Packing triangles in regular tournaments

Raphael Yuster [*]

Abstract

We prove that a regular tournament with n vertices has more than $\frac{n^2}{11.5}(1 - o(1))$ pairwise arc-disjoint directed triangles. On the other hand, we construct regular tournaments with a feedback arc set of size less than $\frac{n^2}{8}$, so these tournaments do not have $\frac{n^2}{8}$ pairwise arc-disjoint triangles. These improve upon the best known bounds for this problem.

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1   Introduction

All graphs and digraphs considered here are finite and contain no parallel edges or anti-parallel arcs. For standard graph-theoretic terminology the reader is referred to [2]. An *Eulerian orientation* of an undirected graph is an orientation of its edges such that the in-degree of each vertex equals its out-degree. It is well known that a graph G has an Eulerian orientation if and only if every vertex is of even degree. A *tournament* is an orientation of a complete graph. Tournaments play an important role in combinatorics, graph theory, and social choice theory. Properties of Eulerian tournaments have been extensively studied in the literature (see, e.g., [11, 14]). Observe that Eulerian tournaments must have an odd number of vertices and that they are *regular*; the in-degree and out-degree of an n-vertex Eulerian tournament is $(n-1)/2$. Eulerian tournaments are, therefore, the same as regular tournaments.

There are exponentially many non-isomorphic regular tournaments with n vertices [11], but they do all share some properties other than just being regular. Most notably, they all have the same number of *triangles* and the same number of *transitive triples*, where a triangle is a set of three arcs $\{(x, y), (y, z), (z, x)\}$ while a transitive triple is a set of three arcs $\{(x, y), (y, z), (x, z)\}$. Indeed, it is well-known, and easy to prove [5], that the number of transitive triples (and hence triangles) in any tournament is determined by the *score* of the tournament, which is the sorted out-degree sequence. For regular tournaments this amounts to $n(n-1)(n-3)/8$ transitive triples

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[*]Department of Mathematics, University of Haifa, Haifa 31905, Israel.
E–mail: raphy@research.haifa.ac.il

and therefore to $\binom{n}{3} - n(n-1)(n-3)/8 = n(n^2-1)/24$ triangles. Asymptotically, this means that a fraction of $1/4$ of the triples induce triangles while $3/4$ of them induce transitive triples. Stated otherwise, a randomly selected triple induces a triangle with (asymptotic) probability $1/4$. Throughout this paper a triangle is denoted by C_3 and a transitive triple is denoted by TT_3.

An (edge) *triangle packing* of a graph is a set of pairwise edge-disjoint subgraphs that are isomorphic to a triangle. The study of triangle packings in graphs dates back to the classical result of Kirkman [10] who proved that K_n has a triangle packing of size $n(n-1)/6$ whenever $n \equiv 1, 3 \bmod 6$. In design-theoretic terms this is known as a *Steiner triple system.* This clearly implies that for other moduli of n there are packings with $(1 - o_n(1))n^2/6$ triangles, and this is asymptotically tight as such packings cover $(1 - o_n(1))\binom{n}{2}$ edges.

In the directed case, a triangle packing of a tournament requires each subgraph to be isomorphic to C_3. Alternatively, one may ask for a TT_3-packing where each subgraph must be isomorphic to TT_3. Triangle packings and TT_3-packings of digraphs have been studied by several researchers (see, e.g., [4, 13]). For a tournament T, we denote by $\nu_3(T)$ the size of a largest triangle packing and by $\nu_{TT_3}(T)$ the size of the largest TT_3-packing. Observe first that the fact that approximately $1/4$ of the triples of a regular tournament are isomorphic to C_3 implies that $\nu_3(T) \geq (1 - o_n(1))n^2/24$ for a regular tournament T. Indeed, take any optimal triangle packing of K_n (recall that it consists of $(1 - o_n(1))n^2/6$ triangles) and consider the number of undirected triangles that eventually become a C_3 after assigning the orientation to the edges. As packings of K_n are invariant under vertex permutations, the expected number of C_3 in the resulting packing of the tournament is asymptotically $1/4$ of the elements of the (undirected) packing. Therefore, $\nu_3(T) \geq (1 - o_n(1))n^2/24$. On the other hand, we always have the trivial upper bound $\nu_3(T) \leq (1 - o_n(1))n^2/6$.

