

The number of bounded-degree spanning trees

Raphael Yuster *

Abstract

For a graph G , let $c_k(G)$ be the number of spanning trees of G with maximum degree at most k . For $k \geq 3$, it is proved that every connected n -vertex r -regular graph G with $r \geq \frac{n}{k+1}$ satisfies

$$c_k(G)^{1/n} \geq (1 - o_n(1))r \cdot z_k$$

where $z_k > 0$ approaches 1 extremely fast (e.g. $z_{10} = 0.999971$). The minimum degree requirement is essentially tight as for every $k \geq 2$ there are connected n -vertex r -regular graphs G with $r = \lfloor n/(k+1) \rfloor - 2$ for which $c_k(G) = 0$. Regularity may be relaxed, replacing r with the geometric mean of the degree sequence and replacing z_k with $z_k^* > 0$ that also approaches 1, as long as the maximum degree is at most $n(1 - (3 + o_k(1))\sqrt{\ln k/k})$. The same holds with no restriction on the maximum degree as long as the minimum degree is at least $\frac{n}{k}(1 + o_k(1))$.

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1 Introduction

For a graph G , let $c_k(G)$ be the number of spanning trees of G with maximum degree at most k and let $c(G)$ be the number of spanning trees of G . Computationally, these parameters are well-understood: Determining $c(G)$ is easy by the classical Matrix-Tree Theorem which says that $c(G)$ is equal to any cofactor of the Laplacian matrix of G , while determining $c_k(G)$ is NP-hard for every fixed $k \geq 2$. In this paper we look at these parameters from the extremal graph-theoretic perspective. The two extreme cases, i.e. $c(G)$ and $c_2(G)$, are rather well-understood. As for $c(G)$, Grone and Merris [9] proved that $c(G) \leq (n/(n-1))^{n-1}d(G)/2m$ where n and m are the number of vertices and edges of G respectively, and $d(G)$ is the product of its degrees. Note that this upper bound is tight for complete graphs. Alon [1], extending an earlier result of McKay [11], proved that if G is a connected r -regular graph, then $c(G) = (r - o(r))^n$. Alon's method gives meaningful results already for $r = 3$, where the proof yields $(1 - o_n(1))c(G)^{1/n} \geq \sqrt{2}$. Alon's result was extended by Kostochka [10] to arbitrary connected graphs with minimum degree $r \geq 3$. He proved that $c(G) \geq d(G)r^{-nO(\ln r/r)}$ and improved the aforementioned case of 3-regular graphs showing that

*Department of Mathematics, University of Haifa, Haifa 3498838, Israel. Email: raphael.yuster@gmail.com.

$(1 - o_n(1))c(G)^{1/n} \geq 2^{3/4}$ and that the constant $2^{3/4}$ is optimal. We mention also that Greenhill, Isaev, Kwan, and McKay [8] asymptotically determined the expected number of spanning trees in a random graph with a given sparse degree sequence.

The case $c_2(G)$ (the number of Hamilton paths) has a significant body of literature. All of the following mentioned results hold, in fact, for counting the number of Hamilton *cycles*. First, we recall that there are connected graphs with minimum degree $n/2 - 1$ for which $c_2(G) = 0$, so most results concerning $c_2(G)$ assume that the graph is *Dirac*, i.e. has minimum degree at least $n/2$. Dirac's Theorem [6] proves that $c_2(G) > 0$ for Dirac graphs. Significantly strengthening Dirac's theorem, Sárközy, Selkow, and Szemerédi [12] proved that every Dirac graph contains at least $c^n n!$ Hamilton cycles for some small positive constant c . They conjectured that c can be improved to $1/2 - o(1)$. In a breakthrough result, Cuckler and Kahn [4] settled this conjecture proving that every Dirac graph with minimum degree r has at least $(r/e)^n(1 - o(1))n$ Hamilton cycles. This bound is tight as shown by an appropriate random graph. Bounds on the number of Hamilton cycles in Dirac graphs expressed in terms of maximal regular spanning subgraphs were obtained by Ferber, Krivelevich, and Sudakov [7]. Their bound matches the bound of Cuckler and Kahn for graphs that are regular or nearly regular.

