Clumsy packings of graphs

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Abstract

Let G and H be graphs. We say that $\mathcal{P}$ is an H-*packing* of G if $\mathcal{P}$ is a set of edge-disjoint copies of H in G. An H-packing $\mathcal{P}$ is *maximal* if there is no other H-packing of G that properly contains $\mathcal{P}$. Packings of maximum cardinality have been studied intensively, with several recent breakthrough results. Here, we consider minimum cardinality maximal packings. An H-packing $\mathcal{P}$ is *clumsy* if it is maximal of minimum size. Let $\mathrm{cl}(G,H)$ be the size of a clumsy H-packing of G. We provide nontrivial bounds for $\mathrm{cl}(G,H)$, and in many cases asymptotically determine $\mathrm{cl}(G,H)$ for some generic classes of graphs G such as K_n (the complete graph), Q_n (the cube graph), as well as square, triangular, and hexagonal grids. We asymptotically determine $\mathrm{cl}(K_n,H)$ for every fixed non-empty graph H. In particular, we prove that

$$\mathrm{cl}(K_n,H)=\frac{\binom{n}{2}-\mathrm{ex}(n,H)}{|E(H)|}+o(\mathrm{ex}(n,H)),$$

where $\mathrm{ex}(n,H)$ is the extremal number of H.

A related natural parameter is $\mathrm{cov}(G,H)$, that is the smallest number of copies of H in G (not necessarily edge-disjoint) whose removal from G results in an H-free graph. While clearly $\mathrm{cov}(G,H)\leqslant \mathrm{cl}(G,H)$, all of our lower bounds for $\mathrm{cl}(G,H)$ apply to $\mathrm{cov}(G,H)$ as well.

Mathematics Subject Classifications: 05C70, 05C35

1 Introduction

Let G and H be graphs. We say that $\mathcal{P}$ is an *H-packing* of G if $\mathcal{P}$ is a set of edge-disjoint subgraphs of G each isomorphic to H. We shall refer to subgraphs isomorphic to H as *copies* of H. An H-packing $\mathcal{P}$ is *maximal* if there is no other H-packing in G that properly contains $\mathcal{P}$. An H-packing $\mathcal{P}$ of G is *perfect* if every edge of G belongs to an element of $\mathcal{P}$. In the case when there is a perfect H-packing of G, we say that H *decomposes* G. The case of perfect H-packings has been extensively studied. A seminal result of Wilson [20] asserts that perfect H-packings of K_n always exist when certain (obviously necessary) divisibility conditions hold. Wilson's result has been extended from K_n to graphs with a sufficiently large minimum degree starting with Gustavsson [10] and culminating in results of Keevash [13] and Glock et al. [9]. In general, however, deciding if G has a perfect H-packing, or computing $\mathrm{pp}(G,H)$, the maximum cardinality of an H-packing of G, is NP-hard for every connected H with at least three edges, [7].

Here, we consider minimum cardinality maximal packings. An H-packing $\mathcal{P}$ is *clumsy* if it is maximal of minimum size. Let $\mathrm{cl}(G,H)$ be the size of a clumsy H-packing of G. Thus, we are interested in the smallest *covering* of all the copies of H in G with edge-disjoint copies of H. Let $\mathrm{cov}(G,H)$ be the smallest cardinality of a set of (not necessarily edge-disjoint) copies of H in G such that each copy of H in G has an edge in some element of the set. Thus, we have $\mathrm{pp}(G,H) \geqslant \mathrm{cl}(G,H) \geqslant \mathrm{cov}(G,H)$. The notion of a clumsy packing was introduced by Gyárfás et al. [11] for domino packings. It was further extended to general polyominoes by Walzer et al. [19].

Our main results provide upper and lower bounds, and in many cases asymptotically determine $\mathrm{cl}(G,H)$ and $\mathrm{cov}(G,H)$ for major generic classes of graph G such as K_n (the complete graph), Q_n (the cube graph), as well as square, triangular, and hexagonal grids. To state our results we need to recall some notation. We assume that the graphs under consideration are non-empty unless otherwise stated. We denote the number of edges of a graph G by $||G||$. Note that $\mathrm{pp}(G) = ||G||/||H||$ if a perfect H-packing of G exists. The extremal number $\mathrm{ex}(G,H)$ is the largest number of edges in a subgraph of G that contains no copy of H and $\mathrm{ex}(n,H) = \mathrm{ex}(K_n,H)$. Using these notations, a trivial lower bound for $\mathrm{cov}(G,H)$, and thus for $\mathrm{cl}(G,H)$ is therefore

$$\frac{||G|| - \mathrm{ex}(G,H)}{||H||} \leqslant \mathrm{cov}(G,H) \leqslant \mathrm{cl}(G,H) \tag{1}$$

since we must cover at least $||G|| - \mathrm{ex}(G,H)$ edges of G with copies of H. Our first main result concerns the case $\mathrm{cl}(K_n,H)$.

Theorem 1. *Let H be any fixed non-empty graph. Then,*

$$\mathrm{cl}(K_n,H) = \frac{||K_n|| - \mathrm{ex}(n,H)}{||H||} + o(\mathrm{ex}(n,H))\ .$$

If H is complete or bipartite, then the $o(\mathrm{ex}(n,H))$ term is only linear in n and if H is a forest, the $o(\mathrm{ex}(n,H))$ term is a constant.

Note that by (1), the same results hold for $\mathrm{cov}(G,H)$ instead of $\mathrm{cl}(G,H)$. The proof of Theorem 1 appears in Section 2.

The asymptotic expression for $\mathrm{ex}(n,H)$, given by Erdős-Stone Theorem when $\chi(H) \geqslant 3$, allows us to obtain the asymptotic ratio between the sizes of clumsy and perfect packings.

Corollary 2. *For a fixed H with $\chi(H) \geqslant 3$,*

$$\lim_{n\to\infty} \frac{\mathrm{cl}(K_n,H)}{\mathrm{pp}(K_n,H)} = \frac{1}{\chi(H)-1}\ .$$

Our next result concerns another well-studied generic graph, the hypercube Q_n. In this setting, it is most natural to evaluate $\mathrm{cl}(Q_n,Q_d)$ where $d \geqslant 2$ is fixed. Unlike the case of Theorem 1 where the true asymptotic is determined and the main goal is to keep the set of uncovered edges (which is small in some cases) as large as possible, in the hypercube setting we are only able to obtain upper and lower bounds and these do not coincide.

Theorem 3. *For a fixed integer d, $d \geqslant 2$,*

$$\Omega\left(\frac{\log d}{d\ 2^d}\right) \leqslant \liminf_{n\to\infty} \frac{||Q_d||}{||Q_n||}\mathrm{cl}(Q_n,Q_d) \leqslant \limsup_{n\to\infty} \frac{||Q_d||}{||Q_n||}\mathrm{cl}(Q_n,Q_d) \leqslant \frac{\sqrt{2\pi}}{\sqrt{d}}(1+o_d(1))$$

and for $d=2$,

$$0.3932 \leqslant \liminf_{n\to\infty} \frac{||Q_2||}{||Q_n||}\mathrm{cl}(Q_n,Q_2) \leqslant \limsup_{n\to\infty} \frac{||Q_2||}{||Q_n||}\mathrm{cl}(Q_n,Q_2) \leqslant \frac{2}{3}\ .$$

It is worth noting that Offner [18] proved that $\frac{||H||}{||Q_n||}\mathrm{pp}(Q_n,H) = 1-o(1)$ for any fixed subgraph H of a hypercube, so the ratios in Theorem 3 also serve as ratios between $\mathrm{pp}(Q_n,Q_d)$ and $\mathrm{cl}(Q_n,Q_d)$. The proof of Theorem 3 appears in Section 3.