As usual in extremal graph theory, we are interested in closing the gap between the upper and lower bound in the worst case. More formally, let $\nu_3(n)$ denote the minimum of $\nu_3(T)$ ranging over all regular tournaments with n vertices (recall that n is odd). So the trivial bounds above imply that

$$\frac{n^2}{24}(1 - o_n(1)) \leq \nu_3(n) \leq \frac{n^2}{6}(1 - o_n(1)) .$$

Some exact values for small n are easy to compute. Clearly, $\nu_3(3) = 1$ and it is easy to see that $\nu_3(5) = 2$. It is also not difficult to somewhat improve both the upper and lower trivial bounds, but determining the right order of $\nu_3(n)$ is an open problem. In this paper we prove new nontrivial upper and lower bounds for $\nu_3(n)$.

Theorem 1.1
$$\frac{n^2}{11.5}(1 - o_n(1)) \leq \nu_3(n) \leq \frac{n^2 - 1}{8} .$$

In fact, the constant 11.5 can be replaced with the slightly smaller value of $1/\ln(12/11)$. The proof of the lower bound in Theorem 1.1 is more involved than the proof of the upper bound. The upper bound follows from a construction of a regular tournament with a relatively small *feedback arc set*, which is a set of arcs whose removal makes a digraph acyclic. Recall that $\beta(G)$ denotes the smallest

size of a feedback arc set of a digraph G. Clearly, for any digraph G we have $\beta(G) \geq \nu_3(G)$ since any feedback arc set must contain an arc from each triangle. Thus, our upper bound follows from the existence of a regular tournament T with n vertices and with $\beta(T) \leq (n^2 - 1)/8$.

The proof of the lower bound in Theorem 1.1 is based upon examining a *fractional relaxation* of the triangle packing problem. We obtain a lower bound for the fractional version of the problem and then utilize a result of Nutov and Yuster [12], based on a technique of Haxell and Rödl [6], enabling us to deduce the same bound for the *integral* version, with only a minor loss in the error term. Recently, several papers have made use of the relationship between the integral and fractional solutions in dense settings [7, 8, 16, 18, 19]. All of these results, however, construct a global fractional packing by gluing together many fractional packings of small (constant) edge-disjoint subgraphs. We cannot use this approach as small subgraphs of a regular tournament *do not* usually induce regular tournaments. Our approach is a global one. We construct a fractional packing for our given tournament that does not decompose into disjoint fractional packings of small subgraphs.

The rest of this paper is organized as follows. In Section 2, we define the fractional relaxation of the problem, and show how a solution to the integral version is deduced (asymptotically) from a solution to the fractional version. Section 3 consists of the proof of the lower bound in Theorem 1.1. Section 4 constructs the example yielding the upper bound of Theorem 1.1. Section 5 contains some concluding remarks.

2   Integer versus fractional triangle packings

We start this section by defining the fractional relaxation of the triangle packing problem, and define the parameter $\nu_3^*(n)$ that is the fractional analogue of $\nu_3(n)$. We then utilize a result of Nutov and Yuster to obtain that $\nu_3^*(n) \leq \nu_3(n) + o(n^2)$. This, in effect, enables us to consider only the fractional parameter.