In this paper we consider $c_k(G)$ for fixed $k \geq 3$. Observe first that $c_k(G)^{1/n} \leq c(G)^{1/n} < d(G)^{1/n}$ (by simple counting or by the aforementioned result [9]). Thus, we shall express the lower bounds for $c_k(G)^{1/n}$ in our theorems in terms of constant multiples of $d(G)^{1/n}$. Notice also that if G is r -regular, then $d(G)^{1/n} = r$.

Our first main result concerns connected regular graphs. It is not difficult to prove that every connected r -regular graph with $r \geq n/(k + 1)$ has $c_k(G) > 0$ (this also holds for $k = 2$ [3]). We prove that $c_k(G)$ is, in fact, already very large under this minimum degree assumption. To quantify our lower bound we define the following functions of k .

$$f_k = 1 - \frac{1}{e} \sum_{i=0}^{k-3} \frac{1}{i!}, \quad g_k = \frac{2}{e(k-1)!}.$$

$$z_k = \begin{cases} 0.0494, & \text{for } k = 3 \\ 0.1527, & \text{for } k = 4 \\ (1 - (k + 1)(f_k + g_k))^{g_k} (1 - g_k)^{1-g_k} g_k^{g_k}, & \text{for } k \geq 5. \end{cases}$$

It is important to observe that z_k approaches 1 extremely quickly, as Table 1 shows.

Theorem 1.1. *Let $k \geq 3$ be given. Every connected n -vertex r -regular graph G with $r \geq \frac{n}{k+1}$ satisfies*

$$c_k(G)^{1/n} \geq (1 - o_n(1))r \cdot z_k.$$

The requirement on the minimum degree in Theorem 1.1 is essentially tight. In Subsection 4.3 we

k	5	6	7	8	9	10	11
z_k	0.843148	0.962200	0.991935	0.998565	0.999783	0.999971	0.999997

Table 1: The value of z_k for some small k .

show that for every $k \geq 2$ and for infinitely many n , there are connected r -regular graphs G with $r = \lfloor n/(k+1) \rfloor - 2$ for which $c_k(G) = 0$. In light of this construction, it may be of some interest to determine whether Theorem 1.1 holds with $n/(k+1) - 1$ instead of $n/(k+1)$. Furthermore, as our proof of Theorem 1.1 does not work for $k = 2$, we raise the following interesting problem.

Problem 1.2. *Does there exist a positive constant z_2 such that every connected n -vertex r -regular graph G with $r \geq \frac{n}{3}$ satisfies*

$$c_2(G)^{1/n} \geq (1 - o_n(1))r \cdot z_2 .$$

One may wonder whether the regularity requirement in Theorem 1.1 can be relaxed, while still keeping the minimum degree at $n/(k+1)$. It is easy to see that a bound on the *maximum* degree cannot be entirely waved. Indeed, consider a complete bipartite graph with one part of order $(n-2)/k$. It is connected, has minimum degree $(n-2)/k > n/(k+1)$, maximum degree $n - (n-2)/k$ but it clearly does not have any spanning tree with maximum degree at most k . However, if we place just a modest restriction on the maximum degree, we can extend Theorem 1.1. Let

$$z_k^* = \left(1 - \frac{1}{7k}\right)^{1 - \frac{1}{7k}} \left(\frac{1}{9k}\right)^{\frac{1}{7k}} .$$

It is easy to see that z_k^* approaches 1. For example, $z_{20}^* > 0.956$.

Theorem 1.3. *There exists a positive integer k_0 such that for all $k \geq k_0$ the following holds. Every connected n -vertex graph G with minimum degree at least $\frac{n}{k+1}$ and maximum degree at most $n(1 - 3\sqrt{\ln k/k})$ satisfies*

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n} \cdot z_k^* .$$

Finally, we obtain a lower bound on $c_k(G)$ where we have no restriction on the maximum degree of G . Analogous to Dirac's theorem, Win [13] proved that every connected graph with minimum degree $(n-1)/k$ has $c_k(G) > 0$ (see also [5] for an extension of this result). Clearly, the requirement on the minimum degree is tight as the aforementioned example of a complete bipartite graph shows that there are connected graphs with minimum degree $(n-2)/k$ for which $c_k(G) = 0$. We prove that for all $k \geq k_0$, if the minimum degree is just slightly larger, then $c_k(G)$ becomes large.