Our third main result consists of constructions of clumsy packings of grid graphs corresponding to the regular tessellations of the plane. There are only three regular tessellations of the plane: the triangular, the square, and the hexagonal, each corresponding to an infinite graph whose vertices correspond to respective k-gons ($k \in \{3,4,6\}$), and edges correspond to pairs of k-gons sharing sides. We construct clumsy C_k-packings for each of these graphs and compute the exact limit ratio of the covered edges in these packings which turns out to be $2/(k+1)$. In particular, for a square grid graph Gr_n, our results here imply that $\mathrm{cl}(Gr_n,C_4) = \frac{n^2}{5}(1+o(1))$. The main result here, stated as Theorem 13, appears in Section 4.

Finally, it is interesting to point out that $\mathrm{cl}(G,H)$ is *not* a monotone graph parameter, which might explain some of the difficulties that arise in its determination. Indeed, consider the graph G' which is the union of $k+1$ graphs $F_0,\dots,F_k$, each isomorphic to C_k such that F_i is edge-disjoint from F_j, $1 \leqslant i < j \leqslant k$, and F_0 shares its i^{th} edge with F_i, when we arbitrarily label the edges of F_0 with $1,\dots,k$. Let $H = C_k$ and let G be obtained from G' by deleting an edge from F_0. Then we see that $\mathrm{cl}(G',H) = 1$ as $\{F_0\}$ forms a clumsy packing. On the other hand $\mathrm{cl}(G,H) = k-1$.

2 Clumsy packings in K_n

Prior to proving Theorem 1 we need to recall some facts and definitions. For a graph H, let $gcd(H)$ denote the greatest common divisor of its vertex degrees. A graph G is called *H-divisible* if $gcd(H)$ divides $gcd(G)$ and $||H||$ divides $||G||$. For example, K_n is K_m-divisible if $n \equiv 1 \bmod m(m-1)$. Clearly, a necessary condition for a perfect H-packing of G is that G is H-divisible. A seminal result of Wilson [20] asserts that for every fixed graph H, if n sufficiently large and K_n is H-divisible, then K_n has a perfect H-packing. Recall also that the Turán graph $T(n,k)$ is the complete k-partite graph on n vertices whose parts form an equitable partition (so the size of each part is either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$). Turán's Theorem asserts that $\mathrm{ex}(n, K_{k+1}) = ||T(n,k)||$ and that $T(n,k)$ is the unique extremal K_{k+1}-free graph with n vertices.

The lower bound on $\mathrm{cl}(K_n, H)$ in Theorem 1 follows from (1). To prove the upper bound, we split the proof of Theorem 1 into four parts, depending on the structure of H. The parts correspond to the cases $H = K_m$, $\chi(H) \geqslant 3$, H is bipartite but not a forest, and H is a forest. While the first two cases are rather standard, the proofs of the bipartite and forest cases are more involved, especially since we know neither the structure nor the asymptotic value of $\mathrm{ex}(n,H)$ in these cases.

2.1 $H = K_m$

Lemma 4. *For fixed $m \geqslant 2$, $\mathrm{cl}(K_n, K_m) \leqslant (||K_n|| - \mathrm{ex}(n, K_m))/||K_m|| + O(n)$. Moreover, for n sufficiently large, $\mathrm{cl}(K_n, K_m) = (||K_n|| - \mathrm{ex}(n, K_m))/||K_m||$ if n is divisible by $m-1$ and $K_{n/(m-1)}$ is K_m-divisible.*

Proof. We assume $m \geqslant 3$ as the case $m = 2$ trivially holds. We shall construct a maximal H-packing with the desired number of copies of K_m. Assume first that n is divisible by $m-1$ and $K_{n/(m-1)}$ is K_m-divisible. Given $G = K_n$, the Turán graph $T(n, m-1)$ is a spanning subgraph of G. Denote the partite sets of $T(n, m-1)$ by $V_1, \ldots, V_{m-1}$. If n is sufficiently large, we can use Wilson's Theorem to find a perfect K_m-packing of each V_i. The union of these $m-1$ perfect packings is a K_m-packing of size $(||K_n|| - \mathrm{ex}(n, K_m))/||K_m||$, as required. It is a maximal packing since each copy of K_m in K_n contains an edge induced by one of V_i's.

Next assume that n is not of the aforementioned form. Let $n' < n$ be the largest integer such that $m-1$ divides n' and $K_{n'/(m-1)}$ is K_m-divisible. For example, every n' of the form $n' \equiv m-1 \bmod m(m-1)^2$ satisfies these conditions. Hence $n - n' < m(m-1)^2$. Now let $G = K_n$ and let $G' = K_{n'}$ be a subgraph of G. By the previous paragraph, if n is sufficiently large (and thus n' is sufficiently large) there is a maximal K_m-packing $\mathcal{P}$ of G' of size $(||K_{n'}|| - \mathrm{ex}(n', K_m))/||K_m|| \leqslant (||K_n|| - \mathrm{ex}(n, K_m))/||K_m||$. However, there may now be copies of K_m in G that are not covered by $\mathcal{P}$. Each such copy must contain an edge incident to one of the $n - n'$ vertices of $V(G) \setminus V(G')$. As there are at most $(n-n')n < m(m-1)^2 n$ such edges, one can greedily add edge-disjoint copies of K_m to $\mathcal{P}$ to obtain a maximal packing of G consisting of less than $|\mathcal{P}| + m(m-1)^2 n$ elements which is less than $(||K_n|| - \mathrm{ex}(n, K_m))/||K_m|| + m(m-1)^2 n$.

One can improve the error term $m(m-1)^2 n$ to a better linear term using a result of Caro and Yuster [6] asserting that for any sufficiently large ℓ and a fixed m, K_ℓ contains at least

$$\left\lfloor \frac{\ell}{m} \left\lfloor \frac{\ell-1}{m-1} \right\rfloor \right\rfloor - 1$$

pairwise edge-disjoint copies of K_m. Thus there is a packing of K_m's in K_ℓ covering at least

$$\frac{m(m-1)}{2}\left(\left\lfloor \frac{\ell}{m} \left\lfloor \frac{\ell-1}{m-1} \right\rfloor \right\rfloor - 1 \right) \geqslant \frac{\ell(\ell-1)}{2} - \frac{(m-2)\ell}{2} - m(m-1)$$

edges of K_ℓ where the latter inequality follows from

$$\begin{aligned}
\left(\left\lfloor \frac{\ell}{m} \left\lfloor \frac{\ell-1}{m-1} \right\rfloor \right\rfloor - 1 \right) &\geqslant \frac{1}{m}\left(\ell \cdot \frac{\ell - m + 1}{m-1} - m \right) - 1 \\
&= \frac{1}{m}\left(\ell \cdot \frac{(\ell-1) + (-m+2)}{m-1} - m \right) - 1 \\
&= \frac{\ell(\ell-1)}{m(m-1)} + \frac{\ell(-m+2)}{m(m-1)} - 2 \,.
\end{aligned}$$

Let $G = K_n$ and $V(G) = V_1 \cup \cdots \cup V_{m-1}$, where the parts form an equitable partition. Let $|V_i| = \ell_i$, $i = 1, \ldots, m-1$. Let $\mathcal{P}_i$ be a densest K_m-packing of $G[V_i]$ with K_m for $i = 1, \ldots, m-1$. Let $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{m-1}$. We see that the set of edges covered by $\mathcal{P}$ is included in the set of edges of the complement of the complete $(m-1)$-partite graph with parts $V_1, \ldots, V_{m-1}$, i.e., the Turán graph $T(n, m-1)$. Thus $|\mathcal{P}| \leqslant (||K_n|| - \mathrm{ex}(n, K_m))/||K_m||$. Let the set of edges in $G[V_i]$'s that are not covered by $\mathcal{P}$ be denoted E'. Then

$$|E'| \leqslant \sum_{i=1}^{m-1} \left(\frac{\ell_i(\ell_i - 1)}{2} - \left(\frac{\ell_i(\ell_i-1)}{2} - \frac{(m-2)\ell_i}{2} - m(m-1) \right) \right) = \frac{(m-2)n}{2} + m(m-1)^2 \,.$$

We can greedily extend $\mathcal{P}$ to a maximal K_m-packing of G by adding at most $|E'|$ elements to cover each edge of E' when possible. The resulting maximal packing will have size at most

$$|\mathcal{P}| + \frac{(m-2)n}{2} + m(m-1)^2 \leqslant \frac{||K_n|| - \mathrm{ex}(K_n, K_m)}{||K_m||} + \frac{(m-2)n}{2} + m(m-1)^2 \,. \quad \square$$

It is worth noting that Lemma 4 implies that for a fixed m,

$$\lim_{n\to\infty} \frac{\mathrm{cl}(K_n, K_m)}{\mathrm{pp}(K_n, K_m)} = \frac{1}{m-1} \,.$$

2.2 $\chi(H) \geqslant 3$

Lemma 5. *Let H be a fixed graph with $\chi(H) \geqslant 3$. Then,* $\mathrm{cl}(K_n, H) \leqslant (||K_n|| - \mathrm{ex}(n, H))/||H|| + o(n^2)$. *In particular,* $\mathrm{cl}(K_n, H) \leqslant (||K_n|| - \mathrm{ex}(n, H))/||H|| + o(\mathrm{ex}(n, H))$.

