Colorful monochromatic connectivity

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Abstract

An edge-coloring of a connected graph is *monochromatically-connecting* if there is a monochromatic path joining any two vertices. How "colorful" can a monochromatically-connecting coloring be? Let $mc(G)$ denote the maximum number of colors used in a monochromatically-connecting coloring of a graph G. We prove some nontrivial upper and lower bounds for $mc(G)$ and relate it to other graph parameters such as the chromatic number, the connectivity, the maximum degree, and the diameter.

1 Introduction

An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (*MC-coloring*, for short) if there is a monochromatic path joining any two vertices. How "colorful" can an MC-coloring be? This question is the natural opposite of the recently well-studied problem of rainbow-connecting colorings [1, 2, 3, 4], where in the latter we seek to find an edge-coloring with minimum number of colors so that there is a rainbow path joining any two vertices.

Let $mc(G)$ denote the maximum number of colors used in an MC-coloring of a graph G. A straightforward lower bound for $mc(G)$ is $m - n + 2$ (throughout this paper n and m denote the number of vertices and edges, respectively). Simply color the edges of a spanning tree with one color, and the remaining edges may be assigned other distinct colors. In particular, $mc(G) = m - n + 2$ whenever G is a tree. However, some graphs can be colored with more colors. Indeed, in the extremal case one has $mc(K_n) = m = \binom{n}{2}$, and clearly $G = K_n$ is the only graph having $mc(G) = m$.

While trees have $mc(G) = m - n + 2$, our first result shows that there are dense graphs that still meet the lower bound.

Theorem 1 *Let G be a connected graph with $n > 3$. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

a. $\overline{G}$ (the complement of G) is 4-connected.

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b. *G is triangle-free.*

c. $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. *In particular, this holds if* $\Delta(G) \le (n+1)/2$, *and this also holds if* $\Delta(G) \le n - 2m/n$.

d. $Diam(G) \ge 3$.

e. *G has a cut vertex.*

Notice that none of the five properties in Theorem 1 imply any other property in the list. Moreover, each of the stated conditions is sharp. The proof of Theorem 1 and constructions demonstrating the sharpness of the conditions are given in Section 2.

The chromatic number dictates an upper bound on $mc(G)$. In Section 3 it is proved that $mc(G) \le m - n + \chi(G)$. An important ingredient is proving that $mc(G) = m - n + r$ when G is a complete r-partite graph. Likewise, the connectivity dictates an upper bound on $mc(G)$. It is proved that if the connectivity is r, then $mc(G) \le m - n + r + 1$. We summarize these upper bounds in the following theorem.

Theorem 2 *Let G be a connected graph.*

a. $mc(G) \le m - n + \chi(G)$.

b. *If G is not r-connected, then* $mc(G) \le m - n + r$.

Theorem 2 is proved in Section 3. Notice that Theorem 2 implies that $mc(G) \le m - n + \delta(G) + 1$. Section 3 also contains a characterization of graphs for which $mc(G) = m - n + \delta(G) + 1$.

2 Graphs attaining the lower bound

An important property of an extremal MC-coloring (a coloring that uses $mc(G)$ colors) is that each color forms a *tree*. Indeed, if there is an MC-coloring that has a monochromatic cycle, it is possible to choose any edge on this cycle and color it with a fresh color while still maintaining an MC-coloring. Likewise, if the subgraph formed by the edges having a given color is disconnected, then a fresh color can be assigned to all the edges of some component while still maintaining an MC-coloring. For the rest of this paper we will use this fact without further mentioning it. For a color c, denote by T_c the tree consisting of the edges colored c. We call T_c the *color tree* of the color c. The color c is *nontrivial* if T_c has at least two edges. Otherwise, c is *trivial*. A nontrivial color tree with m edges is said to *waste* $m-1$ colors. The following lemma shows that one can always find an extremal MC-coloring where, for any two nontrivial colors c and d, the corresponding trees T_c and T_d intersect in at most one vertex. Such an extremal coloring is called *simple*.

