Large Monotone Paths in Graphs with Bounded Degree

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Abstract

We prove that for every $\epsilon > 0$ and positive integer r, there exists $\Delta_0 = \Delta_0(\epsilon)$ such that if $\Delta > \Delta_0$ and $n > n(\Delta, \epsilon, r)$ then there exists a packing of K_n with $\lfloor (n-1)/\Delta \rfloor$ graphs, each having maximum degree at most Δ and girth at least r, where at most ϵn^2 edges are unpacked. This result is used to prove the following: Let f be an assignment of real numbers to the edges of a graph G. Let $\alpha(G, f)$ denote the maximum length of a monotone simple path of G with respect to f. Let $\alpha(G)$ be the minimum of $\alpha(G, f)$, ranging over all possible assignments. Now let α_Δ be the maximum of $\alpha(G)$ ranging over all graphs with maximum degree at most Δ. We prove that $\Delta + 1 \geq \alpha_\Delta \geq \Delta(1 - o(1))$. This extends some results of Graham and Kleitman [6] and of Calderbank, Chung and Sturtevant [4] who considered $\alpha(K_n)$.

1 Introduction

All graphs considered here are finite, undirected and have no loops or multiple edges. For the standard terminology used the reader is referred to [3]. An *edge-ordered graph* is an ordered pair (G, f), where $G = G(V, E)$ is a graph and f is an assignment of real weights to the edges. A *monotone path of length* k in (G, f) is a simple path with k edges, and with nondecreasing edge weights. Given a graph G denote by $\alpha(G)$ the minimum over all edge orderings of the maximum length of a monotone path (note that we can assume f is bijective and that the weights are the integers $1, \ldots, |E|$). Denote by $\alpha'(G)$ the minimum over all edge orderings of the maximum length of a monotone trail (in a trail vertices may appear more than once; a simple cycle is also considered a trail in our definition). Clearly, $\alpha(G) \leq \alpha'(G)$.

The problem of estimating $\alpha(K_n)$ was raised first by Chvátal and Komlós [5]. Graham and Kleitman [6] proved that:

$$\frac{1}{2}(\sqrt{4n-3} - 1) < \alpha(K_n) < \frac{3}{4}n.$$

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The upper bound was improved by Calderbank, Chung and Sturtevant [4], showing,

$$\alpha(K_n) \le \left(\frac{1}{2} + o(1)\right) n.$$

They also conjectured that this is the right order of magnitude of $\alpha(K_n)$. However, no improvement upon the Graham-Kleitman lower bound is known.

There are very few results regarding $\alpha(G)$ for general graphs G. Bialostocki and Roditty [2] have characterized all the graphs G with $\alpha(G) \le 2$. In fact, they showed that if $\alpha(G) \ge 3$ then either G is an odd cycle of length at least 5, or G contains as a subgraph one of six fixed graphs. Roditty, Shoham and Yuster gave upper and lower bounds for $\alpha(G)$ and $\alpha'(G)$ for graphs G belonging to several well-known graph families.

In this paper we consider graphs with maximum degree Δ and ask how large can $\alpha(G)$ be. It is very easy to show that $\alpha(G) \le \Delta + 1$ (cf. Lemma 3.1). We may therefore define α_Δ as the maximum value of $\alpha(G)$ taken over all graphs with maximum degree Δ. Trivially, $\alpha_1 = 1$ and $\alpha_2 = 3$ (every odd cycle with 5 or more vertices shows this). Despite the $\Delta + 1$ upper bound, it is not easy to construct a matching lower bound. In this paper we prove that $\alpha_\Delta \ge \Delta(1 - o(1))$. To summarize, we have:

Theorem 1.1

$$\Delta + 1 \ge \alpha_\Delta \ge \Delta(1 - o(1)).$$

The proof of Theorem 1.1 is based, together with some additional ideas, on a general result concerning packings of complete graphs. Recall that a *packing* of K_n is a collection of graphs, sharing the same vertex set of K_n, and which are edge-disjoint. A *decomposition* of K_n is a packing that uses every edge of K_n. Recall that the *girth* of a graph G, denoted $g(G)$ is the length of the smallest cycle in G. Given a parameter Δ, and assuming Δ divides $n-1$, it is well-known for which values of Δ it is possible to decompose K_n into Δ-regular graphs. It merely follows from the classic fact that if n is odd then K_n can be decomposed into Hamiltonian cycles, and if n is even K_n has chromatic index $n-1$ (see, e.g., [3]). However, the graphs in such a decomposition may (and sometimes will) have small cycles. If we insist that the graphs in the decomposition have girth at least r, we must have n depend upon Δ. This, however, is not sufficient. There are examples of pairs Δ and r, and arbitrary large n, where Δ divides $n-1$, but it is impossible to decompose K_n into Δ-regular graphs with girth at least r. Thus, the best we could hope for is to pack K_n with $(n-1)/\Delta$ graphs, each having maximum degree at most Δ, and girth at least r, and such that the fraction of unpacked edges is very small. Indeed, this can be done, as stated in the following theorem:

Theorem 1.2 *Let $\epsilon > 0$ and let $r \ge 3$ be a positive integer. There exists $\Delta_0 = \Delta_0(\epsilon)$ such that for every $\Delta > \Delta_0$ and for every $n > N(\Delta, \epsilon, r)$, the complete graph K_n can be packed with*

$t = \lfloor (n-1)/\Delta \rfloor$ *graphs* $H_1, \ldots, H_t$, *where* $\Delta(H_i) \le \Delta$ *and* $g(H_i) \ge r$ *for* $i = 1, \ldots, t$. *Furthermore, at most* ϵn^2 *edges of* K_n *are unpacked.*

The proof of Theorem 1.2 is based on probabilistic arguments, and is presented in the next section. Section 3 contains the proof of Theorem 1.1, that is based on Theorem 1.2 together with some additional lemmas. The final section contains concluding remarks and open problems.

2 Proof of Theorem 1.2

We shall assume, without loss of generality, that $\rho = \epsilon/3$ is a rational number of the form $\rho = 1/u$. We pick

$$\Delta_0 = \frac{H(\rho - \rho^2)}{2(1-\rho)\rho^4}$$

where $H(\beta) = -\beta \log_2 \beta - (1-\beta)\log_2(1-\beta)$ is the entropy function. Now let $\Delta > \Delta_0$. We define the function $N = N(\Delta, \rho, r)$ as follows:

$$N = \max\left\{ \left(2\rho^4 - \frac{H(\rho-\rho^2)}{\Delta(1-\rho)}\right)^{-2}, \frac{2\Delta^{r-1}}{(1-\rho)\rho} \right\}$$

Note that the definition of Δ_0 and the facts that $\Delta > \Delta_0$ imply that $\gamma = 2\rho^4 - \frac{H(\rho-\rho^2)}{\Delta(1-\rho)} > 0$. Furthermore, the fact that for $n \ge 102$ the inequality $(n-1)/\log_2(10n) > \sqrt{n}$ holds and the definition of N imply that for every $n > N$:

$$n \left(2^{-\gamma}\right)^{n-1} < 0.1. \tag{1}$$

Now, let $n > N$. We may assume, without loss of generality, that $n-1$ is an integer multiple of $\Delta(1-\rho)$ and also of Δ/ρ. (This can be assumed, since for every n, we can delete constantly many vertices from K_n, the constant depending only on Δ and ρ, and remain with n' vertices, where n' satisfies these divisibility constraints. We have only deleted $\Theta(n)$ edges in this way, that will not be packed, so the result stays intact.) Let

$$k = \frac{n-1}{\Delta(1-\rho)}$$

and let $p = 1/k$. We let each edge of K_n choose a color from the set $\{1, \ldots, k\}$, with uniform distribution. All the $\binom{n}{2}$ choices are independent. For each $i = 1, \ldots, k$ let G_i denote the spanning subgraph of K_n consisting of all the edges that selected the color i. Our first goal is to show that the total number of cycles whose length is less than r, in all the graphs $G_1, \ldots, G_k$, is not too large. Let Z denote the random variable equal to this total. The following claim shows that with relatively high probability, Z is not too large.