Proof. Let $\chi(H) = r \geqslant 3$. The Erdős-Stone Theorem implies that $\text{ex}(n, H) = ||T(n, r - 1)||(1 + o(1))$.

Let $G = K_n$ and let $G' = T(n, r - 1)$ be a spanning subgraph of G. As in Lemma 4, we use Wilson's Theorem to find an H-packing $\mathcal{P}$ of the complement of G' in G (i.e. the vertex-disjoint cliques induced by the $r - 1$ parts) which cover all but $O(n)$ edges of this complement. Indeed, let $r = gcd(H)$ and $h = ||H||$. For each component (that is a clique) of the complement of G', we can delete at most $2rh$ vertices and edges incident to them so that the number of remaining vertices in the clique is congruent to 1 modulo $2rh$. Then the remaining clique has degree divisible by $2rh$, thus in particular divisible by r. Thus the number of edges in the remaining clique is divisible by h, that implies that the clique is divisible by H. Note that we deleted at most a linear in n number of edges during this process.

Let E^* denote the uncovered edges of the complement. Notice that since G' contains no copy of H, any copy of H consisting only of edges of G that is not covered by $\mathcal{P}$ must contain an edge from E^*. We can thus extend $\mathcal{P}$ to a maximal H-packing of G using at most $|E^*| = O(n)$ additional elements. The size of this maximal H-packing is therefore at most

$$\frac{||K_n|| - ||T(n, r-1)||}{||H||} + O(n) \leqslant \frac{||K_n|| - \text{ex}(n, H)}{||H||} + o(n^2) \ . \qquad \square$$

It is worth noting that Lemma 5 implies that for a fixed H with $\chi(H) \geqslant 3$,

$$\lim_{n \to \infty} \frac{\text{cl}(K_n, H)}{\text{pp}(K_n, H)} = \frac{1}{\chi(H) - 1} \ .$$

2.3 $\chi(H) = 2$ and H is not a forest

For this case we prove the following lemma which, in turn, relies on several breakthrough results [4, 10, 13] that imply that for any fixed graph H, an H-divisible graph with sufficiently many vertices and sufficiently large minimum degree has a perfect H-packing.

Lemma 6. *Let H be a fixed graph. Then there exist $\delta = \delta(H) > 0$, $C = C(H)$, and $N = N(H)$ such that the following holds. If G is a graph with $n > N$ vertices and minimum degree at least $(1 - \delta)n$, then G has an H-packing which covers all but at most Cn edges of G.*

Proof. By any one of the results [4, 10, 13], for every fixed H there exist $\epsilon = \epsilon(H) > 0$ and $N_1 = N_1(H)$ such that any H-divisible graph G with $n > N_1$ vertices and minimum degree at least $(1 - \epsilon)n$ has a perfect H-packing. We can assume that $\epsilon < 1/3$.

Let $\delta = \epsilon/3$ and for notational convenience, let $gcd(H) = r$ and $||H|| = h$. Let s be the smallest even integer larger than $6rh/\epsilon$. Let $N = \max\{N_1, \lceil 6s/\epsilon \rceil\}$. Let G be a graph with $n > N$ vertices and minimum degree $(1 - \delta)n$. If G were H-divisible, we would be done as G would have a perfect H-packing. Unfortunately, this might not be the case.

Let $V = V(G) = \{v_1, \ldots, v_n\}$. Consider a set S of new vertices, $V \cap S = \varnothing$, such that $|S| = s$, where s is as defined above. Recall that s is even, $s > 6rh/\epsilon$. We shall construct a new graph on a vertex set $V \cup S$ so that V induces G and so that the new graph is H-divisible. More specifically, we shall construct this new graph in such a way that all its vertex degrees are divisible by $2rh$. Then clearly each degree is divisible by r and the number of edges is divisible by h.

We shall define a graph G' whose vertex set is $V \cup S$, $G'[V] = G$, $G'[S] = K_s$, and the adjacencies between S and V are defined by the following procedure.

We define these adjacencies in n steps where initially before the first step, we take all ns possible edges between S and V and in each step we delete a few of them. Let d_i denote the degree of v_i in G' before the first step (so $d_i = \deg_G(v_i) + s$). Let $b_i \equiv d_i \pmod{2rh}$ so that $0 \leqslant b_i < 2rh$. In the first step we arbitrarily remove b_1 edges between v_1 and S. So, after this removal, the degree of v_1 becomes $d_1 - b_1 \equiv 0 \pmod{2rh}$, and some vertices of S (that is, precisely b_1 vertices of S) have degree equal to $n+s-2$ while the other $s-b_1$ vertices of S have degree equal to $n+s-1$. In a general step i, we remove b_i edges between v_i and S so that the b_i endpoints of these edges in S are the ones that presently have the highest degree. Notice that at any point in this process, the degrees of any two vertices of S differ by at most 1. After this process ends, the resulting G' has the following property. The degree of each $v_i \in V$ in G' is $d_i - b_i \equiv 0 \pmod{2rh}$, and for some q, each vertex from S has degree either q or $q+1$ in G'.

Let us next estimate q. The total number of edges removed in the aforementioned process is $\sum_{i=1}^{n} b_i < 2rhn$. Thus, the number of non-neighbors of each vertex of S is at most $\lceil 2rhn/s \rceil$ implying that $q \geqslant n+s-1-\lceil 2rhn/s \rceil$. Let S_1 be the set of vertices of S with degree $q+1$ in G' and let S_0 be the set of vertices of S with degree q in G'. Recall that $|S_0| + |S_1| = s$ is even. We claim that both $|S_0|$ and $|S_1|$ are even. Assume otherwise, then both are odd. But since all the degrees of all $v \in V$ in G' are $0 \pmod{2rh}$ and in particular even, we have that G' has an odd number of vertices with odd degree, a contradiction. So, we have that both S_1 and S_0 are even. Take an arbitrary perfect matching in $G'[S_1]$ (recall, $G'[S] = K_s$ so this can be trivially done) and remove it. Thus in the resulting new graph G'' all the degrees of the vertices of S are precisely q and we have not changed the degrees of the other vertices. Now, suppose $q \equiv t \pmod{2rh}$ where $0 \leqslant t < 2rh$. Take t pairwise edge-disjoint perfect matchings of S in G'' and remove them from G'' (this can easily be done greedily since after removing each perfect matching the minimum degree the subgraph induced by S is larger than $s/2$ since $s > 4rh$). The resulting graph G^* now has all of its degrees $0 \pmod{2rh}$, so G^* is divisible by H.