Let R_+ denote the set of nonnegative reals. A *fractional triangle packing* of a digraph G is a function ψ from the set $\mathcal{F}_3$ of copies of C_3 in G to R_+, satisfying $\sum_{e \in X \in \mathcal{F}_3} \psi(X) \leq 1$ for each arc $e \in E(G)$. Letting $|\psi| = \sum_{X \in \mathcal{F}_3} \psi(X)$, the *fractional triangle packing number*, denoted $\nu_3^*(G)$, is defined to be the maximum of $|\psi|$ taken over all fractional triangle packings ψ. Since a triangle packing is also a fractional triangle packing (by letting $\psi = 1$ for elements of $\mathcal{F}_3$ in the packing and $\psi = 0$ for the other elements), we always have $\nu_3^*(G) \geq \nu_3(G)$. However, the two parameters may differ. In particular, they may differ for regular tournaments.

Consider, for example, the 5-vertex regular tournament obtained by the following orientation of K_5 on the vertex set $\{1, 2, 3, 4, 5\}$. Orient a Hamilton cycle $(1, 2, 3, 4, 5)$ and another Hamilton cycle as $(1, 4, 2, 5, 3)$. Clearly, $\nu_3(T) = 2$. On the other hand, we may assign each of the five triangles $(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 1), (5, 1, 2)$ the value $1/2$ thereby obtaining a fractional triangle packing of value 2.5.

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A result of Nutov and Yuster [12] asserts that the integral and fractional parameters differ by $o(n^2)$. In fact, their result is more general. Let $\mathcal{S}$ be any given (finite or infinite) family of digraphs. For a digraph G, let $\nu_{\mathcal{S}}(G)$ denote the maximum number of arc-disjoint copies of elements of $\mathcal{S}$ that can be found in G, and let $\nu_{\mathcal{S}}^*(G)$ denote the respective fractional variant. The following is proved in [12].

Theorem 2.1 *For any given family S of digraphs, if G is an n-vertex digraph then $\nu_{\mathcal{S}}^*(G) - \nu_{\mathcal{S}}(G) = o(n^2)$.*

We note that an *undirected* version of Theorem 2.1 has been proved by Yuster [17] extending an earlier result of Haxell and Rödl [6] (considering single element families) who were the first to prove this interesting relationship between integral and fractional packings. The proof of Theorem 2.1 makes use of the *directed* version of Szemerédi's regularity lemma [15] that has been used implicitly in [3] and proved in [1].

By considering the single-element family $\mathcal{S} = \{C_3\}$ we obtain the following immediate corollaries of Theorem 2.1.

Corollary 2.2 *If T is an n-vertex regular tournament then $\nu_3^*(T) - \nu_3(T) = o(n^2)$.*

Let $\nu_3^*(n)$ be the minimum possible value of $\nu_3^*(T)$ ranging over all n-vertex regular tournaments T. Obviously, $\nu_3^*(n) \geq \nu_3(n)$. Together with Corollary 2.2 we have:

Corollary 2.3 $\nu_3^*(n) \geq \nu_3(n) \geq \nu_3^*(n) - o(n^2)$.

3 Proof of the lower bound

In this section we prove the following theorem that, together with Corollary 2.3, yields the lower bound in Theorem 1.1.

Theorem 3.1 *A regular tournament T with n vertices has $\nu_3^*(T) \geq (1 - o_n(1)) \ln(12/11) n^2$.*

For a vertex v let $D^+(v)$ denote the set of out-neighbors of v and let $D^-(v)$ be the set of in-neighbors of v. Any vertex v of a regular tournament thus has $|D^+(v)| = |D^-(v)| = (n-1)/2$.

Observe that an arc (u, v) of a regular tournament appears in at most $(n-1)/2$ triangles. Indeed, any C_3 containing (u, v) must contain another arc (v, w) where $w \in D^+(v)$. The following proposition gives a naive lower bound for $\nu_3^*(T)$.

Proposition 3.2 *Let T be a regular tournament with n vertices. Then, $\nu_3^*(T) \geq n(n+1)/12$.*

Proof. Recall that the number of C_3 in T is precisely $n(n^2-1)/24$. We define a fractional triangle packing by assigning each C_3 the value $2/(n-1)$. This constitutes a valid fractional triangle packing

as each arc is contained in at most $(n-1)/2$ triangles. The overall value of this fractional packing is, therefore,

$$\frac{n(n^2-1)}{24} \cdot \frac{2}{n-1} = \frac{n(n+1)}{12} .$$

∎

The lower bound in Proposition 3.2 is optimal for regular tournaments on 3 or 5 vertices. Our main result shows that this lower bound can be considerably improved for all sufficiently large regular tournaments.