Lemma 2.1 *Every connected graph G has a simple extremal MC-coloring.*

Proof: Consider an extremal MC-coloring with the most number of trivial colors. We prove that this coloring must be simple. Let c and d be two nontrivial colors such that T_c and T_d contain k common vertices, with $k \geq 2$. Let e_c and e_d denote the number of edges in T_c and T_d, respectively. The subgraph H consisting of the edges of $T_c \cup T_d$ is connected; its number of vertices is $(e_c+1)+(e_d+1)-k$, which equals e_c+e_d+2-k. Now, instead of coloring the edges of H with c and d, color a spanning tree of H with c, and give each of the remaining $k-1$ edges of H fresh new colors. Clearly, the new coloring is also an MC-coloring. Now, if $k > 2$ this new coloring uses more colors than our original one, contradicting the fact that our original coloring was extremal. If $k = 2$ this new coloring uses the same number of colors as our original one but has more trivial colors, contradicting the assumption. ■

The proof of Theorem 1 is split into four parts. In Theorem 3 it is proved that $mc(G) = m-n+2$ when $\overline{G}$ is 4-connected. In Theorem 4 it is proved that $mc(G) = m-n+2$ when G is triangle-free. In Theorem 5 it is proved that $mc(G) = m-n+2$ when $\Delta(G) < n-(2m-3(n-1))/(n-3)$. The proof of the fact that $mc(G) = m-n+2$ when $\mathrm{diam}(G) \geq 3$ is given in Proposition 6. Part (e) of Theorem 1 is a special case of the second part of Theorem 2, whose proof is given in Section 3.

Theorem 3 *If $\overline{G}$ is 4-connected, then $mc(G) = m-n+2$.*

Proof: Let f be a simple extremal MC-coloring of G. Suppose that f consists of k nontrivial color trees, denoted $T_1, \ldots, T_k$, and set $t_i = |V(T_i)|$. As T_i has $t_i - 1$ edges, it wastes $t_i - 2$ colors. Hence, it suffices to prove that $\sum_{i=1}^{k}(t_i - 2) \geq n-2$. The fact that $\overline{G}$ is connected implies that each vertex must appear in some nontrivial color tree. Hence, if $k = 1$, then $t_1 = n$ and we are done, so assume $k \geq 2$.

Partition the set of nontrivial color trees into two parts, those with $t_i \geq 4$ (large trees) and those with $t_i = 3$ (small trees, isomorphic to a star with two leaves). Assume that there are ℓ large trees and $s = k-\ell$ small trees, and that $T_1, \ldots, T_\ell$ are the large trees. Note that no T_i is spanning.

Partition $V(G)$ into three types of vertices as follows: V_1 consists of all the vertices that appear only in small trees; V_2 consists of all the vertices that appear in at least two large trees, or appear in a large tree and also as a leaf of a small tree; V_3 consists of all the vertices that appear in precisely one large tree and possibly in small trees, but only as non-leaves. Indeed, $V_1 \cup V_2 \cup V_3 = V(G)$, since each vertex appears in some nontrivial tree.

Each $v \in V_1$ appears in at least 4 small trees as a leaf. This follows from the fact that $\overline{G}$ has minimum degree at least 4, and hence v has at least 4 nonneighbors in G. A small tree in which v is a leaf can monochromatically connect v to precisely one nonneighbor. In particular, it follows that the number, s, of small trees satisfies $s \geq 2|V_1|$.

We also claim that each large tree T_i contributes at least 4 vertices to V_2. If $V(T_i) \subset V_2$, then we are done. Otherwise, let $v \in V(T_i) \cap V_3$. All of the neighbors of v in $\overline{G}$ are in T_i. This means that

$V(T_i) \cap V_2$ disconnects $\overline{G}$. Indeed, $V(T_i) \cap V_2$ separates $V(T_i) \cap V_3$ from the set of vertices outside T_i (recall that T_i is not spanning). Since $\overline{G}$ is 4-connected, this implies that $|V(T_i) \cap V_2| \geq 4$.

For a large tree, let $|V(T_i) \cap V_2| = m_i$. Summing the orders of the large trees, we have

$$\sum_{i=1}^{\ell} t_i = |V_3| + \sum_{i=1}^{\ell} m_i = n - |V_1| - |V_2| + \sum_{i=1}^{\ell} m_i \; .$$

Consider first the case where $|V_2| \leq (\sum_{i=1}^{\ell} m_i)/2$. Recalling that $s \geq 2|V_1|$, we have

$$\sum_{i=1}^{k}(t_i - 2) = (\sum_{i=1}^{\ell} t_i) - 2\ell + s = n - |V_1| - |V_2| + (\sum_{i=1}^{\ell} m_i) - 2\ell + s \geq$$

$$n - \frac{s}{2} - \frac{\sum_{i=1}^{\ell} m_i}{2} + (\sum_{i=1}^{\ell} m_i) - 2\ell + s = n + \frac{s}{2} + \frac{\sum_{i=1}^{\ell} m_i}{2} - 2\ell \geq n + \frac{s}{2} + \frac{4\ell}{2} - 2\ell = n + \frac{s}{2} > n - 2$$

as required.