Let us next estimate the minimum degree of G^* which has $n+s$ vertices. The degree in G^* of every vertex $v_i \in V$ is $\deg_G(v_i) + s - b_i \geqslant \deg_G(v_i) \geqslant (1-\delta)n \geqslant (1-\epsilon/2)(n+s)$. In the last inequality we use the fact that $\delta = \epsilon/3$ and n is sufficiently large. The degree of every vertex of S in G^* is $q - t \geqslant n+s-1-\lceil 2rhn/s \rceil - 2rh \geqslant (1-\epsilon/2)(n+s)$, where we have used here that $s \geqslant 6rh/\epsilon$. Therefore, G^* has a perfect H-packing $\mathcal{P}$. The elements of $\mathcal{P}$ that are not entirely contained in G are those that have an edge incident to S. The number of such copies of H is at most $s(n+s)$. Deleting the edges of these copies gives an H-packing of G that covers all but at most $(n+s)s||H|| \leqslant Cn$ edges for an appropriate constant C. □

Lemma 7. *Let H be a bipartite graph that contains a cycle. Then,* $\mathrm{cl}(K_n, H) \leqslant (||K_n|| - \mathrm{ex}(n, H))/||H|| + O(n)$. *In particular,* $\mathrm{cl}(K_n, H) = (||K_n|| - \mathrm{ex}(n, H))/||H|| + o(\mathrm{ex}(n, H))$.

Proof. Suppose that H contains k vertices (so $k \geqslant 4$) and let 2ℓ denote the length of a shortest (hence even) cycle in H, $k \geqslant \ell$. On the one hand, any graph that is $C_{2\ell}$-free is also H-free and on the other hand, any graph that contains $K_{k,k}$ also contains H. By the known lower bounds for $\mathrm{ex}(n, C_{2\ell})$ [16, 17], see also the improved bounds in [15], we have that $\mathrm{ex}(n, H) \geqslant \mathrm{ex}(n, C_{2\ell}) = \Omega(n^{1+\frac{2}{3\ell+3}}) \geqslant \Omega(n^{1+\frac{4}{3k+6}})$.

Let δ, N, C be the constants from Lemma 6, $\delta < 1$, and let $\gamma = \delta/2$. Let $G = K_n$ where $n > N + k(4/\gamma)^k$ and let G' be a spanning subgraph of G which is H-free and has $||G'|| = \mathrm{ex}(n, H)$. Recall that $\mathrm{ex}(n, H) \leqslant \mathrm{ex}(n, K_{k,k}) \leqslant kn^{2-1/k}$ as follows from Zarankiewicz' argument. Let L be the set of vertices of G' whose degree in G' is at least γn and let $S = V(G') \setminus L$ be the remaining vertices. Note that $|L| \leqslant \gamma n/2$ since otherwise $||G'|| \geqslant \gamma^2 n^2/4 > \mathrm{ex}(n, H)$. In particular, we have that $|S| \geqslant n/2$.

We claim that $|L| \leqslant k(4/\gamma)^k$. For consider the bipartite graph B whose parts are L, S and that contains all the edges of G' with one endpoint in L and the other in S. Then $||B|| \geqslant |L|(\gamma n - |L|) \geqslant |L|\gamma n/2$ and since B is $K_{k,k}$-free, it follows from the Kovári-Sós-Túran Theorem [14] that

$$|L|\frac{\gamma}{2}n \leqslant (k-1)^{1/k} n|L|^{1-1/k} + k|L|,$$

which implies that $|L| < k(4/\gamma)^k$.

Let G^* be the complement of $G'[S]$. So, G^* has $|S| = n - |L| \geqslant n - k(4/\gamma)^k > N$ vertices and its minimum degree is at least $n - \gamma n - |L| \geqslant |S|(1-\delta)$. By Lemma 6, G^* contains an H-packing $\mathcal{P}$ that covers all but at most $C|S| \leqslant Cn$ edges of G^*. Since G' is H-free, any copy of H in G which does not have an edge in an element of $\mathcal{P}$ must contain an edge that is either one of these at most Cn uncovered edges, or an edge incident to L. Hence we can augment $\mathcal{P}$ to a maximal H-packing by adding at most $Cn + |L|n$ elements to it. □

2.4 *H* is a forest

We shall need additional results about special packings of trees in dense graphs.

Lemma 8. *Let k be a fixed positive integer and N be an integer, $n > 18(k-1)$. Let T be a forest on k vertices. Then if F is a graph on n vertices with minimum degree at least $2n/3$ then F contains n edge-disjoint copies of T such that for each vertex of T, its n respective images in the n copies are distinct. In particular, each vertex of F belongs to exactly k copies of T.*

Proof. We shall use induction on k with a trivial basis $k = 1$. Let T be a forest on k vertices and T' is the forest obtained from T by removing a leaf v adjacent to a vertex v'. Note that if there is no such leaf, then T has no edges and the result follows trivially. Then by induction hypothesis there is a set $\mathcal{T}'$ of n pairwise edge-disjoint copies of T' in F such that in particular the images of v' are distinct. Let F' be a graph obtained from F by deleting the edges of copies of T' from $\mathcal{T}'$ as well as by deleting all those edges of F that join an image of v' to the vertices of its copy of T', for each copy of T' from $\mathcal{T}'$. We see that each vertex of F is an image of each vertex of T' in some copy of T'. Each vertex of F belongs to $k-1$ copies of T' playing a role of different vertices of T'. Thus the number of deleted edges that were incident to each vertex is at most $\sum_{v \in V(T')} \deg(v) + (k-1) = 3(k-1)$. Therefore the minimum degree of F' is at least $2n/3 - 3(k-1) \geqslant n/2$. By Dirac's theorem, we see that there is a Hamiltonian cycle in F'. Extend each copy of T' from $\mathcal{T}'$ to a copy of T by picking a neighbor of the image of v' in that copy on this cycle such that distinct vertices get distinct neighbors. These newly picked neighbors serve as images of v in respective copies of T. □

Lemma 9. *Let H be a forest with at least one edge. Then,* $\mathrm{cl}(K_n, H) \leqslant (||K_n|| - \mathrm{ex}(n,H))/||H|| + O(1)$. *In particular,* $\mathrm{cl}(K_n, H) = (||K_n|| - \mathrm{ex}(n,H))/||H|| + o(\mathrm{ex}(n,H))$.

Proof. Assume that the number of vertices of H is $k \geqslant 3$ and H has at least two edges. If H has one edge, and $n \geqslant |V(H)|$, we have $\mathrm{ex}(n,H) = 0$ and $\mathrm{cl}(K_n,H) = ||K_n|| = (||K_n|| - \mathrm{ex}(n,H))/||H||$. Let $G = K_n$ and let G' be a spanning subgraph of G which is H-free and satisfies $||G'|| = \mathrm{ex}(n,H)$. By the simple bounds on $\mathrm{ex}(n,H)$ we have that $\lfloor n/2 \rfloor \leqslant ||G'|| \leqslant kn$. Indeed, for the lower bound note that a matching does not contain H. For the upper bound, note that a graph on kn edges with n sufficiently large has a sufficiently large subgraph with minimum degree k, that in turn contains H. Let L be the set of vertices of G' of degree at least $n/4$, let S be the set of remaining vertices, and let $|L| = \ell$. Since $(n/4)|L| \leqslant 2||G'|| \leqslant 2kn$ we have that $|L| \leqslant 8k$. It follows that $\delta(\overline{G'}[S]) \geqslant n - 1 - n/4 - |L| \geqslant 2n/3$ (assuming $n \geqslant 12(8k+1)$) . Unlike the case for bipartite graphs that are not forests, we shall first completely cover the edges of $\overline{G'}$ between L and S, then we shall (almost completely) pack the remaining edges of $\overline{G'}[S]$.