Theorem 1.4. *There exists a positive integer k_0 such that for all $k \geq k_0$ the following holds. Every*

connected n -vertex graph G with minimum degree at least $\frac{n}{k}(1 + 3\sqrt{\ln k/k})$ satisfies

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n} \cdot z_k^* .$$

Using Szemerédi’s regularity lemma, it is not too difficult to prove a version of Theorem 1.4 that works already for $k \geq 3$ and where $c_k(G)$ is exponential in n . However, the bound we can obtain by that method, after taking its n -th root, is not a positive constant multiple of $d(G)^{1/n}$. We do conjecture that the error term in the minimum degree assumption can be eliminated.

Conjecture 1.5. *Let $k \geq 3$. There is a constant $z_k^\dagger > 0$ such that every connected n -vertex graph G with minimum degree at least $\frac{n}{k}$ satisfies*

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n} \cdot z_k^\dagger$$

where $\lim_{k \rightarrow \infty} z_k^\dagger = 1$.

All of our theorems are based on two major ingredients. The first ingredient consists of proving that G has many spanning forests, each with only a relatively small number of component trees, and each having maximum degree at most k . However, the proof of this property varies rather significantly among the various theorems and cases therein. We combine the probabilistic model of Alon [1] for showing that there are many *out-degree one* orientations with certain properties, together with a novel nibble approach to assemble edges from *several* out-degree one orientations. The second ingredient consists of proving that each of the large spanning forests mentioned above has small “edit distance” from a spanning tree with maximum degree at most k . Once this is established, it is not difficult to deduce that G has many spanning trees with maximum degree at most k .

In Section 2 we prove the edit-distance property. In Section 3 we introduce out-degree one orientations and the *multi-stage model* which is the basis for our nibble approach. In Section 4 we consider regular graphs and prove Theorem 1.1. In Section 5 we prove Theorems 1.3 and 1.4.

Throughout the paper we assume that the number of vertices of the host graph, always denoted by n , is sufficiently large as a function of all constants involved. Thus, we refrain from repeatedly mentioning this assumption. We also ignore rounding issues (floors and ceilings) whenever these have no effect on the final statement of our results. We use the terminology *G -neighbor* of a vertex v to refer to a neighbor of v in G , as opposed to a neighbor of v in spanning tree or a spanning forest of G . The notation $d(v)$ always denotes the degree of v in G . Other notions that are used are standard, or defined upon their first use.

2 Extending a bounded degree forest

The edit distance between two graphs on the same vertex set is the number of edges in the symmetric difference of their edge sets. In this section we prove that the edit distance between a bounded degree spanning forest and a bounded degree spanning tree of a graph is proportional to the number of components of the forest, whenever the graph is connected and satisfies a minimum degree condition.

Lemma 2.1. *Let $k \geq 3$ and let G be a connected graph with n vertices and minimum degree at least $n/(k+1)$. Suppose that F is a spanning forest of G with $m < n-1$ edges and maximum degree at most k . Furthermore, assume that F has at most t vertices with degree k where $t \leq n/(6.8k)$. Then there exists a spanning forest F^* of G with $m+1$ edges that contains at least $m-3$ edges of F . Furthermore, F^* has maximum degree at most k and at most $t+4$ vertices with degree k .*

Proof. For a forest (or tree) with maximum degree at most k , its W -vertices are those with degree k and its U -vertices are those with degree less than k . Denote the tree components of F by T_1, \dots, T_{n-m} . Let $U_i \neq \emptyset$ denote the U -vertices of T_i and let W_i denote the W -vertices of T_i . We distinguish between several cases as follows:

- (a) There is some edge of G connecting some $u_i \in U_i$ with some $u_j \in U_j$ where $i \neq j$.
- (b) Case (a) does not hold but there is some T_i with fewer than $n/(k+1)$ vertices.
- (c) The previous cases do not hold but there is some edge of G connecting some $u_i \in U_i$ to a vertex in a different component of F .
- (d) The previous cases do not hold.

Case (a). We can add to F the edge $u_i u_j$ obtaining a forest with $m+1$ edges which still has maximum degree at most k . The new forest has at most $t+2$ W -vertices since only u_i and u_j increase their degree in the new forest.

Case (b). Let u_i be some vertex with degree 1 or 0 in T_i (note that it is possible that T_i is a singleton so that the degree of its unique vertex is indeed 0 in T_i). Since T_i has fewer than $n/(k+1)$ vertices, and since u_i has minimum degree at least $n/(k+1)$ in G , we have that u_i has at least two G -neighbors that are not in T_i . Let w_1, w_2 denote such neighbors. Notice that w_1, w_2 are W -vertices of F as we assume Case (a) does not hold.