Claim 1:

$$\Pr\left[Z > 2k\Delta^r\right] < 0.5.$$

Proof: For $r > s \geq 3$, there are exactly $\binom{n}{s}\frac{(s-1)!}{2}$ cycles of length s in K_n. Each such cycle appears in G_i with probability exactly p^s. The expected number of cycles of length s in G_i is, therefore

$$\binom{n}{s}\frac{(s-1)!}{2}p^s = \frac{n(n-1)\cdots(n-s+1)}{2sk^s} =$$

$$\frac{n(n-1)\cdots(n-s+1)(1-\rho)^s\Delta^s}{2s(n-1)^s} \leq \frac{(n(1-\rho))^s\Delta^s}{2s(n-1)^s} < \Delta^s.$$

Thus, the expected number of cycles of length less than r in G_i is at most $\Delta^3 + \cdots + \Delta^{r-1} < \Delta^r$. Summing for each i we get that the expectation of Z is at most $k\Delta^r$. Claim 1 now follows from Markov's inequality. □

Our next goal is to show that for every subset S of indices from $\{1, \ldots, k\}$, having some specified size, and for every vertex $v \in V(K_n)$, the probability that a relatively large number of edges adjacent to v have chosen colors only from S, is small. This is established in the following claim.

Claim 2: With probability at least 0.9, for every subset $S \subset \{1, \ldots, k\}$ with $|S| = \frac{\rho(n-1)}{\Delta}$, and for every $v \in V(K_n)$, the number of edges adjacent to v that chose a color from S is less than ρn.

Proof: We fix a subset S of $\frac{\rho(n-1)}{\Delta}$ colors, and fix a vertex v. Each edge adjacent to v chooses a color from S with probability

$$p|S| = \frac{1}{k}\frac{\rho(n-1)}{\Delta} = \frac{\Delta(1-\rho)}{n-1}\frac{\rho(n-1)}{\Delta} = \rho(1-\rho).$$

Thus, if we denote by X the random variable that counts the number of edges adjacent to v that chose a color from S, we have that X has the binomial distribution $B(n-1, \rho(1-\rho))$ (we use here the fact that edges choose colors independently). The expectation of X is, therefore, $E[X] = (n-1)\rho(1-\rho)$. We can use standard large deviation results to estimate the probability that X deviates from its mean by some absolute value a. The large deviation result of Chernoff (cf. [1] Appendix A) states that:

$$\Pr\left[X - E[X] > a\right] < \exp\left(-\frac{2a^2}{n-1}\right).$$

Putting $a = \rho^2(n-1)$ we get that:

$$\Pr\left[X > (n-1)\rho\right] < \exp\left(-\frac{2\rho^4(n-1)^2}{n-1}\right) = \left(\exp(-2\rho^4)\right)^{n-1}$$

The number of possible choices for the set S is

$$\binom{k}{\frac{\rho(n-1)}{\Delta}}.$$

The number of possible choices for the vertex v is n. It therefore suffices to prove that

$$n\binom{k}{\frac{\rho(n-1)}{\Delta}}\left(\exp(-2\rho^4)\right)^{n-1} < 0.1. \qquad (2)$$

We shall use the well-known entropy inequality:

$$\binom{k}{\beta k} < \left(2^{H(\beta)}\right)^k$$

which is valid for every $0 < \beta < 1$ (for which βk is an integer). Using this inequality, together with (1) we have:

$$n\binom{k}{\frac{\rho(n-1)}{\Delta}}\left(\exp(-2\rho^4)\right)^{n-1} = n\binom{k}{(1-\rho)\rho k}\left(\exp(-2\rho^4)\right)^{n-1} <$$
$$n\left(2^{H(\rho-\rho^2)}\right)^k\left(\exp(-2\rho^4)\right)^{n-1} < n\left(2^{H(\rho-\rho^2)}\right)^{\frac{n-1}{\Delta(1-\rho)}}\left(2^{-2\rho^4}\right)^{n-1} =$$
$$n\left(2^{\frac{H(\rho-\rho^2)}{\Delta(1-\rho)}-2\rho^4}\right)^{n-1} = n\left(2^{-\gamma}\right)^{n-1} < 0.1$$