Now, consider H, our given forest, let H' be obtained from H by deleting a leaf u' adjacent to a vertex u. Let H^* be a union of ℓ copies of H' that pairwise share only a vertex corresponding to u. Apply Lemma 8 to find $|S|$ edge-disjoint copies of H^* in $F = \overline{G'}[S]$ such that each vertex of S serves as an image of u in some copy of H^*. For a vertex $s \in S$, let H^*_s be the copy of H^* such that s serves as an image of u. Let X_s be the set of all vertices in L such that s is not adjacent to its members in G', let $d_s = |X_s|$, note that $0 \leqslant d_s \leqslant \ell = |L|$. Delete $\ell - d_s$ copies of H' from H^*_s and add the pairs $\{sx : x \in X_s\}$ to the edge set of the resulting graph, call it H''_s. Note that H''_s is an edge disjoint union of d_s copies of H that covers all edges from s to L in the complement of G'. Thus all graphs H''_s, for $s \in S$ are pairwise edge-disjoint and cover all the edges between S and L in the complement of G'. Let $\mathcal{P} = \{H''_s : s \in S\}$. Let R be a graph with a vertex set S with remaining edges in $\overline{G'}[S]$, i.e., the edges of $\overline{G'}[S]$ that do not belong to any of H''_s, $s \in S$. We see that the minimum degree of R is at least $2n/3 - |H^*|^2 \geqslant 2n/3 - \ell^2 k^2 \geqslant |S|/2$ for n sufficiently large.

By a result of Yuster [21], if H is a tree and $|S|$ is sufficiently large, a graph with $|S|$ vertices and minimum degree at least $|S|/2$ has an H-packing where less than $||H|| < k$ edges remain uncovered. The proof in [21] gives the same result when H is a forest. Another way to see this is that from any forest without isolated vertices, one can construct a tree consisting of two edge-disjoint copies of that forest, and pack with that tree thereby packing with the forest.

Let then $\mathcal{Q}$ be an H-packing of R where less than k edges remain uncovered.

Hence $\mathcal{P} \cup \mathcal{Q}$ is an H-packing of $\overline{G'}$ which covers all edges of $\overline{G'}$ except for at most k edges in $\overline{G'}[S]$ and except uncovered so far edges in $\overline{G'}[L]$. However, there are at most $\binom{\ell}{2} \leqslant 32k^2$ edges induced by L in $\overline{G'}$. Hence, we can obtain a maximal H-packing of $\overline{G'}$ of size at most $|\mathcal{P} \cup Q| + k + 32k^2 \leqslant (||K_n|| - \mathrm{ex}(n, H))/||H|| + k + 32k^2$, as required. □

3 Hypercubes

In this section we prove Theorem 3. We first note that computing $\mathrm{cl}(Q_n, Q_2)$ (and moreover $\mathrm{cl}(Q_n, Q_d)$) is difficult already for very small values of n. While $\mathrm{cl}(Q_3, Q_2) = 2$, $\mathrm{cl}(Q_4, Q_2) = 3$ are trivial, it is only known that $\mathrm{cl}(Q_5, Q_2) \in \{7, 8\}$ [12]. Recall also that Q_n is an n-regular graph with 2^n vertices hence $||Q_n|| = n2^{n-1}$. More generally, we observe the following.

Lemma 10. *The number of copies of Q_d in Q_n is $2^{n-d}\binom{n}{d}$. Each edge of Q_n belongs to $\binom{n-1}{d-1}$ copies of Q_d.*

Proof. Each copy of Q_d can be represented by an n-vector in $\{0, 1, \star\}$ with d entries of $\star$. So, the first part of the lemma follows from the fact that there are $2^{n-d}\binom{n}{d}$ such vectors. If an edge e is fixed, its endpoints differ in exactly one position, say position i. Then the i'th coordinate corresponds to a $\star$ in any copy of Q_d containing e. There are $\binom{n-1}{d-1}$ ways to choose other $\star$ positions and the remaining coordinates must take the respective values of endpoints of e. □

Let $f(n, d) = ||Q_n|| - ex(Q_n, Q_d)$ be the smallest size of an edge subset S of Q_n such that each copy of Q_d in Q_n contains at least one element of S. Identically, $f(n, d)$ is the transversal number of the hypergraph whose vertices are the edges of Q_n and whose edges are the (edges of) the copies of Q_d in Q_n. Let

$$c(d) = \lim_{n\to\infty} \frac{f(n, d)}{||Q_n||} .$$

Alon, Krech, and Szabó [2] proved that for some absolute positive constant C,

$$\Omega\left(\frac{\log d}{d\ 2^d}\right) \leqslant c(d) \leqslant \frac{C}{d^2} . \tag{2}$$

3.1 Lower bound

In this subsection we prove the simple lower bounds stated in Theorem 3. Let $\mathcal{P}$ be a maximal Q_d-packing of Q_n. Since $\mathcal{P}$ is maximal, every copy of Q_d in Q_n contains an edge of a member of $\mathcal{P}$. Hence, by Lemma 10, we are counting $\binom{n}{d}2^{n-d}$ edges in this way, but each edge may be counted many times, as it may appear in $\binom{n-1}{d-1}$ copies of Q_d. Thus, the total number of edges of all elements of $\mathcal{P}$ is at least $\binom{n}{d}2^{n-d}/\binom{n-1}{d-1}$. Since each element of $\mathcal{P}$ consists of $d2^{d-1}$ edges it follows that

$$|\mathcal{P}| \geqslant \frac{\binom{n}{d}2^{n-d}}{\binom{n-1}{d-1}d2^{d-1}} = \frac{2^{n-2d+1}n}{d^2} .$$

To improve this lower bound by a factor of $\log d$ we use (2). Indeed, if $\mathcal{P}$ is the smallest possible set of Q_d's in Q_n that contains an edge of each Q_d of Q_n (namely $|\mathcal{P}| = \mathrm{cov}(Q_n, Q_d)$), then the set of all edges of members of $\mathcal{P}$ forms a transversal of Q_d's in Q_n. By (2), $||Q_d|| \cdot |\mathcal{P}| \geqslant f(n,d) \geqslant \Omega\left(\frac{\log d}{d\ 2^d}\right)||Q_n||$, thus

$$\mathrm{cl}(Q_n, Q_d) \geqslant \mathrm{cov}(Q_n, Q_d) = |\mathcal{P}| \geqslant \Omega\left(\frac{\log d}{d\ 2^d}\right)\frac{||Q_n||}{||Q_d||} .$$

To get a lower bound on $\mathrm{cl}(Q_n, Q_2)$, we use a result of Baber [3] stating that

$$\mathrm{ex}(Q_n, Q_2) \leqslant 0.6068||Q_n||(1 + o(1)).$$

Thus by (1), we have

$$\mathrm{cl}(Q_n, Q_2) \geqslant \frac{(||Q_n|| - \mathrm{ex}(Q_n, Q_2))}{||Q_2||} \geqslant 0.3932\frac{||Q_n||}{||Q_2||}(1 - o(1)) .$$

We note that Erdős conjectured that $\mathrm{ex}(Q_n, Q_2) = \frac{1}{2}||Q_n||(1+o(1))$, so if true, the constant 0.3932 in the last inequality can be replaced by $\frac{1}{2}$.

3.2 Upper bound

In this subsection we prove the upper bounds stated in Theorem 3. For $i = 0, \ldots, n$ we denote by V_i the set of vertices of Q_n with i one's in their vector representation. We say that vertices or respective vectors from V_i have weight i. For $i = 1, \ldots, n$, let L_i be the set of edges of Q_n with endpoints in $V_{i-1} \cup V_i$. We call L_i the i^{th} *edge layer* of Q_n. We provide constructions of maximal Q_d-packings $\mathcal{P}$ of Q_n such that the edges of $\mathcal{P}$ cover almost completely every $(d-1)^{\mathrm{st}}$ layer of Q_n, for $d \geqslant 3$, and for $d = 2$, these edges cover two out of every three consecutive layers of Q_n almost completely.

Let $I = [n/2 - \sqrt{n \log n}, n/2 + \sqrt{n \log n}]$. Since the properties of binomial distribution give us that $\sum_{i \notin I} |V_i| = o(|Q_n|)$, we have that $\sum_{i \notin I} |L_i| = o(||Q_n||)$, thus we shall focus on the middle layers L_i, $i \in I$ and later consider any maximal packing of the remaining layers. We denote the edge set in these *middle layers* by $M = \bigcup_{j \in I} L_j$.

