Dominating a family of graphs with small connected subgraphs

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Abstract

Let $F = \{G_1, \ldots, G_t\}$ be a family of n-vertex graphs defined on the same vertex-set V, and let k be a positive integer. A subset of vertices $D \subset V$ is called an (F,k)-*core* if for each $v \in V$ and for each $i = 1, \ldots, t$, there are at least k neighbors of v in G_i which belong to D. The subset D is called a *connected* (F,k)-*core*, if the subgraph induced by D in each G_i is connected. Let δ_i be the minimum degree of G_i and let $\delta(F) = \min_{i=1}^{t} \delta_i$. Clearly, an (F,k)-core exists if and only if $\delta(F) \geq k$, and a connected (F,k)-core exists if and only if $\delta(F) \geq k$ and each G_i is connected. Let $c(k,F)$ and $c_c(k,F)$ be the minimum size of an (F,k)-core and a connected (F,k)-core, respectively. The following asymptotic results are proved for every $t < \ln \ln \delta$ and $k < \sqrt{\ln \delta}$:

$$c(k,F) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)) \qquad c_c(k,F) \leq n\frac{\ln \delta}{\delta}(1 + o_\delta(1)).$$

The results are asymptotically tight for infinitely many families F. The results unify and extend related results on dominating sets, strong dominating sets and connected dominating sets.

1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph-theoretic terminology the reader is referred to [3]. A major area of research in graph theory is the theory of domination. Recently two books [7, 8] have been published that present most of the known results concerning domination parameters. Among the most popular of these parameters are the "connected domination number", the "k-domination number" and the "strong domination number" which are considered in this paper.

A subset D of vertices in a graph G is a *dominating set* if every vertex not in D has a neighbor in D. D is called a *strong dominating set* if every vertex of G has a neighbor in D. If the subgraph induced by D is connected, then D is called a *connected dominating set* or a *connected strong dominating set*, appropriately. D is called a *strong k-dominating set* if every vertex of G has at least k neighbors in D. The analogous definitions of a *k-dominating set*, *connected strong*

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k*-dominating set* and *connected* k*-dominating set* are obvious. The *domination number*, denoted $\gamma(G)$, and the *connected domination number*, denoted $\gamma_c(G)$, are the minimum cardinalities of a dominating set and a connected dominating set, respectively. The analogous parameters for the "strong" versions are $\gamma^*(G)$ and $\gamma_c^*(G)$. The parameters for (connected) k-domination and (connected) strong k-domination are denoted $\gamma(k, G)$, $\gamma_c(k, G)$, $\gamma^*(k, G)$ and $\gamma_c^*(k, G)$.

A graph G has a connected dominating set if and only if G is connected; thus $\gamma_c(G)$ is well-defined on the class of connected graphs. The same is true for connected strong domination (assuming the graph has at least two vertices). In order to have a k-dominating set, or a strong k-dominating set, it is necessary and sufficient that the minimum degree be at least k.

The problem of finding small connected dominating sets and small connected strong dominating sets are a major topic of research in the area of graph algorithms, because such sets correspond to the non-leaves of a spanning tree.

There are several results which estimate some of the above-mentioned graph parameters as a function of the minimum degree of the graph. A well-known result of Lovász [9] (see another proof in [2]) states that $\gamma(G) \le n\frac{1+\ln(\delta+1)}{\delta+1}$ for every n-vertex graph G with minimum degree $\delta > 1$. This result is asymptotically optimal for general graphs G. This was shown by Alon [1] who proved by probabilistic methods that when n is large there exists a δ-regular graph with no dominating set of size less than $(1+o(1))\frac{1+\ln(\delta+1)}{\delta+1}n$. (We mention here that when $\delta \le 3$ exact results were obtained in [10, 11]). Caro [4] has considered k-domination numbers and showed an analog result to the one obtained by Lovász, under the (obviously necessary) assumption that $\delta >> k$. Thus, he showed that $\gamma(k, G) \le n\frac{\ln\delta}{\delta}(1 + o_\delta(1))$. Considering connected domination, Caro, West and Yuster [5] have shown by more complicated arguments that the bound obtained by Lovász also holds in this much more restricted case, namely $\gamma_c(k, G) \le n\frac{\ln\delta}{\delta}(1 + o_\delta(1))$. Their result also supplies a sequential deterministic algorithm which produces a connected dominating set with (at most) this cardinality, in polynomial time. In this paper we present a generalization of all these results which covers, as a special case, all the above-mentioned graph parameters.