as required in (2). This completes the proof of the claim. □

For an edge $(u,v) \in K_n$ denote by $c(u,v)$ the color that was chosen by (u,v). By definition, (u,v) appears in $G_{c(u,v)}$. We call the edge (u,v) *bad* if at least one of its endpoints has degree larger than Δ in $G_{c(u,v)}$. Note that if (u,v) is bad, say, because of u, then all edges incident with u in $G_{c(u,v)}$ are also bad, and there are at least $\Delta+1$ such edges. Using Claim 2, we can prove that the number of bad edges is very small, with high probability. This is proved in the following claim:

Claim 3: With probability at least 0.9, there are at most ρn^2 bad edges.

Proof: It suffices to show that with probability at least 0.9, for every vertex v there are at most ρn bad edges incident with v. Assume, therefore, that a vertex v has more than ρn bad edges incident with v. By the remark prior to the claim, this means that there is a set S of $\lceil(\rho n)/(\Delta+1)\rceil$ colors, and a set of at least ρn edges adjacent to v, that chose colors only from the set S. Since

$$\lceil\frac{\rho n}{\Delta+1}\rceil \le \frac{\rho(n-1)}{\Delta}$$

we know from Claim 2, that with probability at least 0.9 this is impossible for every set S and every vertex v. This completes the proof of Claim 3. □

We are now ready to complete the proof of Theorem 1.2. According to Claims 1 and 3, with probability at least 0.4, in a random coloring of the edges there are at most $2k\Delta^r$ cycles whose length is shorter than r, in all the graphs $G_1,\ldots,G_k$, and there are at most ρn^2 bad edges. We therefore fix such a coloring. We pick from each cycle (in any of the graphs $G_1,\ldots,G_k$) whose length is shorter than r, an arbitrary edge. Denote by F_1 the set of edges picked. Let F_2 denote the set of bad edges. We obviously have $|F_1 \cup F_2| \le \rho n^2 + 2k\Delta^r$. For $i = 1,\ldots,k$ let H_i be the spanning subgraph of G_i obtained by deleting the edges of G_i belonging to $F_1 \cup F_2$. Clearly, $\Delta(H_i) \leq \Delta$ and $g(H_i) \geq r$. Finally, note that the total number of graphs, k, is slightly larger than the requirement in the statement of the theorem. We shall consider only the graphs $H_1, \ldots, H_{\lfloor (n-1)/\Delta \rfloor}$. Let F_3 denote the set of edges of the graphs $H_{1+\lfloor (n-1)/\Delta \rfloor}, \ldots, H_k$. Since $e(H_i) \leq \Delta n/2$ we have:

$$|F_3| \leq \frac{\Delta n}{2}\left(k - \lfloor \frac{n-1}{\Delta} \rfloor\right) \leq \frac{\Delta n}{2}\left(\frac{n-1}{(1-\rho)\Delta} - \frac{n-1}{\Delta} + 1\right) =$$
$$\frac{\Delta n}{2}\left(\frac{n-1}{\Delta}\frac{\rho}{1-\rho} + 1\right) \leq n^2 \frac{\rho}{2(1-\rho)} + \frac{\Delta n}{2} < \frac{3}{4}\rho n^2 + \frac{1}{4}\rho n^2 \leq \rho n^2.$$

In the last inequality we used the fact that $\rho \leq 1/3$ and $n > N \geq \frac{2\Delta}{\rho}$. It remains to show that $|F_1 \cup F_2 \cup F_3| \leq \epsilon n^2$. Indeed, using the fact that $n > N \geq \frac{2\Delta^{r-1}}{(1-\rho)\rho}$ and $\rho = \epsilon/3$ we have:

$$|F_1 \cup F_2 \cup F_3| \leq \rho n^2 + 2k\Delta^r + \rho n^2 = 2\rho n^2 + 2\frac{n-1}{(1-\rho)\Delta}\Delta^r \leq 3\rho n^2 = \epsilon n^2.$$

□

3 Proof of Theorem 1.1

Before proving Theorem 1.1 we need several lemmas. The upper bound in Theorem 1.1 is very easy, and is established in the following lemma.