We consider first the case $d = 2$ and later see how to generalize our arguments to an arbitrary d. Let $M = M_0 \cup M_1 \cup M_2$, where $e \in M_i$ if and only if $e \in L_j$, $j \equiv i \pmod 3$.

Lemma 11. *Let $j \in I$. There is a Q_2-packing, denoted $\mathcal{P}_j$, such that each member of $\mathcal{P}_j$ contains at least one edge of $L_j \cup L_{j+1}$ and such that $\mathcal{P}_j$ covers all but at most $O(n^{-1/3}(|L_j| + |L_{j+1}|))$ edges of $L_j \cup L_{j+1}$.*

Proof. Let H_j be the hypergraph whose vertices correspond to the edges of $L_j \cup L_{j+1}$ and whose hyperedges are four-element subsets forming a copy of Q_2. Since any two copies of Q_2 intersect in at most one edge, H_j is simple (linear). Note that the degree of an element of L_j in H_j is $n-j$ since it appears in precisely $n-j$ Q_2's having all of their edges $L_j \cup L_{j+1}$. Indeed, suppose this element is the edge $e=(u,v) \in L_j$ where u is a vector of weight $j-1$ and v is a vector of weight j. Then a vertex x of a Q_2 containing e and which is adjacent to u must be also of weight j but distinct from v, so there are $n-j$ options to choose x. The fourth vertex of this Q_2 is now completely determined. Similarly, the degree of an element of L_{j+1} in H_j is j. To see this, suppose this element is the edge $e=(u,v) \in L_{j+1}$ where u is a vector of weight j and v is a vector of weight $j+1$. Then a vertex x of a Q_2 containing e and which is adjacent to v must be also of weight j but distinct from u, so there are j options to choose x. The fourth vertex of this Q_2 is now completely determined. We see, using that $j \in I$, that the absolute difference between the degrees of any two vertices of H_j is at most $|n-2j| \leqslant 4\sqrt{n \log n}$.

A result of Alon, Kim, and Spencer [1], implies that if a 4-uniform hypergraph H has minimum degree at least $D - O(\sqrt{D \log D})$, where D is the maximum degree, then there is a matching in the hypergraph covering all but at most $|V(H)|O(D^{-1/3})$ vertices. Since the maximum degree of H_j is $\max\{j, n-j\} \geqslant n/2$, we see that there is a matching of H_j covering all but $(|L_j| + |L_{j+1}|)O(n^{-1/3})$ vertices. This matching corresponds to a Q_2-packing, call it $\mathcal{P}_j$, whose elements cover all but $O(n^{-1/3}(|L_j| + |L_{j+1}|))$ edges of $L_j \cup L_{j+1}$. □

Let $\mathcal{P}' = \bigcup_{j \in I,\ j \equiv 0 \pmod 3} \mathcal{P}_j$. Then $\mathcal{P}'$ is a Q_2-packing that covers all but $o(|M_0 \cup M_1|)$ edges of $M_0 \cup M_1$, and does not cover any edge from M_2. Let F denote the set of $o(|M_0 \cup M_1|)$ uncovered edges of $M_0 \cup M_1$. Now augment $\mathcal{P}'$ to a maximal Q_2-packing $\mathcal{P}$ of Q_n. We claim that each element of $\mathcal{P} \setminus \mathcal{P}'$ contains an edge from $F \cup (E(Q_n) \setminus M)$. Indeed this just follows from the obvious fact that each Q_2 contains edges from precisely two consecutive layers, hence each Q_2 in $\mathcal{P} \setminus \mathcal{P}'$ has an edge which is not from M_2, thus from $F \cup (E(Q_n) \setminus M)$.

But now, since $|F \cup (E(Q_n) \setminus M)| = o(|M_0 \cup M_1|) + o(||Q_n||) = o(||Q_n||)$, it follows that

$$\mathrm{cl}(Q_n, Q_2) \leqslant |\mathcal{P}| = \frac{2}{3}\frac{|M|}{||Q_2||} + o(||Q_n||) \leqslant \frac{2}{3}\frac{||Q_n||}{||Q_2||}(1+o(1)) \, .$$

Next consider the case $d \geqslant 3$. We shall apply a similar idea as in the case $d=2$, by first finding a packing $\mathcal{P}'$ of the middle layers with copies of Q_d. Let M_0 be the union of L_j's such that $L_j \subseteq M$ and $j \equiv 0 \pmod d$. First we find a packing $\mathcal{P}'$ such that M_0 is covered almost completely, then we augment this packing with a few copies of Q_d so that the resulting packing is maximal. In what follows we assume that d is odd. For d even the argument is very similar. Notice that each copy of Q_d has edges from precisely d consecutive layers, so when d is odd, the *middle layer* of a Q_d is well-defined. A *maximum co-degree* of a hypergraph is the largest number of hyperedges whose interection has size at least two.

Lemma 12. *Let $j \in I$ and $d \geqslant 3$. There is a Q_d-packing denoted $\mathcal{P}_j$, such that each member of $\mathcal{P}_j$ contains at least one edge of L_j in its middle layer and such that $\mathcal{P}_j$ covers all but at most $o(|L_j|)$ edges of L_j.*

Proof. Let H_j be the hypergraph whose vertices are the edges of L_i, $i \in J$, where $J = [j-(d-1)/2, j+(d-1)/2]$ and whose hyperedges are $d2^{d-1}$-element subsets forming a copy of Q_d. We see that H_j is $r = d2^{d-1}$-uniform and by symmetry, all vertices from the same layer L_i have the same degree. Let the maximum degree of H_j be D, and denote the degree of a vertex from L_i in H_j by d_i. Observe also that $D = d_j$.

We shall construct an r-uniform hypergraph H' containing H_j as a spanning subhypergraph such that H' is almost regular, i.e., has degrees D or $D-1$ and such that $E(H') - E(H_j)$ forms a simple hypergraph. Let $H' = H_j \cup \bigcup_{i \in J} H'_i$, where H'_i is a simple r-uniform hypergraph satisfying $V(H'_i) = L_i$, and all of the degrees of H'_i are either $D - d_i$ or $D - d_i - 1$. Note that since $d_j = D$ we have that H'_j is an empty hypergraph. A result of Bollobás [5] asserts that such an H'_i exists if 1) $x(D-d_i) + y(D-d_i-1)$ is divisible by r, where x and y are the numbers of vertices of degree $D - d_i$ and $D - d_i - 1$, respectively, 2) $|E(H'_i)|$ approaches infinity as $|V(H_i)|$ approaches infinity. The second condition is clearly satisfied when $D > d_i$. To see that the first condition is satisfied for some x and y, $x + y = |V(H'_i)|$, observe that $x(D-d_i) + y(D-d_i-1) = (D-d_i)|V(H'_i)| - y$, so we can choose y to be an integer between 0 and r such that $(D-d_i)|V(H'_i)| - y$ is divisible by r. We have that $|V(H'_i)| = |L_i| = \binom{n}{i} i$. So, H' is a hypergraph whose vertices have degrees D or $D-1$ and whose co-degree is at most the co-degree of H_j.

Next we shall compare the degree D of H' and its maximum co-degree coD. We shall view the vertices of Q_n as subsets of $[n]$. For a vertex $x \subseteq [n]$, let $Up(x)$ and $Down(x)$ be the up-set and down-set of x, respectively, i.e., the set of all supersets of x and the set of all subsets of x. Let V_k be the k^{th} vertex layer of Q_n, $k = 0, \ldots, n$.

The degree of a vertex $e = xy$, $x \subseteq y$ in H_j corresponds to the number of copies of Q_d's containing e and having middle layer in L_j. The number of ways to choose the maximal element of such a Q_d is equal to $u = |Up(y) \cap V_{j+(d-1)/2}| \geqslant cn^k$. The number of ways to choose the minimal element of such a Q_d is equal to $d = |Down(x) \cap V_{j-(d-1)/2-1}| \geqslant c'n^{d-1-k}$, where c, c' are constants depending on d and k is the distance in Q_n between the vertex-layer containing y and the vertex layer $V_{j+(d-1)/2}$. Then the degree of e in H_j is at least $cc'n^{d-1}$.

