 \leq b \leq d$, we can replace the multigraph R which has b vertices with degree d, with a *graph* R' with $b + 2h/d$ vertices, which is d-regular, (as shown there in the case $b = 1$). Thus, in this case, $CO(H, K_n) = C(H, K_n) + 1$. This solves and sharpens the problem posed by Etzion, mentioned in the introduction. In fact, by modifying the proof of Lemma 2.3, we can guarantee that G has an H-decomposition in which every copy of H contains *exactly one* edge from R. This, in turn, shows that an optimal 2-overlap covering with $CO(H, K_n)$ copies can be obtained with the additional property that every copy in the covering has at most one edge which is covered twice. (See [2, 12] which deal with this type of covering). This can be done by defining the graph R' to be the multigraph obtained from the graph R by replacing each edge with two multiple edges. Now, construct G, as in Lemma 2.3, which contains R', has an H-decomposition, and every copy of H in the decomposition contains exactly one edge from R'. Now, as in the proof of Theorem 1.1, the graph $K_n \setminus (G \setminus R') \setminus R$, satisfies Gustavsson's Theorem, and its H-decomposition, together with the H-decomposition of G, is a covering with $CO(H, K_n)$ members, where each member has at most one edge which is covered twice (in fact, only the edges of R are covered twice). Note the interesting fact that there are infinitely many values of d and n, in which d is even, $n - 1 = 0 \bmod d$, $b = 2 \leq d$, and thus every realization of $C(H, K_n)$ contains an edge which is covered d times (since in this case R is a multigraph with 2 vertices having d multiple edges connecting them). However, since $CO(H, K_n) = C(H, K_n) + 1$ in this case, it follows that at a price of one more copy of H, one can obtain a covering with overlap 2, in which every copy contains at most one edge which is covered twice.

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