# The order of monochromatic subgraphs with a given minimum degree

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**Abstract**

Let $G$ be a graph. For a given positive integer $d$, let $f_G(d)$ denote the largest integer $t$ such that in every coloring of the edges of $G$ with two colors there is a monochromatic subgraph with minimum degree at least $d$ and order at least $t$. Let $f_G(d) = 0$ in case there is a 2-coloring of the edges of $G$ with no such monochromatic subgraph. Let $f(n, k, d)$ denote the minimum of $f_G(d)$ where $G$ ranges over all graphs with $n$ vertices and minimum degree at least $k$. In this paper we establish $f(n, k, d)$ whenever $k$ or $n-k$ are fixed, and $n$ is sufficiently large. We also consider the case where more than two colors are allowed.

## 1 Introduction

All graphs considered in this paper are finite, simple and undirected. For standard terminology used in this paper see [6]. It is well known that in any coloring of the edges of a complete graph with two colors there is a monochromatic connected spanning subgraph. This folkloristic Ramsey-type fact, which is straightforward to prove, has been generalized in many ways, where one shows that some given properties of a graph $G$ suffice in order to guarantee a large monochromatic subgraph of $G$ with related given properties in any two (or more than two) edge-coloring of $G$. See, e.g., [2, 3, 4, 5] for these types of results. In this paper we consider the property of having a certain minimum degree.

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For given positive integers $d$ and $r$, and a fixed graph $G$, let $f_G(d,r)$ denote the largest integer $t$ such that in every coloring of the edges of the graph $G$ with $r$ colors there is a monochromatic subgraph with minimum degree at least $d$ and order at least $t$. If $G$ has an $r$-coloring of its edges with no monochromatic subgraph of minimum degree at least $d$ we define $f_G(d,r)=0$. Let $f(n,k,d,r)$ denote the minimum of $f_G(d)$ where $G$ ranges over all graphs with $n$ vertices and minimum degree at least $k$. The main results of our paper establish $f(n,k,d,2)$ whenever $k$ or $n-k$ are fixed, and $n$ is sufficiently large. In particular, we prove the following results.

**Theorem 1.1** *(i) For all $d \geq 1$ and $k \geq 4d-3$,*

$$f(n,k,d,2) \geq \frac{k-4d+4}{2(k-3d+3)}n + \frac{3d(d-1)}{4(k-3d+3)}. \tag{1}$$

*(ii) For all $d \geq 1$ and $k \leq 4d-4$, if $n$ is sufficiently large then $f(n,k,d,2) \leq d^2-d+1$. In particular, $f(n,k,d,2)$ is independent of $n$.*

**Theorem 1.2** *For all $d \geq 1$, $r \geq 2$ and $k > 2r(d-1)$, there exists a constant $C$ such that*

$$f(n,k,d,r) \leq n\frac{k-2r(d-1)}{r(k-(r+1)(d-1))} + C.$$

*In particular, $f(n,k,d,2) \leq \frac{k-4d+4}{2(k-3d+3)}n + C$.*

Notice that Theorem 1.1 and Theorem 1.2 show that for fixed $k$, $f(n,k,d,2)$ is determined up to a constant additive term. The theorems also show that $f(n,k,d,2)$ transitions from a constant to a value linear in $n$ when $k=4d-3$.

The following theorem determines $f(n,k,d,2)$ whenever $k$ is very close to $n$.

**Theorem 1.3** *Let $d$ and $k$ be positive integers. For $n$ sufficiently large, $f(n,n-k,d,2) = n-2d-k+3$.*

The next section presents our main results. The final section contains some concluding remarks. Throughout the rest of this paper, we use the term *$k$-subgraph* to denote a subgraph with minimum degree at least $k$.

## 2 Results

We need the following lemmas. The first one is well-known (see, e.g., [1] page xvii).