Consider next the case where $|V_2| > (\sum_{i=1}^{\ell} m_i)/2$. For $v \in V_2$, let $x(v)$ be the number of large trees v appears in, and let $y(v)$ be the number of small trees in which v is a leaf. By the definition of V_2, $x(v)+y(v) \geq 2$. On the other hand, $\sum_{v \in V_2} x(v) = \sum_{i=1}^{\ell} m_i$. Hence, $\sum_{v \in V_2} y(v) \geq 2|V_2| - (\sum_{i=1}^{\ell} m_i)$. Since each $v \in V_1$ appears in at least 4 small trees as a leaf and since each small tree has two leaves, the number of small trees is at least $2|V_1| + |V_2| - (\sum_{i=1}^{\ell} m_i)/2$. Hence, $|V_2| \leq s - 2|V_1| + (\sum_{i=1}^{\ell} m_i)/2$. We thus have

$$\sum_{i=1}^{k}(t_i - 2) = (\sum_{i=1}^{\ell} t_i) - 2\ell + s = n - |V_1| - |V_2| + (\sum_{i=1}^{\ell} m_i) - 2\ell + s \geq$$

$$n - s - \frac{\sum_{i=1}^{\ell} m_i}{2} + |V_1| + (\sum_{i=1}^{\ell} m_i) - 2\ell + s = n + \frac{\sum_{i=1}^{\ell} m_i}{2} - 2\ell + |V_1| \geq n + \frac{4\ell}{2} - 2\ell + |V_1| = n + |V_1| > n - 2 \; ,$$

as required. ■

One cannot hope to strengthen Theorem 3 by replacing the 4-connectedness requirement of $\overline{G}$ with 3-connectedness. We construct a graph G such that $\overline{G}$ is 3-connected and $mc(G) \geq m - n + 4$. The complement $\overline{G}$ of our graph G is an edge-disjoint union of 8 copies of $K_6 - C_6$, obtained by the following construction. Number the vertices of $\overline{G}$ by $\{0, \ldots, 35\}$. Start by placing 6 copies of $K_6 - C_6$ on the vertices $\{5i, 5i+1, \ldots, 5i+5\}$ for $i = 0, \ldots, 5$ (in this numbering vertex 30 is vertex 0). For each copy, the "missing" C_6 is $(5i, 5i+2, 5i+4, 5i+1, 5i+5, 5i+3)$. Place two additional copies of $K_6 - C_6$ as follows: the first copy is on the vertices $\{4, 14, 24, 31, 33, 35\}$, where the missing C_6 is $(4, 35, 14, 31, 24, 33)$; the second copy is on the vertices $\{9, 19, 29, 30, 32, 34\}$, where the missing C_6 is $(9, 30, 19, 32, 29, 34)$. The graph $\overline{G}$ is depicted in Figure 1. It is easy to verify that it is 3-connected. Consider its complement G. Each of the 8 copies of $K_6 - C_6$ in $\overline{G}$ becomes a C_6 in G, so one can

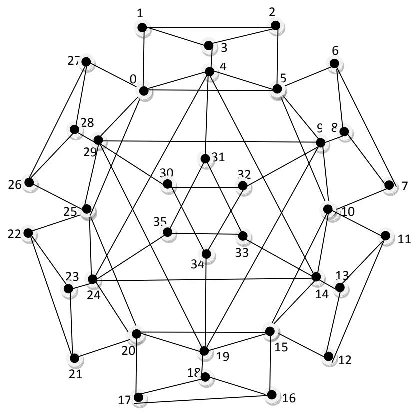

Figure 1: The 3-connected complement of a graph with $mc(G) \geq m - n + 4$.

pick a path on six vertices in each of them. Color each of these paths monochromatically, with a distinct color for each path, using altogether 8 colors. The other edges of G receive fresh distinct colors. This is an MC-coloring, and the number of wasted colors is 32, which equals $n - 4$. Hence, $mc(G) = m - n + 4$, as required.

Theorem 4 *If G is K_3-free, then $mc(G) = m - n + 2$.*

Proof: Let f be a simple extremal MC-coloring of G. Suppose that f consists of k nontrivial color trees, denoted $T_1, \ldots, T_k$, where $t_i = |V(T_i)|$. As T_i has $t_i - 1$ edges, it wastes $t_i - 2$ colors. Hence, it suffices to prove that $\sum_{i=1}^{k}(t_i - 2) \geq n - 2$. If G has a vertex with degree $n - 1$, then G must be a star and we are done. Otherwise, each vertex appears in at least one of the T_i's.