Lemma 3.1 *Let G be a graph with maximum degree Δ. Then, $\alpha'(G) \leq \Delta + 1$. In particular, $\alpha(G) \leq \Delta + 1$ and $\alpha_\Delta \leq \Delta + 1$.*

Proof: By Vizing's Theorem (cf. [3]), the edges of G can be decomposed into $k \leq \Delta + 1$ matchings $H_1, \ldots, H_k$. Now, consider an edge ordering f of G where for any two edges e_1 and e_2, if $f(e_1) < f(e_2)$ then $e_1 \in H_i$ implies $e_2 \in H_j$ where $j \geq i$. Clearly, every monotone trail contains at most one edge from each H_i. Thus, $\alpha'(G) \leq k$. □

Recall that the *arboricity* of a graph G, denoted $a(G)$, is the minimum number of spanning subforests of G that, together, cover all the edges of G. A well-known theorem of Nash-Williams [7] asserts that $a(G)$ is the maximum possible value of $\lceil e(H)/(v(H)-1) \rceil$ where H ranges over the subgraphs of G. A graph G is called *d-degenerate* if it has an acyclic orientation where no outdegree is greater than d. The *degeneracy* of G, denoted $d(G)$, is the smallest d for which G is d-degenerate. It is easy to see that d-degeneracy is a hereditary property, and that $e(G) \leq (|G|-1)d(G)$. These observations, together with the above-mentioned theorem of Nash-Williams show that $a(G) \leq d(G)$. In [8] it is shown that $\alpha'(G) \leq 3a(G)$. (This follows from the fact that for any tree T, $\alpha'(T) \leq 3$.) Hence, we have the following:

Corollary 3.2 *If G is d-degenerate then $\alpha'(G) \leq 3d$.* □

We will need the following well-known upper bound on the degeneracy:

Lemma 3.3 *If G has m edges then $d(G) \le \sqrt{2m}$.*

Proof: It suffices to show that the vertices of G can be ordered $v_1, \ldots, v_n$ such that each v_i has at most $x = \lfloor \sqrt{2m} \rfloor$ neighbors appearing after v_i in the ordering. Indeed, let v_1 be the vertex with smallest degree in G. Deleting v_1 from G, let v_2 be the vertex with smallest degree in the resulting graph, and so on. We claim that the ordering obtained in this way has the desired property. Assume otherwise, and let v_t be the first vertex violating the property. The subgraph of G induced by $v_t, \ldots, v_n$ has minimum degree at least $x+1$. This subgraph has, therefore, at least $(x+1)(x+2)/2 > m$ edges, a contradiction. $\Box$

A crucial ingredient used in the proof of Theorem 1.1 is a result concerning $\alpha'(K_n)$. Unlike the big gap between the upper and lower bounds for $\alpha(K_n)$, mentioned in the introduction, the situation for $\alpha'(K_n)$ is much simpler, and easier. In fact, $\alpha'(K_n)$ was completely determined by Graham and Kleitman (cf. [5]). They have shown the following:

Lemma 3.4 (Graham and Kleitman) $\alpha'(K_n) = n-1$ *unless* $n = 3, 5$ *in which case* $\alpha'(K_3) = 3$ *and* $\alpha'(K_5) = 5$. $\Box$

A *decomposition* of a graph G is a collection of edge-disjoint spanning subgraphs of G, that covers every edge of G.