**Lemma 2.1** *For every $m \geq k$, every graph with $m$ vertices and more than $(k-1)m-\binom{k}{2}$ edges contains a $k$-subgraph. Furthermore, there are graphs with $m$ vertices and $(k-1)m-\binom{k}{2}$ edges that have no $k$-subgraph.* □

**Lemma 2.2** *Let $G$ be a graph and let $X$ be the set of vertices of $G$ that are not in any $k$-subgraph of $G$. If $|X| \geq k$ then*

$$\sum_{x \in X} d_G(x) \leq 2(k-1)|X| - \binom{k}{2}.$$

**Proof** Assume the lemma is false. Put $x = |X|$ and let $S \subset V(G) \setminus X$ denote the set of vertices of the graph $G$ that have at least one neighbor in $X$. Put $s = |S|$. Notice that there are at most $(k-1)x - \binom{k}{2}$ edges in $G[X]$ (the subgraph induced by $X$), and hence, if $z$ denotes the number of edges between $X$ and $S$ then, by the assumption on the sum of degrees in $X$ we have

$$z \geq \sum_{x \in X} d_G(x) - 2\left[(k-1)x - \binom{k}{2}\right] > \binom{k}{2}.$$

We distinguish between two cases. Assume first that $s \geq k$. We create a new graph $H$, which is obtained from $G$ by removing all the edges of $G[S]$ and adding a set $M$ of edges between vertices of $S$ such that $H[S]$ has $(k-1)s - \binom{k}{2}$ edges and no $k$-subgraph. Such an $M$ exists by Lemma 2.1. Now, the sum of the degrees of the subgraph of $H$ on $X \cup S$ is greater than

$$2(k-1)x - 2\binom{k}{2} + 2z + 2(k-1)s - 2\binom{k}{2} \geq 2(k-1)(x+s) - k(k-1).$$

Hence, this subgraph which has $x + s$ vertices, has more than $(k-1)(x+s) - \binom{k}{2}$ edges and therefore contain a $k$-subgraph, $P$. Clearly, $P$ contains at least one vertex of $X$. Now, revert from $H$ to $G$ by deleting $M$ and adding the original edges with both endpoints in $S$. Also, add to $P$ all other vertices of $V(G) \setminus (X \cup S)$ and all their incident edges. Notice that the obtained subgraph is a $k$-subgraph of $G$ that contains a vertex of $X$, a contradiction. Now assume $s < k$ (clearly $s \geq 1$). We can repeat the same argument where instead of $M$ we use a complete graph on $S$, and similar computations hold. □

**Proof of Theorem 1.1, part (i).** The theorem is trivial for $d = 1$ so we assume $d \geq 2$. Let $G = (V, E)$ have $n$ vertices and minimum degree at least $k$, and consider some fixed red-blue coloring of $G$. Let $B$ (resp. $R$) denote the set of vertices of $G$ that are not on any blue (resp. red) $d$-subgraph but are on some red (resp. blue) $d$-subgraph. Let $C$ denote the set of vertices of $G$ that are neither in a red $d$-subgraph nor in a blue $d$-subgraph. Put $|R| = r$, $|B| = b$, $|C| = c$. Clearly, there is a monochromatic subgraph of order at least $(n-c)/2$. Hence, if $c < d$ the theorem trivially holds since the r.h.s. of (1) is always at most $(n-d+1)/2$. We may therefore assume $c \geq d$. For each $v \in B \cup C$ (resp. $v \in R \cup C$) let $b(v)$ (resp. $r(v)$) denote the number of blue (resp. red) edges incident with $v$ and that are not on any blue (resp. red) $d$-subgraph. By Lemma 2.2 applied to the graph spanned by blue edges on $B \cup C$ (resp. red edges on $R \cup C$),

$$\sum_{v \in B \cup C} b(v) \leq 2(d-1)(b+c) - \binom{d}{2}, \qquad \sum_{v \in R \cup C} r(v) \leq 2(d-1)(r+c) - \binom{d}{2}.$$

Notice that, trivially, for each $v \in C$, $b(v) + r(v) = deg(v) \geq k$. Put

$$b_c = \sum_{v \in C} b(v), \qquad r_c = \sum_{v \in C} r(v).$$

Thus, $b_c + r_c \geq kc$. By Lemma 2.1, the subgraph induced by $C$ contains at most $(d-1)c - \binom{d}{2}$ blue edges and at most $(d-1)c - \binom{d}{2}$ red edges. Hence, this subgraph contributes to the sum of $b(v)$ at most $2(d-1)c - d(d-1)$ and to the sum of $r(v)$ at most $2(d-1)c - d(d-1)$. Hence, the sum of $b(v)$ (resp. $r(v)$) on the vertices of $B$ (resp. $R$) must be at least $b_c - 2(d-1)c + d(d-1)$ (resp. $r_c - 2(d-1)c + d(d-1)$). It follows that:

$$2(d-1)(b+c) - \binom{d}{2} \geq \sum_{v \in B \cup C} b(v) \geq b_c + (b_c - 2(d-1)c + d(d-1)),$$

$$2(d-1)(r+c) - \binom{d}{2} \geq \sum_{v \in R \cup C} r(v) \geq r_c + (r_c - 2(d-1)c + d(d-1)).$$