Let $F = \{G_1, \ldots, G_t\}$ be a family of graphs which share the same vertex set V. A subset of vertices $D \subset V$ is called an (F, k)*-core* if D is a strong k-dominating set of each graph in F. We call D a *connected* (F, k)*-core* if D is a connected strong k-dominating set of each graph in F. Let $c(k, F)$ and $c_c(k, F)$ denote the minimum cardinality of an (F, k)-core, and a connected (F, k)-core, respectively. Clearly, $c(k, F)$ can be defined if and only if each graph in F has minimum degree at least k, and $c_c(k, F)$ can be defined if and only if each graph in F is connected and has minimum degree at least k. We prove the following general result:

Theorem 1.1 *Let k, t and δ be positive integers satisfying $k < \sqrt{\ln\delta}$ and $t < \ln\ln\delta$. Let F be a family of t graphs on the same n-vertex set. Assume that every graph in F has minimum degree at least δ. Then:*

$$c(k, F) \le n\frac{\ln\delta}{\delta}(1 + o_\delta(1)).$$

If all graphs in F are connected then:

$$c_c(k,F) \le n\frac{\ln\delta}{\delta}(1+o_\delta(1)).$$

Note that the lower bound mentioned by Alon shows, in particular, that the bounds obtained in Theorem 1.1 are asymptotically optimal. Moreover, by considering the case $t=1$ (i.e. $F=\{G\}$) we have that Theorem 1.1 contains, as a special case, all the above-mentioned results. The result of Lovász on $\gamma(G)$ is obtained (in the asymptotic sense) by taking $k=1$ and using the fact $\gamma(G) \le \gamma^*(G) = \gamma^*(1,G) = c(1,\{G\})$. Caro's result on $\gamma(k,G)$ is obtained by using the fact that $\gamma(k,G) \le \gamma^*(k,G) = c(k,\{G\})$. The Caro, West and Yuster result on $\gamma_c(G)$ is obtained by taking $k=1$ and using $\gamma_c(G) \le \gamma_c^*(G) = \gamma_c^*(1,G) = c_c(1,\{G\})$.

Our proof of Theorem 1.1 uses a probabilistic approach similar to the proof of the Lovász bound in [2]. However, the proof here is slightly more complicated since we also need to satisfy the connectivity and the commonality requirements. The proof is presented in the next section.

2 Proof of the main result

We begin with a lemma that sharpens a result of Duchet and Meyniel [6], who proved that $\gamma(G) \le \gamma_c(G) \le 3\gamma(G)-2$.

Lemma 2.1 *Let G be a connected graph. If X is a strong k-dominating set of G that induces a subgraph with s components, then there exists a connected strong k-dominating set of G, containing X, whose cardinality is at most $|X|+2s-2$. In particular,*

$$\gamma^*(k,G) \le \gamma_c^*(k,G) \le 3\gamma^*(k,G)-2.$$

Proof: It suffices to show that whenever $s>1$, we can find at most two vertices in $V \setminus X$ such that adding them to X decreases the number of components by at least one. Partition X into parts X_1 and X_2 such that X_1 and X_2 have no edge connecting them. Let $x_1 \in X_1$ and $x_2 \in X_2$ be two vertices whose distance in G is the smallest possible. The distance between x_1 and x_2 is at most 3, because otherwise, there is a vertex (in the middle of a shortest path from x_1 to x_2) that has distance at least 2 from both X_1 and X_2 and has no neighbor in X, contradicting the fact that X is, in particular, a dominating set. $\Box$