Consider first the case where every vertex appears in at least two distinct nontrivial color trees. In this case we have $\sum_{i=1}^{k} t_i \geq 2n$. So, if $k \leq n/2 + 1$, we have $\sum_{i=1}^{k}(t_i - 2) \geq 2n - 2k \geq n - 2$, and we are done. Assume therefore that $k > n/2 + 1$. Since G is triangle-free, it contains at most $n^2/4$ edges. Hence, $\overline{G}$ contains at least $\binom{n}{2} - n^2/4$ edges. Since T_i can monochromatically connect at most $\binom{t_i - 1}{2}$ pairs of nonneighbors in G, we must have

$$\sum_{i=1}^{k} \binom{t_i - 1}{2} \geq \binom{n}{2} - n^2/4 \,. \tag{1}$$

Assume that $\sum_{i=1}^{k}(t_i - 1) < n - 2 + k$. As each T_i is nontrivial, we have $t_i - 1 \geq 2$. By straightforward convexity, the expression $\sum_{i=1}^{k} \binom{t_i - 1}{2}$, subject to $t_i - 1 \geq 2$, is maximized when $k - 1$ of the t_i's equal 3 and one of the t_i's, say t_k, is as large as it can be, namely $t_k - 1$ is the largest integer smaller than

$n-2+k-2(k-1)=n-k$. Hence, $t_k-1=n-k-1$. Even in this extremal case, we have

$$\sum_{i=1}^{k}\binom{t_i-1}{2} \le (k-1)+\binom{n-k-1}{2} < \binom{n}{2}-n^2/4\ ,$$

where we have used the fact that $k>n/2+1$. As the last inequality contradicts (1), our assumption that $\sum_{i=1}^{k}(t_i-1)<n-2+k$ is false, and hence $\sum_{i=1}^{k}(t_i-2)\ge n-2$ as required.

It may now be assumed that there are vertices that appear in unique nontrivial color trees. Assume first that v appears only in T_i and u appears only in T_j, with $j\neq i$. If T_i and T_j are disjoint, then let $w\in V(T_j)$ be a neighbor of u. Now $\{v,u,w\}$ induces a triangle in G, a contradiction. So T_i and T_j are not disjoint, and since f is simple, $|V(T_i)\cap V(T_j)|=1$. Let w be their unique common vertex. If w is not a neighbor of u in T_j, then there is some other neighbor of u in T_j, say u'. Now, as before, $\{v,u,u'\}$ induces a triangle, a contradiction. It may now be assumed that w is a neighbor of u, and, symmetrically, it may be assumed that w is a neighbor of v. Now, $\{u,v,w\}$ induces a triangle, a contradiction.

It may now be assumed that all the vertices that appear in a single nontrivial color tree are all in the same tree, say T_1. Let $v\in V(T_1)$ appear only in T_1, and let $w\in V(T_1)$ be a neighbor of v. Let $u\in V(T_2)$, where $u\notin V(T_1)$. If u and w are adjacent, then $\{v,w,u\}$ induces a triangle, which is a contradiction. Otherwise, there is some other tree T_j that contains both u and w. This implies that $V(T_1)\cap V(T_j)=\{w\}$. Let $u'\in V(T_j)$ be a neighbor of u. Now $\{v,u,u'\}$ induces a triangle, which is a contradiction, and we are done. ■

One cannot hope to strengthen Theorem 4 by replacing the triangle-freeness requirement with K_4-freeness. The graph $K_1\vee P_{n-1}$ composed of a path with $n-1$ vertices and an additional vertex connected to all the vertices of the path is a K_4-free graph with $mc(G)\ge m-n+3$ (in fact, $mc(G)=m-n+3$ in this case). Color the $n-2$ edges of the path with a single color and each of the remaining $n-1$ edges with distinct colors. This is an MC-coloring using $m-n+3$ colors.

Theorem 5 *If* $\Delta(G)<n-\frac{2m-3(n-1)}{n-3}$, *then* $mc(G)=m-n+2$.

Proof: Let $d=\Delta(G)$. The case $d=2$ of the theorem is trivial, so assume $d\ge 3$.

Let f be a simple extremal MC-coloring of G with the maximum possible number of trivial colors, and assume that no color tree is spanning (otherwise $mc(G)=m-n+2$ and we are done). For every color tree T and for every vertex $v\notin T$, we have $deg(v)\ge|T|$. Consider the monochromatic paths (including single edges) from v to the $|T|$ vertices of T. These paths are internally vertex disjoint since f is simple. Hence, $deg(v)\ge|T|$. In particular, no tree has more than d vertices.