Summing the two last inequalities we have:

$$2(d-1)(b+r) - d(d-1) + 4(d-1)c \geq (2k - 4(d-1))c + 2d(d-1).$$

Thus, $r + b \geq (k - 4d + 4)c/(d-1) + 3d/2$. On the other hand $r + b + c \leq n$. It follows that

$$c \leq \frac{d-1}{k-3d+3}n - \frac{3d(d-1)}{2(k-3d+3)}, \qquad \frac{r+b}{2} + c \leq \frac{k-2d+2}{2(k-3d+3)}n - \frac{3d(d-1)}{4(k-3d+3)}.$$

It follows that there is either a red or a blue monochromatic $d$-subgraph of order at least

$$\frac{k-4d+4}{2(k-3d+3)}n + \frac{3d(d-1)}{4(k-3d+3)}.$$

□

**Proof of Theorem 1.1, part (ii).** It suffices to prove the theorem for $k = 4d - 4$. We first create a specific graph $H$ on $n$ vertices. Place the $n$ vertices in a sequence $(v_1, \ldots, v_n)$ and connect any two vertices whose distance is at most $d - 1$. Hence, all the vertices $\{v_d, \ldots, v_{n-d+1}\}$ have degree $2(d-1)$. The first $d$ and last $d$ vertices have smaller degree. To compensate for this we add the following $\binom{d}{2}$ edges. For all $i = 1, \ldots, d-1$ and for all $j = i, \ldots, d-1$ we add the edge $(v_i, v_{jd+1})$. For example, if $d = 3$ we add $(v_1, v_4)$, $(v_1, v_7)$ and $(v_2, v_7)$. Notice that these added edges are indeed new edges. The resulting graph $H$ has $n$ vertices and $(k-1)n$ edges. Furthermore, all the vertices have degree $2(d-1)$ except for $v_{jd+1}$ whose degree is $2(d-1) + j$ for $j = 1, \ldots, d-1$ and $v_{n-d+1+j}$ whose degree is $2(d-1) - j$ for $j = 1, \ldots, d-1$. Also notice that any $d$-subgraph of $H$ may only contain the vertices $\{v_1, \ldots, v_{d^2-d+1}\}$. Thus, the order of any $d$-subgraph of $H$ is at most $d^2 - d + 1$. The crucial point to observe is that the vertices of excess degree, namely $\{v_{d+1}, v_{2d+1}, \ldots, v_{d^2-d+1}\}$ form an independent set. Hence, for $n$ sufficiently large,

$K_n$ contains two edge disjoint copies of $H$ where in the second copy, the vertex playing the role of $v_{jd+1}$ plays the role of the vertex $v_{n-d+1+j}$ in the first copy, for $j = 1, \ldots, d-1$, and vice versa. In other words, there exists a $4(d-1)$-regular graph with $n$ vertices, and a red-blue coloring of it, such that the red subgraph and the blue subgraph are each isomorphic to $H$. In particular, there is no monochromatic $d$-subgraph with more than $d^2 - d + 1$ vertices. $\square$

**Proof of Theorem 1.2.** The theorem is trivial for $d = 1$ so we assume $d \geq 2$. It clearly suffices to prove the theorem for $n = (m + d)r$ where $m$ is an arbitrary element of some fixed infinite *arithmetic* sequence whose difference and first element are only functions of $d, k$ and $r$. Let $m$ be a positive integer such that