Now we upper bound the co-degree of two vertices of H_j: $xy, x'y'$, $x \subseteq y$ and $x' \subseteq y'$. We therefore need to find the number of copies of Q_d containing xy and $x'y'$ and having middle layer in L_j. The number of ways to choose the maximal element of such a Q_d is equal to $|Up(y \cup y') \cap V_{j+(d-1)/2}| \leqslant c''n^k$, where k is the distance in Q_n between the vertex-layer containing $y \cup y'$ and the vertex layer $V_{j+(d-1)/2}$. The number of ways to choose the minimal element of such a Q_d is equal to $|Down(x \cap x') \cap V_{j-(d-1)/2-1}| \leqslant c'''n^{k'}$, where k' is is the distance in Q_n between the vertex-layer containing $x \cap x'$ and the vertex layer $V_{j-(d-1)/2-1}$. Then the co-degree of xy and $x'y'$ is at most $c''c'''n^{k+k'}$. We see that $x \cap x' \subseteq y \cup y'$ and $|(y \cup y') - (x \cap x')| \geqslant 2$, so $k' \leqslant d-2-k$. Thus $coD \leqslant Cn^{d-2}$, for a constant C. Since any two hyperedges from $E(H') - E(H_j)$ intersect in at most one vertex, the maximum co-degree of H' is also at most Cn^{d-2}.

We need a result of Frankl and Rödl [8] on near perfect matchings of uniform hypergraphs. They have proved that for an integer $r \geqslant 2$ and a real $\beta > 0$ there exists $\mu = \mu(r, \beta) > 0$ such that if the r-uniform hypergraph L has the following properties for some t: (i) The degree of each vertex is between $(1-\mu)t$ and $(1+\mu)t$, (ii) the maximum co-degree is at most μt, then L has a matching of size at least $(|V(L)|/r)(1-\beta)$. Applying their result to our hypergraph H' (which is almost regular) we obtain that it has a matching W that covers all but $o(|V(H')|) = o(|L_j|)$ vertices of H'. In particular, we see that W covers all, but at most $o(|L_j|)$ vertices from L_j. Since all edges from H' that contain vertices from L_j are all from H_j, we see that the set W' of hyperedges of W containing vertices from L_j corresponds to a set of pairwise edge-disjoint copies of Q_d with middle layer in L_j. These copies cover all but $o(|L_j|)$ edges of L_j. Let $\mathcal{P}_j$ be a Q_d-packing corresponding to the hyperedges of W'. We have that $|\mathcal{P}_j| = (|L_j|/m_d)(1+o(1))$, where m_d is the number of edges in the middle layer of Q_d:

$$m_d = \binom{d}{(d-1)/2}(d+1)/2 \approx (d/2)2^d/\sqrt{\pi d/2} = 2^{d-1/2}\sqrt{d}/\sqrt{\pi}. \qquad \square$$

The rest of the construction is done as in the case $d = 2$. Let $\mathcal{P}' = \cup_{j \in I, j \equiv 0 \pmod d} \mathcal{P}_j$. Let $M = M_0 \cup \cdots \cup M_{d-1}$, where $e \in M_i$ if and only if $e \in L_j$, $j \equiv i \pmod d$. Thus, $\mathcal{P}'$ covers all but $o(|M_0|)$ edges of M_0. Let F denote these $o(|M_0|)$ uncovered edges of M_0. Now augment $\mathcal{P}'$ to a maximal Q_d-packing $\mathcal{P}$ of Q_n. We claim that each element of $\mathcal{P} \setminus \mathcal{P}'$ contains an edge from $F \cup (E(Q_n) \setminus M)$. Indeed this just follows from the obvious fact that each Q_d contains edges from precisely d consecutive layers, hence each Q_d in $\mathcal{P} \setminus \mathcal{P}'$ has an edge which is not from $M_1 \cup \cdots \cup M_{d-1}$, thus from $F \cup (E(Q_n) \setminus M)$.

But now, since $|F \cup (E(Q_n) \setminus M)| = o(|M_0|) + o(||Q_n||) = o(||Q_n||)$, it follows that

$$\mathrm{cl}(Q_n, Q_d) \leqslant |\mathcal{P}| = \frac{1}{d}\frac{||Q_n||}{m_d} + o(||Q_n||) = \frac{\sqrt{2\pi}}{\sqrt{d}(1-o_d(1))}\frac{||Q_n||}{||Q_d||}(1+o_n(1)) \ .$$

4 Regular planar tilings

In this section we consider the regular tessellations (tilings) of the Euclidean plane. It is well-known that there are only three such tilings. The triangular tiling R_3, the square tiling R_4, and the hexagonal (honeycomb) tiling R_6. Viewed as infinite graphs, the vertices and edges of R_k $(k = 3, 4, 6)$ are those of the regular k-gons comprising it.

To naturally define clumsy packing and perfect packing of R_k, we consider parametrized finite subgraphs of R_k. Assume that the edges of R_k have unit length and that there is an edge of R_k connecting the origin $(0, 0)$ and $(1, 0)$. This uniquely defines all the Euclidean points of the vertices of R_k. For an integer n, let $R_k(n)$ be the induced subgraph of R_k on vertices inside $[0, n) \times [0, n)$. So, for example $R_4(n)$ is just the square $n \times n$ grid Gr_n. Let

$$\mathrm{cl}(R_k) = \lim_{n\to\infty} \frac{k \cdot \mathrm{cl}(R_k(n), C_k)}{||R_k(n)||} \qquad \mathrm{pp}(R_k) = \lim_{n\to\infty} \frac{k \cdot \mathrm{pp}(R_k(n), C_k)}{||R_k(n)||} \ .$$

The fact that these limits exist will follow in particular from the proof below. So, in $\mathrm{cl}(R_k)$ and $\mathrm{pp}(R_k)$ we want to measure the "fraction" of edges of R_k that are covered by the "smallest" (resp. "largest") maximal packing of R_k. Note that it is straightforward that R_3 has a perfect triangle packing and that R_4 has a perfect C_4-packing hence $\mathrm{pp}(R_3) = \mathrm{pp}(R_4) = 1$. Clearly, R_6 does not have a perfect C_6-packing as it is 3-regular, but it is a straightforward exercise to pack R_6 with C_6 such that the unpacked edges form a perfect matching, hence $\mathrm{pp}(R_6) = 2/3$. In the next theorem we determine $\mathrm{cl}(R_k)$.

Theorem 13. $\mathrm{cl}(R_k) = \frac{2}{k+1}$.

Proof. Consider first the case of R_3. The pattern on the right side of Figure 1 shows how to obtain a maximal triangle packing of R_3 where the ratio between covered and uncovered edges is $\frac{1}{2}$. More formally, this pattern shows that $\frac{\mathrm{cl}(R_3(n),C_3)}{||R_3(n)||} \leqslant \frac{1}{6} + o_n(1)$ implying that $\limsup_{n\to\infty} \frac{3\cdot\mathrm{cl}(R_3(n),C_3)}{||R_3(n)||} \leqslant \frac{1}{2}$. Consider the subgraph H of R_3 shown on the left side of Figure 1. Observe that H has three internal edges and 6 boundary edges. Clearly, there is a covering $\mathcal{C}$ of R_3 with copies of H such that the internal edges of each copy are pairwise edge-disjoint, while the boundary edges are shared between two copies in $\mathcal{C}$. Consider some maximal C_3-packing $\mathcal{P}$ of R_3 and consider some $H \in \mathcal{C}$. We weigh the number of edges of H covered by $\mathcal{P}$ by giving each covered internal edge of H a weight 1 and each covered boundary edge the weight $\frac{1}{2}$. We claim that the weight of each $H \in \mathcal{C}$ is at least 3. Indeed, if the internal triangle of H is in $\mathcal{P}$, we are done. Otherwise, at least one of the internal edges is covered, which means that there is a non-internal triangle of H in $\mathcal{P}$ which consists of one internal edge and two boundary edges. This already yields a weight of 2. But then the other two non-internal triangles of H must intersect elements of $\mathcal{P}$ as well, so each gives an additional weight of at least $\frac{1}{2}$. Now, since the weight of each $H \in \mathcal{C}$ is at least 3 and since its total weight to the edge count is 6 as it has three internal edges and six boundary edges (so $6 \times \frac{1}{2} + 3 \times 1 = 6$), we have that $\frac{\mathrm{cl}(R_3(n),C_3)}{||R_3(n)||} \geqslant \frac{1}{6} - o_n(1)$ implying that $\liminf_{n\to\infty} \frac{3\cdot\mathrm{cl}(R_3(n),C_3)}{||R_3(n)||} \geqslant \frac{1}{2}$.