Suppose now that $mc(G)\ge m-n+3$. Clearly, in this case, f contains at most $n-3$ nontrivial trees. Let $T_1,\ldots,T_k$ denote all the nontrivial trees in f. We claim that $\sum_{i=1}^{k}|E(T_i)|\le n+k-3$. Suppose $\sum_{i=1}^{k}|E(T_i)|\ge n+k-2$. Consider the subgraph G' consisting of the union of the T_i and suppose that it has r components. Take a spanning tree in each component and give its edges a new color. Also give all the other edges of G' fresh distinct colors. The new coloring of G' uses at least $k-2+2r \geq k$ colors. So, it either uses more than k colors, or uses k colors but more trivial colors, contradicting the assumption on f. Thus, $\sum_{i=1}^{k} |E(T_i)| \leq n+k-3$. Each T_i can monochromatically connect at most $\binom{|E(T_i)|}{2}$ independent pairs of vertices. The total number of independent pairs to be monochromatically connected is $\binom{n}{2} - m$, so

$$\binom{n}{2} - m \leq \sum_{i=1}^{k} \binom{|E(T_i)|}{2}.$$

We must optimize $\sum_{i=1}^{k} \binom{|E(T_i)|}{2}$ subject to $|E(T_i)| \leq d-1$ and to $\sum_{i=1}^{k} |E(T_i)| \leq n+k-3$. This quantity reaches its maximum when all but at most one of the $|E(T_i)|$ equal $d-1$; hence $n+k-3 \geq (k-1)(d-1)+q$ for some $q \leq d-1$. This implies that $k-1 \leq (n-2-q)/(d-2)$. It follows that

$$\binom{n}{2} - m \leq \frac{n-2-q}{d-2}\binom{d-1}{2} + \binom{q}{2} = \frac{(n-2)(d-1)}{2} - \frac{q}{2}(d-q) \leq \frac{(n-3)(d-1)}{2}.$$

This implies that $d \geq n - (2m - 3(n-1))/(n-3)$, contradicting the assumption. Hence, we must have $mc(G) = m-n+2$. ■

Theorem 5 gives, in particular, that for $n \geq 3$, if $\Delta(G) \leq (n+1)/2$, then $mc(G) = m-n+2$. The following example shows that this is tight. Let G have $2s-2$ vertices and be formed from $K_{s,s-2}$ by adding a path P_s in the larger side of $K_{s,s-2}$. Color this path with the color 1, and color a star with center at the larger side and $s-2$ leaves in the smaller side with the color 2. The remaining colors are trivial. This is an MC-coloring with $m-n+3$ colors, while the maximum degree is $s = n/2+1$.

This section ends with a short proof showing that diameter at least 3 implies $mc(G) = m-n+2$.

Proposition 6 *If* $\mathrm{diam}(G) \geq 3$*, then* $mc(G) = m-n+2$.

Proof: Suppose u and v are two vertices having distance at least 3. Let X be the set of neighbors of u, and let $Y = V(G) \setminus (X \cup \{u,v\})$. Put $x = |X|$ and $y = n-2-x = |Y|$. Let f be a simple extremal coloring, and let T be the nontrivial color tree containing both u and v. Suppose T contains t edges (note that $t \geq 3$), and thus T contains precisely $t-1$ vertices of $X \cup Y$. Suppose that T contains t_x vertices of X and $t_y = t-1-t_x$ vertices of Y. Thus, T does not cover $x - t_x$ vertices of X and it does not cover $y - t_y$ vertices of Y. The coloring f must therefore contain trees $R_1, \ldots, R_r$ connecting u to the $y - t_y$ uncovered vertices of Y. Since f is simple, any two of these trees intersect only at u. Since each of these trees must contain a vertex of X, the total number of edges of $R_1, \ldots, R_r$ is at least $y - t_y + r$. Similarly, f contains trees $Q_1, \ldots, Q_q$ connecting v to the $x - t_x$ vertices of X (these trees are all distinct from the trees $R_1, \ldots, R_r$ since no other tree but T may contain both u and v). The total number of edges of $Q_1, \ldots, Q_q$ is at least $x - t_x + q$. The sum of the waste of the trees

$T, R_1, \ldots, R_r, Q_1, \ldots, Q_q$ is at least $(t-1)+(y-t_y)+(x-t_x) = y+x$, which equals $n-2$, showing that $mc(G) = m-n+2$, as required. ∎

3 Upper bounds

We have already seen that $mc(G) = m-n+2$ when G is bipartite. More generally, it will be shown that if $\chi(G) = r$, then $mc(G) \leq m-n+r$. We first consider complete r-partite graphs and then turn to other r-chromatic graphs.