We call an arc α-*dense* if it is contained in at least αn triangles. Observe that no arc is $1/2$-dense as any arc (u,v) appears in at most $(n-1)/2$ triangles. We require the following lemma that bounds the number of arcs that are α-dense.

Lemma 3.3 *The number of α-dense arcs is at most $2(1/2-\alpha)n^2$. In particular, the number of triangles that contain such arcs is less than $n^3(1/2-\alpha)$.*

Proof. For a vertex v, we compute the number of α-dense arcs entering it. Let $X \subset N^-(v)$ be the set of vertices x such that (x,v) is α-dense. Consider a vertex x of maximum in-degree in the sub-tournament $T[X]$ induced by X. Since in any tournament with $|X|$ vertices the maximum in-degree is at least $(|X|-1)/2$ we have that x has at least $(|X|-1)/2$ arcs entering it in $T[X]$. On the other hand, as (x,v) is α-dense, we also have that x has at least αn vertices of $N^+(v)$ entering it. Since $N^+(v) \cap X = \emptyset$ we have that the in-degree of x in T is at least $(|X|-1)/2 + \alpha n$. But the in-degree of x in T is $(n-1)/2$ and thus

$$(|X|-1)/2 + \alpha n \leq (n-1)/2 .$$

It follows that $|X| \leq n(1-2\alpha)$. Summing over all $v \in V$ we have that the number of α-dense arcs is at most $n^2(1-2\alpha)$. As each arc is contained in at most $(n-1)/2$ triangles we have that the number of triangles that contain α-dense arcs is at most $n^2(n-1)(1/2-\alpha) < n^3(1/2-\alpha)$. ∎

For an arc e let $f(e)$ denote the number of triangles that contain e. We define a fractional triangle packing ψ by assigning to a triangle X the value

$$\psi(X) = \frac{1}{\max_{e \in X} f(e)} .$$

In other words, we consider the three arcs of X and take the arc e of whose $f(e)$ is maximal, setting $\psi(X)$ to $1/f(e)$. Notice that ψ is a valid fractional triangle packing. Indeed, the sum of the weights of triangles containing any arc e is at most $f(e) \cdot f(e)^{-1} = 1$. Theorem 3.1 is obtained by proving that $|\psi| \geq (1 - o_n(1)) \ln(12/11)n^2$.

Proof of Theorem 3.1: Let k be a positive integer and let $0 < \alpha_0 < 1/2$ be a constant to be chosen later. For some constant $0 < c < 1$ to be chosen later as well, let $\alpha_i = \alpha_0 c^i$ for $i = 0, \ldots, k$. Let

$$E_i = \{e \in E(T) \ : \ f(e) \geq \alpha_i n\} .$$

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So, E_i is the set of all α_i-dense arcs and notice that $E_0 \subset E_1 \subset \cdots \subset E_k$. For $i = 0, \ldots, k$, let S_i denote the set of all triangles that contain an arc from E_i and *do not* contain an arc from E_j where $j < i$. In particular, S_0 is just the set of triangles that contain an arc from E_0. Finally, let S_{k+1} be the triangles that are not in $\cup_{i=0}^k S_i$ and note that $S_0, \ldots, S_{k+1}$ is a partition of the set of all $n(n^2 - 1)/24$ triangles of T.