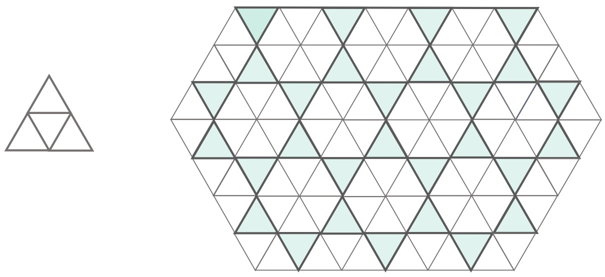

Figure 1: A clumsy triangle packing of R_3 (right) and a gadget subgraph for the lower bound proof (left).

Consider next the case of R_6. The pattern on the right side of Figure 2 shows how to obtain a maximal C_6-packing of R_6. Take the packing to consist of the internal C_6 of each colored region. It is easy to verify that the ratio between covered and uncovered edges of this maximal packing is 2/7. More formally, this pattern shows that $\frac{\mathrm{cl}(R_6(n),C_6)}{||R_3(n)||} \leqslant \frac{1}{21}+o_n(1)$ implying that $\limsup_{n\to\infty} \frac{6\cdot\mathrm{cl}(R_R(n),C_6)}{||R_3(n)||} \leqslant \frac{2}{7}$. Consider now the subgraph H of R_6 shown on the left side of Figure 2. Observe that H has 12 internal edges and 18 boundary edges. As the left side of Figure 2 shows, there is a covering $\mathcal{C}$ of R_6 with copies of H such that the internal edges of each copy are pairwise edge-disjoint, while the boundary edges are shared between two copies in $\mathcal{C}$. Consider some maximal C_6-packing $\mathcal{P}$ of R_6 and consider some $H \in \mathcal{C}$. We weigh the number of edges of H covered by $\mathcal{P}$ by giving each covered internal edge of H a weight 1 and each covered boundary edge the weight $\frac{1}{2}$. We claim that the weight of each $H \in \mathcal{C}$ is at least 6. Indeed, if the internal C_6 of H is in $\mathcal{P}$, we are done. Otherwise, at least one of the internal edges of the internal C_6 is covered, which means that there is a non-internal C_6 of H, call it $X \in \mathcal{P}$ which consists of three internal edges of H and three boundary edges of H. This already yields a weight of 4.5. If $\mathcal{P}$ contains an additional non-internal C_6 of H, then we get a weight of 9 and we are done. Otherwise, the three non-internal C_6 of H which are edge-disjoint from X each contribute at least $\frac{1}{2}$ as $\mathcal{P}$ is a maximal packing, hence overall weight at least 6 as claimed. Now, since the weight of each $H \in \mathcal{C}$ is at least 6 and since its total weight to the edge count is 21 as it has 12 internal edges and 18 boundary edges (so $18 \times \frac{1}{2} + 12 \times 1 = 21$), we have that $\frac{\mathrm{cl}(R_6(n),C_6)}{||R_6(n)||} \geqslant \frac{1}{21} - o_n(1)$ implying that $\liminf_{n\to\infty} \frac{6\cdot\mathrm{cl}(R_6(n),C_6)}{||R_6(n)||} \geqslant \frac{2}{7}$.

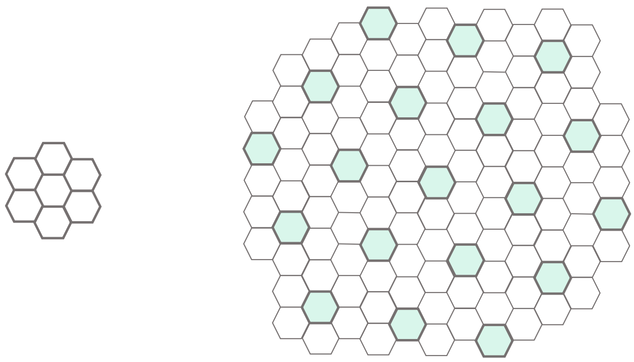

Figure 2: A clumsy C_6 packing of R_6 (right) and a gadget subgraph for the lower bound proof (left).

Consider next the case of $R_4(d)$-packings of R_4. The idea in this case is close to the one on clumsy packings of polyominoes [19]. The pattern on the right side of Figure 3 shows how to obtain a maximal $R_4(d)$-packing of R_4. In fact, the figure specifically shows the case $d = 4$ but the generalization is obvious. Notice also that $R_4(2) = C_4$. Observing the proportion of the edges of the packing in each column and each row of R_4, we have

$$\begin{aligned} \mathrm{cl}(R_4(n), R_4(d)) &\leqslant \frac{d(d-1)}{2(d-1)(2d-2)+1}\frac{||R_4(n)||}{||R_4(d)||}(1+o(1)) \\ &= \frac{d^2-d}{4d^2-8d+5}\frac{||R_4(n)||}{||R_4(d)||}(1+o(1)). \end{aligned}$$

For the lower bound, assume that $\mathcal{P}$ is a maximal $R_4(d)$-packing of $R_4(n)$. We see that each copy of $R_4(d)$ in $R_4(n)$ shares an edge with a copy of an element of $\mathcal{P}$. From the left side of Figure 3 we see marked all the positions of the lower left corner of a copy of $R_4(d)$ that shares an edge with the marked copy of $R_4(d)$. The number of such positions is $(2d-1)^2-4$. Therefore, if x is the total number of $R_4(d)$'s in $R_4(n)$, then $x \leqslant ((2d-1)^2-4)\mathcal{P}(1+o(1))$. Since $x = n^2(1-o(1))$, we have that

$$\begin{aligned}
\mathrm{cl}(R_4(n), R_4(d)) &\geqslant |\mathcal{P}| \\
&\geqslant \frac{n^2}{((2d-1)^2-4)}(1+o(1)) \\
&= \frac{||R_4(n)||}{2((2d-1)^2-4)}(1+o(1)) \\
&= \frac{(d-1)2d}{2((2d-1)^2-4)}\frac{||R_4(n)||}{||R_4(d)||}(1+o(1)) \\
&= \frac{d^2-d}{4d^2-4d-3}\frac{||R_4(n)||}{||R_4(d)||}(1+o(1)).
\end{aligned}$$

Note that the upper and the lower bounds match for $d=2$, giving the claimed value $\mathrm{cl}(R_4) = 2/5$. □

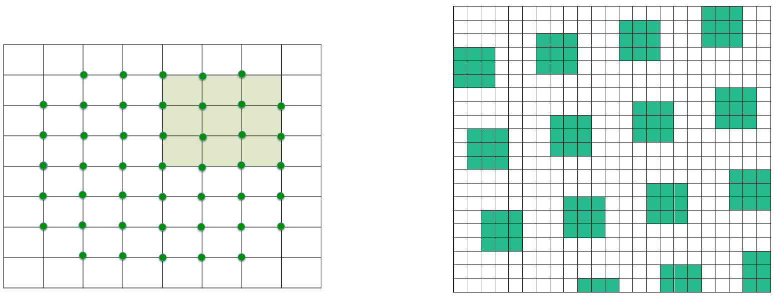

Figure 3: Clumsy packing of $R_4(d)$ in R_4 (right) and the lower bound argument (left).

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