Theorem 7 *If G is a complete r-partite graph, then $mc(G) = m-n+r$.*

Proof: The case $r = 2$ is a special case of Theorem 4, so assume $r \geq 3$. Consider a simple extremal MC-coloring. We claim that no color tree T_c can have vertices in more than two vertex classes. Suppose that $V(T_c)$ intersects t vertex classes, say $V_1, \ldots, V_t$, and that $t \geq 3$. Let $P_i = V(T_c) \cap V_i$ and $p_i = |P_i|$.

Observe that T_c has $(\sum_{i=1}^t p_i) - 1$ edges, and since the coloring is simple, all other edges of G induced by $\bigcup_{i=1}^t P_i$ are of trivial trees. Overall, $\bigcup_{i=1}^t P_i$ contains $(\sum_{1 \leq i < j \leq t} p_i p_j) - (\sum_{i=1}^t p_i) + 2$ colors. We change the coloring induced by $\bigcup_{i=1}^t P_i$. One vertex from P_i will be adjacent to all vertices of P_{i+1} by a fresh color, call it c_i, for $i = 1, \ldots, t$ (cyclically, that is, a vertex of P_t is adjacent to all other vertices of P_1 by color c_t). All other edges induced by $\bigcup_{i=1}^t P_i$ receive trivial colors. The new coloring is also an MC-coloring, but it now uses $(\sum_{1 \leq i < j \leq t} p_i p_j) - (\sum_{i=1}^t p_i) + t$ colors, contradicting the assumption that our original coloring is extremal.

So, if f is a simple extremal MC-coloring, then each color tree intersects precisely two vertex classes. We further claim that it is possible to find such an f in which each color tree is a star. Suppose that some T_c intersects vertex classes V_1 and V_2 with at least two vertices in each, say $p_i \geq 2$ vertices in V_i. Let P_i denote the corresponding sets of vertices in these classes. Since $r \geq 3$, there is a vertex v in V_3, and all the edges from v to $P_1 \cup P_2$ are of different colors, as f is simple. Recolor G so that a vertex from P_1 is adjacent to all vertices of P_2 by one color and v is adjacent to all vertices of P_1 by another color, say color d. Observe that the new tree T_d is a star whose center is v. All the other edges induced by $P_1 \cup P_2 \cup \{v\}$ are colored with other distinct colors.

We have obtained a new extremal coloring f' (notice that the number of colors used by f and f' is the same). However, the number of color trees that intersect two vertex classes in at least two vertices has been reduced. Furthermore, each color tree still has vertices in exactly two vertex classes. Observe, however, that f' need not be simple. It may be the case that T_d, which is a star having vertices in $V_1 \cup V_3$, intersects other original color trees (trees that are not affected by the change from f to f') in at least two vertices, one of which is v and the other is in V_1. Observe that any such original color tree has vertices in $V_1 \cup V_3$ and is necessarily not a star. Pick one such original color tree T_a. Exactly as in the proof of Lemma 2.1, consider the union of the edge sets of T_a and T_d, and color a spanning tree of this union with color a, and all other edges induced by the vertices of this spanning tree with distinct colors. The new coloring is also extremal, since it uses at least as many colors as f', and has more trivial color trees than f'. Now, the new T_a is still not a star, and only T_a can intersect any other color tree in at least two vertices, so we can continue this process of growing T_a until the coloring becomes simple. Observe that during this process we have not increased the number of color trees that intersect two vertex classes in at least two vertices, and that the resulting simple extremal coloring has fewer such trees than f had. Repeating the process results in a coloring in which each color tree is a star, all of whose leaves are in the same vertex class. Hence, assume f is such an MC-coloring.

Now, assume that G has q vertex classes $V_1, \dots, V_q$ with at least two vertices each, and the remaining $r - q$ vertex classes are singletons. For $1 \le i \le q$, in order to monochromatically connect the $\binom{|V_i|}{2}$ distinct pairs of vertices of V_i, we need a set of nontrivial stars in f, say, $T_{i,1}, \dots, T_{i,t_i}$, to achieve this goal. Suppose $T_{i,j}$ has $e_{i,j}$ edges. We need, therefore, that $\sum_{j=1}^{t_i} \binom{e_{i,j}}{2} \ge \binom{|V_i|}{2}$, and the goal is to minimize $\sum_{j=1}^{t_i}(e_{i,j} - 1)$. We claim that $\sum_{j=1}^{t_i}(e_{i,j} - 1) \ge |V_i| - 1$. Indeed, as all $\binom{|V_i|}{2}$ pairs must be monochromatically connected, no $T_{i,j}$ is leaf-disjoint from all other stars. So we may re-order the trees so that for $2 \le j \le t_i$, $T_{i,j}$ shares a leaf with some $T_{i,j'}$ for $j' < j$. This implies that as we go sequentially for j from 2 to t_i, each $T_{i,j}$ covers at most $e_{i,j} - 1$ yet uncovered vertices of V_i. As eventually all $|V_i|$ vertices must be covered, the claim follows. Consequently, the number of trees used by our coloring is at most $m - (\sum_{i=1}^{q} |V_i|) + q = m - (n - (r - q)) + q$, which equals $m - n + r$. Notice that this bound is always realizable since one may take T_i as a star whose center is in V_{i+1} (cyclically). Hence, $mc(G) = m - n + r$. ■