For $i = 0, \ldots, k$, all the elements of $S_0 \cup \cdots \cup S_i$ contain edges that are α_i-dense and therefore by Lemma 3.3 we have that for $i = 0, \ldots, k$:

$$t_i = |\cup_{j=0}^i S_i| < n^3 \left(\frac{1}{2} - \alpha_i \right) . \tag{1}$$

By the definition of t_i we have that for $i = 1, \ldots, k$, $|S_i| = t_i - t_{i-1}$ and that $|S_0| = t_0$. Thus, we also have that

$$|S_{k+1}| = \frac{n(n^2 - 1)}{24} - t_k .$$

For $i = 1, \ldots, k + 1$, all the elements of S_i receive weight that is greater than $1/(\alpha_{i-1}n)$. Indeed, consider $X \in S_i$. We know that it does *not* contain an arc from E_j for $j < i$. So the maximum value of $f(e)$ for an arc e of X is smaller than $\alpha_{i-1}n$. By the definition of ψ we therefore have that $\psi(X) > 1/(\alpha_{i-1}n)$. For elements $X \in S_0$ we use the trivial bound $\psi(X) > 2/n$. Summing up the weights of all the triangles of T we find that:

$$|\psi| \geq t_0 \cdot \frac{2}{n} + \sum_{i=1}^k (t_i - t_{i-1}) \frac{1}{\alpha_{i-1}n} + \left(\frac{n(n^2 - 1)}{24} - t_k \right) \frac{1}{\alpha_k n} .$$

Rearranging the terms we have:

$$|\psi| \geq \frac{n^2 - 1}{24\alpha_k} - \frac{t_0}{n} \left(\frac{1}{\alpha_0} - 2 \right) - \sum_{i=1}^k \frac{t_i}{n} \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}} \right) .$$

Using (1) we have that:

$$|\psi| \geq \frac{n^2 - 1}{24\alpha_k} - n^2 \left(\frac{1}{2} - \alpha_0 \right) \left(\frac{1}{\alpha_0} - 2 \right) - \sum_{i=1}^k n^2 \left(\frac{1}{2} - \alpha_i \right) \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}} \right) .$$

As we need to prove that $|\psi| \geq (1 - o_n(1)) \ln(12/11)n^2$, we conclude from the last inequality that we must show that there exist choices of k, α_0 and c such that

$$\frac{1}{24\alpha_k} - \left(\frac{1}{2} - \alpha_0 \right) \left(\frac{1}{\alpha_0} - 2 \right) - \sum_{i=1}^k \left(\frac{1}{2} - \alpha_i \right) \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}} \right)$$

gets arbitrarily close to $\ln(12/11)$. The last expression is identical to

$$k + 2 - 2\alpha_0 + \frac{1}{24\alpha_k} - \frac{1}{2\alpha_k} - \sum_{i=1}^k \frac{\alpha_i}{\alpha_{i-1}} .$$

Recalling that $\alpha_i = \alpha_0 c^i$ and denoting the last expression as $g(k, c, \alpha_0)$ we get that:

$$g(k, c, \alpha_0) = k + 2 - 2\alpha_0 - \frac{11}{24\alpha_0 c^k} - ck . \qquad (2)$$

Let us first find the value of α_0 that maximizes $g(k, c, \alpha_0)$ subject to a given c and k. As $f'_{\alpha_0} = -2 + 11/(24c^k\alpha_0^2)$ the maximum is obtained when

$$\alpha_0 = \sqrt{\frac{11}{48}} c^{-k/2} .$$

Plugging the value of α we get that at this point $g(k, c)$ (now a function of two variables) is

$$g(k, c) = k + 2 - ck - c^{-k/2}\sqrt{11/3} .$$

The value of c that maximizes $g(k, c)$ subject to a given k is therefore

$$c = (11/12)^{1/(k+2)} .$$

So for a given k, $g(k, c, \alpha_0)$ is maximized when $c = (11/12)^{1/(k+2)}$ and $\alpha_0 = \frac{1}{2}(11/12)^{1/(k+2)}$, so indeed $0 < c < 1$ and $0 < \alpha_0 < 1/2$. Plugging these values we get that at this point $g(k)$ (now a function of a single variable) is