Proof of Theorem 1.1: We shall prove the (obviously more difficult) connected (F,k)-core version of the theorem, for $t = \lfloor \ln\ln\delta \rfloor$ and $k = \lfloor \sqrt{\ln\delta} \rfloor$. Fix $0<\epsilon<1/2$. We shall prove that, for sufficiently large δ, every $F=\{G_1,\ldots,G_t\}$ (the graphs sharing the same vertex set V) has an (F,k)-core of size at most $(1+\epsilon)n\frac{\ln\delta}{\delta}$.

Let $p=(1+\frac{\epsilon}{2})\frac{\ln\delta}{\delta}$ and let X be a random subset of V, where each vertex is chosen independently with probability p. Let Y be the set of vertices in V that have fewer than k neighbors in X in one of the graphs $G_1, \ldots, G_t$. Note that $X \cup Y$ is a k-dominating set for each G_i (although not necessarily a *strong* one). So let Z be a minimal set containing k neighbors of every vertex $y \in Y$ in each G_i; thus $|Z| \leq kt|Y|$. Then $X \cup Y \cup Z$ is strongly k-dominating in each G_i. Let $H_i = G_i[X \cup Y \cup Z]$ (the subgraph of G_i induced by $X \cup Y \cup Z$), and let c_i denote the number of components of H_i. According to Lemma 2.1, we can add at most $2c_i - 2$ vertices to $X \cup Y \cup Z$ and obtain a connected strong k-dominating set of G_i. It follows that there exists a connected (F, k)-core whose size is less than

$$w = |X| + |Y| + |Z| + 2\sum_{i=1}^{t} c_i.$$

We shall estimate the expectations of the summands. Obviously, $E\left[|X|\right] = pn = (1 + \frac{\epsilon}{2})n \ln \delta / \delta$. By examining any δ neighbors of a vertex v in G_i we see that the probability that v is adjacent to fewer than k vertices of X in G_i is at most

$$\sum_{i=0}^{k-1} \binom{\delta}{i} p^i (1-p)^{\delta - i} < \sum_{i=0}^{k-1} (\delta p)^i e^{-p(\delta - k)} = O\left(k(2\ln\delta)^k \delta^{-(1+\epsilon/2)}\right),$$

which is at most $O\left(\delta^{-(1+\frac{\epsilon}{4})}\right)$, so

$$E[|Y|] = O\left(nt\delta^{-(1+\frac{\epsilon}{4})}\right) = o\left(\frac{n}{\delta}\right)$$

and since $|Z| \leq kt|Y|$ we also have $E(|Y| + |Z|) = o(n/\delta)$. Finally, we estimate $E[c_i]$. Every vertex of $X \setminus Y$ has at least k neighbors in X, and hence belongs to a component of H_i of order at least $k+1$, so

$$c_i \leq \frac{1}{k+1}(|X| + |Y| + |Z|) + |Y| + |Z|$$

and thus

$$E[c_i] \leq \frac{pn}{k+1} + o\left(\frac{n}{\delta}\right) = o\left(n\frac{\ln\delta}{\delta \ln\ln\delta}\right).$$

We therefore have:

$$E[2\sum_{i=1}^{t} c_i] = o\left(n\frac{\ln\delta}{\delta}\right)$$

and hence, by linearity of expectation, $E[w] = (1 + \frac{\epsilon}{2} + o(1))n \ln\delta/\delta$, which implies that there is an (F, k)-core of size at most $(1+\epsilon)n\ln\delta/\delta$ for δ sufficiently large. $\square$

Acknowledgment

The authors wish to thank Teresa W. Haynes for valuable references, and the referee for supplying us with a much shorter proof than the original one.

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