$$y = m\frac{(d-1)(r-1)}{k-(r+1)(d-1)}$$

is an integer. Whenever necessary we shall assume $m$ is sufficiently large. We shall create a graph with $n = (m+d)r$ vertices, minimum degree at least $k$, having an $r$-coloring of its edges with no monochromatic subgraph larger than the value stated in the theorem. Let $A_1, \ldots, A_r$ be pairwise disjoint sets of vertices of size $y$ each. Let $B_1, \ldots, B_r$ be pairwise disjoint sets of vertices (also disjoint from the $A_i$) of size $x = m + d - y$ each. The vertex set of our graph is $\cup_{i=1}^{r}(A_i \cup B_i)$. The edges of $G$ and their colors are defined as follows. In each $B_i$ we place a graph of minimum degree at least $k - (r-1)(d-1)$, and color its edges with the color $i$. In each $A_i$ we place a $(d-1)$-degenerate graph with the maximum possible number of vertices of degree $2(d-1)$. It is easy to show that such graphs exists with precisely $d$ vertices of degree $d-1$ and the rest are of degree $2(d-1)$. Denote by $A_i'$ the $y - d$ vertices of $A_i$ with degree $2(d-1)$ in this subgraph and put $A_i'' = A_i \setminus A_i'$. Color its edges with the color $i$. Now for each $j \neq i$ we place a bipartite graph whose sides are $A_i$ and $A_j \cup B_j$ and whose edges are colored $i$. The degree of all the vertices of $A_j \cup B_j$ in this subgraph is $d-1$, the degrees of all the vertices of $A_i'$ are at least $(k-(r+1)(d-1))/(r-1)$ and the degrees of all vertices of $A_i''$ in this subgraph are at least $(k - r(d-1))/(r-1)$. This can be done for $m$ sufficiently large since

$$(y-d)\left\lceil\frac{k-(r+1)(d-1)}{r-1}\right\rceil + d\left\lceil\frac{k-r(d-1)}{r-1}\right\rceil \leq (d-1)(m+d).$$

Notice that when $m$ is sufficiently large we can place all of these $r(r-1)$ bipartite subgraphs such that their edge sets are pairwise disjoint (an immediate consequence of Hall's Theorem).

By our construction, the minimum degree of the graph $G$ is at least $k$. Furthermore, any monochromatic subgraph with minimum degree at least $d$ must be completely placed within some $B_i$. It follows that

$$f(n,k,d,r) \leq x = m + d - m\frac{(d-1)(r-1)}{k-(r+1)(d-1)} = n\frac{k-2r(d-1)}{r(k-(r+1)(d-1))} + C.$$

$\square$

**Proof of Theorem 1.3.** Suppose $n \geq R(4d+2k-5, 4d+2k-5)$ where $R(a,b)$ is the usual Ramsey number. Let $G$ be a a graph with $\delta(G) = n-k$ and fix a red-blue coloring of $G$. Add edges to $G$ in order to obtain $K_n$. Note that at most $k-1$ new edges are incident with each vertex. Color the new edges arbitrarily using the colors red and blue. The obtained complete graph contains either a red or blue $K_{4d+2k-5}$ . Deleting the new edges we get a monochromatic subgraph of $G$ on $4d+2k-5$ vertices and minimum degree at least $4d+k-4 \geq 4d-3 \geq d$. Now consider the largest monochromatic subgraph $Y$ with minimum degree at least $d$. Hence, $|Y| \geq 4d+2k-5$. Assume, w.l.o.g., that $|Y|$ is red. If $|Y| \leq n-2d-k+2$, then define $X$ to be a set of $2d+k-2$ vertices in $V \setminus Y$. We call a vertex $y \in Y$ *bad* if it has $d$ "red" neighbors in $X$. Let $B$ denote the subset of bad vertices in $Y$. Since the number of red edges between $X$ and $B$ is at most $|X|(d-1)$ we have $|B|d \leq |X|(d-1)$. Hence, $|B| < |X| = 2d+k-2 \leq 4d+2k-5 \leq |Y|$. In particular, $|B| \leq 2d+k-3$. Consider the bipartite blue graph on $X$ versus $Y \setminus B$. Its order is $|X|+|Y|-|B| > |Y|$. Furthermore, we claim that it has minimum degree at least $d$. This is true because each $y \in Y \setminus B$ has at least $|X|-(d-1)-(k-1) = d$ blue neighbors in $|X|$ and each vertex in $X$ is adjacent to at least $|Y|-|B|-(d-1)-(k-1) \geq 4d+2k-5-(2d+k-3)-(d-1)-(k-1) = d$ vertices in $Y \setminus B$. Thus, $X \cup (Y \setminus B)$ contradicts the maximality of $Y$. So, we must have $|Y| \geq n-2d-k+3$, as required. Clearly the value $n-2d-k+3$ is sharp for large $n$. Take a red $K_{n-2d-k+3}$ on vertices $v_1, \ldots, v_{n-2d-k+3}$ and a blue $K_{2d+k-3}$ on vertices $u_1, \ldots, u_{2d+k-3}$. Put $A = \{v_1, \ldots, v_{2d+k-3}\}$. Connect with $d-1$ blue edges the vertex $u_i$ to the vertices $v_i, \ldots, v_{i+d-2(\bmod 2d+k-3)}$ , and connect with $d-1$ red edges the vertex $u_i$ to the vertices $v_{i+d-1}, \ldots, v_{i+2d-3(\bmod 2d+k-3)}$. There are no edges between $u_i$ and $v_{i+2d-2}, \ldots, v_{i+2d+k-4(\bmod 2d+k-3)}$ . The rest of the edges between the $u_i$ and $v_j$ for $j \geq 2d+k-2$ are colored blue. It is easy to verify that this graph is $(n-k)$-regular and contain no blue nor red $d$-subgraph with more than $n-2d-k+3$ vertices. □