$$g(k) = (k + 2)\left(1 - \left(\frac{11}{12}\right)^{1/(k+2)}\right) .$$

By l'Hôpital's 's Rule,

$$\lim_{k \to \infty} g(k) = \ln(12/11) .$$

■

It is possible to slightly improve the lower bound constant of $\ln(12/11)$ in the proof of Theorem 3.1. Indeed, in the proof of Theorem 3.1, when considering α_i-dense arcs that are not α_{i-1} dense, we may use the fact that each such arc lies on less than $\alpha_{i-1}n$ triangles, so the total number of triangles containing such arcs as maximal edges (w.r.t. the function $f(e)$) is only the number of such arcs multiplied by $\alpha_{i-1}n$ instead of multiplying by $(n-1)/2$ as we do in the last line of Lemma 3.3. However this observation has a marginal effect on the lower bound and improves it by less than 0.002, while the analysis becomes considerably more difficult.

4   Proof of the upper bound

We construct an n-vertex regular tournament T_n with a feedback arc set of size $\beta(T_n) \le (n^2 - 1)/8$. In particular, $\nu_3(T_n) \le (n^2 - 1)/8$.

Consider two disjoint sets of vertices $A = \{a_1, \ldots, a_{(n-1)/2}\}$ and $B = \{b_1, \ldots, b_{(n-1)/2}\}$. We construct a regular tournament T_n on the vertex set $A \cup B \cup \{c\}$ as follows. We induce a transitive

tournament on A by orienting (a_i, a_j) whenever $i > j$. We induce a transitive tournament on B by orienting (b_i, b_j) whenever $i > j$. We orient (c, a_i) for all $a_i \in A$ and orient (b_i, c) for all $b_i \in B$. Finally we orient (a_i, b_j) whenever $j \geq i$ and orient (b_j, a_i) whenever $j < i$.

Observe that T_n is indeed a regular tournament, as each vertex has an out-degree and an in-degree of $(n-1)/2$. How many arcs go from B to A? The vertex b_j has precisely $(n-1)/2 - j$ out-neighbors in A which are $\{a_{j+1}, \ldots, a_{(n-1)/2}\}$. Altogether, there are

$$\sum_{j=1}^{(n-1)/2} \frac{n-1}{2} - j = \frac{(n-1)(n-3)}{8}$$

arcs from B to A. Deleting all of these arcs and deleting all of the $(n-1)/2$ arcs entering c we obtain an acyclic digraph. Indeed, in the remaining subgraph, any three vertices that induce three arcs must all be either in A or in B but both of these sets induce transitive tournaments. We have shown that:

$$\beta(T_n) \leq \frac{(n-1)(n-3)}{8} + \frac{n-1}{2} = \frac{n^2 - 1}{8}$$

as required.

5   Concluding remarks

Interestingly, the analogous problem of *vertex triangle packing* of regular tournaments is almost settled. A vertex triangle packing is a set of pairwise vertex-disjoint subgraphs that are isomorphic to a triangle. Given an n-vertex regular tournament, we can hope for a vertex triangle packing of size at most $\lfloor n/3 \rfloor$, and it has been conjectured by the author as well as by Cuckler that this is always achievable. Recently, Keevash and Sudakov [9] proved that there always exists a vertex triangle packing that covers all but at most 3 vertices. Observe that, unlike for vertex triangle packing, the upper bound in Theorem 1.1 shows that there are regular tournaments for which we cannot expect to cover more than a fraction of 3/4 of the arcs even in an optimal edge triangle packing.

The proof of Theorem 1.1 can be extended, with very minor modifications, to *almost regular* tournaments. We say that a tournament is almost regular if the minimum in-degree and the minimum out-degree are both at least $(1/2 - o(1))n$ (more formally, a family of tournaments is almost regular if each tournament in the family has minimum in-degree and minimum out-degree at least $(1/2 - o(1))n$). The bounds in Theorem 1.1 remain intact for almost regular tournaments.

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