Theorem 7 and the following corollary yield the first part of Theorem 2.

Corollary 8 *Any graph G satisfies $mc(G) \le m - n + \chi(G)$.*

Proof: First observe that if G is a spanning subgraph of some graph H, then $mc(H) \ge e(H) - e(G) + mc(G)$. Indeed, let f be an MC-coloring of G realizing $mc(G)$. Color the remaining edges of H with $e(H) - e(G)$ fresh distinct colors, and observe that this is an MC-coloring of H.

Next suppose $\chi(G) = r$. Now G is a connected spanning subgraph of some complete r-partite graph H. By Theorem 7, $mc(H) = e(H) - n + r$. It follows from the observation in the previous paragraph that $mc(G) \le (e(H) - n + r) + e(G) - e(H) = m - n + r$, as required. ■

Notice that Theorem 7 together with the observation in the last corollary can be used to supply a *lower bound* for $mc(G)$.

Corollary 9 *If G contains a spanning complete r-partite graph, then $mc(G) \ge m - n + r$.*

It is not true that graphs having $mc(G) = m - n + r$ must contain spanning complete r-partite graphs. The following proposition shows that there are graphs G whose complements are connected (thus, they do not contain spanning complete partite graphs), yet $mc(G) \geq m - 2n/3$.

Proposition 10 *If G is the complement of the cycle C_n with $n \geq 5$, then $mc(G) \geq m - \lceil 2n/3 \rceil$.*

Proof: We prove it for $n = 0 \bmod 3$; the other cases are similar. Suppose the missing cycle is $(0, 1, \ldots, n-1)$. Construct a coloring consisting of the following $n/3$ nontrivial color trees. Let tree T_i consist of the path $(3i+1, 3i+3, 3i, 3i+2)$ for $i = 0, \ldots, n/3 - 1$ (indices modulo n). This is clearly an MC-coloring. The number of colors used is precisely $m - 2n/3$. ■

The following proves the second part of Theorem 2.

Theorem 11 *If G is not k-connected, then $mc(G) \leq m - n + k$. This is sharp for any k.*

Proof: We assume, equivalently, that G is k-connected and not $(k+1)$-connected, and we prove that then $mc(G) \leq m - n + k + 1$. Let $S = \{v_1, \ldots, v_k\}$ be a set of k vertices disconnecting G. Let u be a vertex in a connected component A of $G \setminus S$.

Let f be a simple extremal coloring of G and consider all the color trees that contain u and vertices of $G \setminus S$ not in A. Notice that all such trees contain at least three vertices. All these trees have u in common, but as f is simple, they intersect only in u, and all of them must use a vertex of S. Hence there are at most k such trees, say $T_1, \ldots, T_q$ where $q \leq k$.

Let $B = A \setminus \cup_{i=1}^{q} V(T_i)$, and let $b = |B|$. Notice that these trees contain at least $n - (k - q) - b$ vertices and use q colors. If $b = 0$ we are done, since we used at least $n - (k - q) - 1$ edges in $T_1, \ldots, T_q$ and hence $mc(G) \leq m - (n - k + q - 1) + q \leq m - n + k + 1$. If $b > 0$, then consider $w \in V(T_1) \setminus (A \cup S)$. The vertex w has to reach those b vertices of B via color trees $R_1, \ldots, R_t$, any pair of them intersecting only in w. Assume R_i contains b_i vertices of B. There must be at least $b_i + 1$ edges in each R_i, because R_i also contains w and a vertex of S. Thus, at least $\sum_{i=1}^{t}(b_i + 1) = b + t$ edges are used and t colors are used. Altogether $n - (k - q) - b - 1$ edges are used with q colors and $b + t$ edges are used with t colors, so $mc(G) \leq m - (n - k + q - b - 1) - (b + t) + q + t = m - n + k + 1$.