Lemma 3.5 *Let $\{H_1, \ldots, H_k\}$ be a decomposition of G. Then, $\alpha'(G) \le \alpha'(H_1) + \ldots + \alpha'(H_k)$.*

Proof: Consider an edge ordering of G similar to the one defined in Lemma 3.1. That is, all the edges of H_1 receive the lowest numbers. All the edges of H_2 receive the next remaining lowest numbers, and so on. Clearly, in every monotone trail of G, the set of edges of the trail belonging to H_i form a subtrail. This yields the desired inequality $\alpha'(G) \le \alpha'(H_1) + \ldots + \alpha'(H_k)$. $\Box$

Proof of Theorem 1.1: Let $\gamma > 0$. We need to show that there exists Δ_0 such that for every $\Delta > \Delta_0$ there exists a graph G with $\Delta(G) \le \Delta$ and with $\alpha(G) \ge (1-\gamma)\Delta$. We shall define $\epsilon = (\gamma/6)^2$, and define $\Delta_0 = \Delta_0(\epsilon)$ as in Theorem 1.2. We also define $r = \Delta + 1$. We now apply Theorem 1.2. (Note that we can define $r = \Delta + 1$ in Theorem 1.2, since the definition of Δ_0 does not depend on r). According to Theorem 1.2, if n is sufficiently large then K_n can be packed with $t = \lfloor (n-1)/\Delta \rfloor$ graphs $H_1, \ldots, H_t$ each having $\Delta(H_i) \le \Delta$ and $g(H_i) > \Delta$. Furthermore, at most ϵn^2 edges are unpacked. Let H_0 denote the spanning subgraph of K_n consisting of the unpacked edges. By definition, $\{H_0, H_1, \ldots, H_t\}$ is a decomposition of K_n. By Lemma 3.3, the degeneracy of H_0 satisfies $d(H_0) \le \sqrt{\epsilon} n$. By Corollary 3.2, $\alpha'(H_0) \le 3\sqrt{\epsilon} n$. By Lemma 3.5: $\alpha'(K_n) \le \sum_{i=0}^{t} \alpha'(H_i)$. By Lemma 3.4 we have:

$$n - 1 \le \alpha'(K_n) \le 3\sqrt{\epsilon} n + \sum_{i=1}^{t} \alpha'(H_i).$$

It follows that for at least one H_i:

$$\alpha'(H_i) \geq \frac{n-1-3\sqrt{\epsilon}n}{t} \geq \frac{n-1-3\sqrt{\epsilon}n}{\frac{n-1}{\Delta}} = \Delta - \frac{3\sqrt{\epsilon}n\Delta}{n-1} \geq (1-\gamma)\Delta.$$

Since $g(H_i) > \Delta$, any trail of length at most Δ is a simple path. It follows that $\alpha(H_i) \geq (1-\gamma)\Delta$. □

4 Concluding remarks and open problems

1. Theorem 1.1 shows the existence of graphs H with $\Delta(H) = \Delta$ and with $\alpha(H) \geq (1-o(1))\Delta$. However, the graphs appearing in the proof have large girth. Having a large girth is, obviously, not a requirement. One can always add, say, a triangle to H, thereby increasing the maximum degree by at most 2 (a constant that does not change the asymptotic lower bound), and not decreasing α.

2. In Theorem 1.2, the value of Δ_0 depends upon $\rho = \epsilon/3$. In fact, Δ_0 is polynomial in ϵ^{-1}. We conjecture, however, that this dependency is not needed:

   Conjecture 4.1 *Let $\epsilon > 0$ and let r and Δ be positive integers. There exists $N = N(\epsilon, \Delta, r)$ such that for every $n > N$ the complete graph K_n can be packed with $t = \lfloor (n-1)/\Delta \rfloor$ graphs $H_1, \ldots, H_t$, where $\Delta(H_i) \leq \Delta$ and $g(H_i) \geq r$ for $i = 1, \ldots, t$. Furthermore, at most ϵn^2 edges of K_n are unpacked.*

   It is well-known that Conjecture 4.1 holds for $\Delta = 1, 2$. This is due to the fact that K_n can be decomposed into at most n perfect matchings and packed with $\lfloor (n-1)/2 \rfloor$ Hamiltonian cycles.

3. It is interesting to determine α_d precisely. It is plausible that $\alpha_d = d+1$ for $d \geq 2$. If true, then Lemma 3.1 shows that the extremal examples must be graphs with chromatic index $d+1$. Characterizing the extremal graphs seems even more difficult.

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