# 3 Concluding remarks

- In the proof of Theorem 1.3 we assume $n \geq R(4d+2k-5, 4d+2k-5)$ and hence $n$ is very large. We can improve upon this to $n \geq \Theta(d+k)$ using the following argument. Let $g(n,m,d,r)$ denote the largest integer $t$ such that in any $r$ coloring of a graph with $n$ vertices and $m$ edges there exists a monochromatic subgraph of order at least $t$ and minimum degree $d$.

**Proposition 3.1**

$$g(n,m,d,r) \geq \sqrt{2\left(m-(d-1)n+\binom{d}{2}\right)/r} \geq \sqrt{2m/r-2dn/r}.$$

**Proof.** Suppose $G$ has $n$ vertices $m$ edges and the edges are $r$-colored. Start deleting edge-disjoint monochromatic $d$-graphs as long as we can. We begin with $m$ edges and when we stop we remain with at most $(d-1)n-\binom{d}{2}$ edges. Hence, there are at least $q = (m-(d-1)n+\binom{d}{2})/r$ edges in one of the monochromatic $d$-graphs. Thus, this monochromatic $d$-graph contains at least $\sqrt{2q}$ vertices as claimed. Notice that this bound is rather tight for $d \le \sqrt{2m/r}-1$. Consider the $n$-vertex graph composed of $r$ vertex-disjoint copies of $K_{\sqrt{2m/r}}$ and $n-\sqrt{2mr}$ isolated vertices (assume all numbers are integers, for simplicity). Then, $e(G) \ge m$ and by coloring each of the $r$ large cliques with different colors we get that any monochromatic $d$-subgraph has at most $\sqrt{2m/r}$ vertices. □

Proposition 3.1 shows that in the proof of Theorem 1.3 we can ensure an initial big monochromatic $d$-subgraph already when $n \ge 7(k+2d)/2 = \Theta(d+k)$.

- In the case where $r \ge 3$ colors are considered and $k > 2r(d-1)$ is fixed, Theorem 1.2 supplies a linear upper bound for $f(n,k,d,r)$. However, unlike the case where only two colors are used, we do not have a matching lower bound. The following recursive argument supplies a linear lower bound in case $k = k(d)$ is sufficiently large. We may assume that $r$ is a power of 2 as any lower bound for $r$ colors implies a lower bound for less colors. Given an $r$-coloring of an $n$-vertex graph $G$, split the colors into two groups of $r/2$ colors each. Now, using Theorem 1.1 we have a subgraph that uses only the colors of one of the groups, and whose minimum degree is $x$, where $x$ is a parameter satisfying $k \ge 4x-3$. The order of this subgraph is at least $n(k-4x+4)/(2(k-3x+3))$. Now we can use the recursion to show that this $r/2$-colored linear subgraph has a linear order subgraph which is monochromatic. $x$ is chosen so as to maximize the order of the final monochromatic subgraph. For example, with $r=4$ we can take $x=4d-3$ and hence $k \ge 16d-15$. For this choice of $x$ (which is optimal for this strategy) we get a monochromatic subgraph of order at least
$$n\frac{(k-4(4d-3)+4)((4d-3)-4d+4)}{(2(k-3(4d-3)+3))(2((4d-3)-3d+3))} = n\frac{k-16d+16}{4d(k-12d+12)}.$$

- Our theorems determine, up to a constant additive term, the value of $f(n,k,d,2)$ whenever $k$ or $n-k$ are fixed and $n$ is sufficiently large. It may be interesting to establish precise values for all $k < n$. Another possible path of research is the extension of the definition of $f(n,k,d,r)$ to $t$-uniform hypergraphs.

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