To see the sharpness of the result, one constructs graphs G with $\kappa(G) = k$ and $mc(G) = m - n + k + 1$. Consider $K_{k-1} \vee P_{n-k+1}$. This graph is k-connected but is not $(k+1)$-connected. Still, it is possible to color the edges of the path P_{n-k+1} with the same color and color the rest of the edges with other distinct colors, obtaining an MC-coloring that uses $m - n + k + 1$ colors. ■

The following proposition is a useful tool for characterizing $mc(G)$ in various special classes of graphs, as demonstrated by the corollaries following its proof. Call a graph *s-perfectly-connected* if it can be partitioned into $s + 1$ parts, $\{v\}, V_1, \ldots, V_s$, such that each V_i induces a connected subgraph, any pair V_i, V_j induces a corresponding complete bipartite graph, and v has precisely one neighbor in each V_i. Notice that such a graph has minimum degree s, and v has degree s.

Proposition 12 *If $\delta(G) = s$, then $mc(G) \leq m - n + s$, unless G is s-perfectly-connected, in which case $mc(G) = m - n + s + 1$.*

Proof: First observe that, according to Proposition 11, $mc(G) \leq m - n + s + 1$, since a graph with minimum degree s is not $(s+1)$-connected.

Let f be a simple extremal coloring and consider the number of colors used on the edges incident to vertex v of minimum degree. If there are two edges incident with v that have the same color, then G has a spanning tree whose edges use at most $s-1$ colors. It follows that $mc(G) \leq m-(n-1)+s-1 = m - n + s$, as required.

So, it may be assumed that for every vertex v of minimum degree s, all the edges incident to it are colored with s distinct colors. Fix such a vertex v, and let $T_1, \ldots, T_s$ be the color trees corresponding to the s colors of the edges incident with v. If $\sum_{i=1}^{s} |E(t_i)| \geq n$, then there are at most $m - n$ edges left; already s colors have been used, and hence $mc(G) \leq m - n + s$. So it can be assumed that $\sum_{i=1}^{s} |E(t_i)| \leq n - 1$, but since $T_1 \cup \cdots \cup T_s$ contains a spanning tree, it must be that $\sum_{i=1}^{s} |E(t_i)| = n - 1$ and that all the T_i share only v in common, but are otherwise disjoint. Notice also that v is a leaf of each of the T_i. So $V_i = V(T_i) \setminus \{v\}$ induces a connected subgraph, and $|V_i| = |E(t_i)|$.

If the edges between V_i and V_j do not form a complete bipartite graph, then the only way to monochromatically connect nonadjacent $x \in V_i$ and $y \in V_j$ is via another nontrivial color tree (that contains at least two edges), distinct from all $T_1, \ldots, T_s$. Hence, $mc(G) \leq m - (n-1) - 2 + s + 1 = m - n + s$. It can therefore be assumed that (V_i, V_j) induces a corresponding complete bipartite graph.

Now the graph has a vertex v of minimum degree s, and $G - v$ is partitioned into subsets $V_1, \ldots, V_s$. Each V_i induces a connected subgraph, and any two of them induce a complete bipartite graph. Hence this graph is s-perfectly-connected. Clearly $mc(G) = m - n + s + 1$, since one can use s nontrivial color trees that span $V_i \cup \{v\}$ for $i = 1, \ldots, s$, and color the other edges trivially. ■

Here are a few applications of Proposition 12.

Corollary 13

a. *For $n \geq 5$, the wheel W_n has $mc(G) = m - n + 3$.*

b. *If G is an outerplanar graph, then $mc(G) = m-n+2$, except that $mc(K_1 \vee P_{n-1}) = m-n+3$.*

c. *If G is a planar graph with minimum degree 3, then $mc(G) \leq m - n + 3$, except that $mc(K_2 \vee P_{n-2}) = m - n + 4$.*

Proof: The wheel W_n has minimum degree 3, and W_n is not 3-perfectly-connected when $n \geq 5$. Hence $mc(G) \leq m - n + 3$. Clearly, $mc(G) \geq m - n + 3$, achieved by using a monochromatic path through the non-central vertices and coloring the rest of the edges trivially.

An outerplanar graph G has a vertex of degree at most 2. Hence $mc(G) \leq m - n + 2$, unless it is 2-perfectly-connected. By the trivial bound in the introduction, $mc(G) \geq m - n + 2$, and hence $mc(G) = m - n + 2$. The only outerplanar graph that is 2-perfectly-connected is $K_1 \vee P_{n-1}$.

If G is planar with minimum degree 3, then $mc(G) \leq m-n+3$ unless G is 3-perfectly-connected. The only planar graph with minimum degree 3 that is 3-perfectly-connected is $K_2 \vee P_{n-2}$. ■

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