Assume first that w_1, w_2 are adjacent in F (in particular, they are in the same component of F). Let F^* be obtained from F by adding both edges $u_i w_1$ and $u_i w_2$ and removing the edge $w_1 w_2$. Note that F^* has $m+1$ edges, has $m-1$ edges of F , and has maximum degree at most k . It also has at most $t+1$ W -vertices as only u_i may become a new vertex of degree k (in fact, the degree of u_i in F^* is at most 3 so if $k > 3$ we still only have t W -vertices in F^*).

We may now assume that w_1, w_2 are independent in F . Removing both of them from F further introduces at least $2k-1$ component trees denoted L_1, \dots, L_s where $s \geq 2k-1$. To see this, observe first that if we remove w_1 , we obtain at least k nonempty components since w_1 has degree k . If

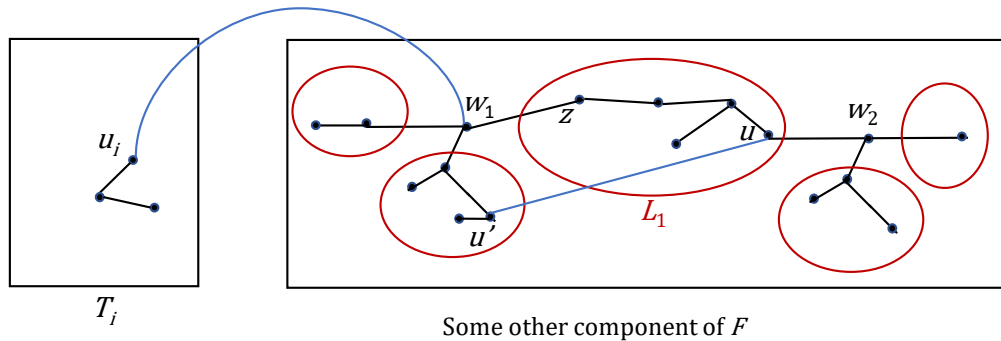


Figure 1: Constructing F^* from F in Case (b) of Lemma 2.1 (here we use $k = 3$). The figure depicts the component T_i containing u_i and some other component containing w_1 (and w_2 in this example). The red ovals depict the various L_j 's obtained when removing w_1 and w_2 . The denoted L_1 contains a vertex u of degree 1 in L_1 (and degree smaller than 3 in F) which has a neighbor u' in G that also has degree smaller than 3 in F . The blue edges represent edges of G that are not used in F . To obtain F^* we add $u_i w_1$, add $u u'$ and remove the edge $w_1 z$.

we then remove w_2 , we either obtain an additional set of k components (if w_2 is not in the same component of w_1 in F) or an additional set of $k - 1$ components (if w_2 is in the same component of w_1 in F).

Each L_j , being a tree, either has at least two vertices of degree 1, or else L_j is a singleton, in which case it has a single vertex with degree 0 in L_j . If L_j is a singleton, then its unique vertex has degree at most 2 in F as it may only be connected in F to w_1 and w_2 . If L_j is not a singleton, then let v_1, v_2 be two vertices with degree 1 in L_j . It is impossible for both v_1, v_2 to have degree at least 3 in F as otherwise they are both adjacent to w_1, w_2 in F , implying that F is not a forest (has a $K_{2,2}$). In any case, we have shown that each L_j (whether a singleton or not) has a vertex which is a U -vertex of F .

Consider now an L_j with smallest cardinality, say L_1 . Its number of vertices is therefore at most

$$\frac{n}{s} \leq \frac{n}{2k-1}. \quad (1)$$

Let u be a vertex of L_1 which is a U -vertex of F . By our minimum degree assumption on G , u has at least $n/(k+1) - (|V(L_1)| - 1)$ neighbors in G that are not in L_1 . By (1) we have that

$$\frac{n}{k+1} - (|V(L_1)| - 1) \geq \frac{n}{k+1} - \frac{n}{2k-1} > \frac{n}{6.8k} \geq t. \quad (2)$$

It follows that u has a G -neighbor u' not in L_1 which is a U -vertex of F . Notice that u and u' must be in the same component of F since we assume Case (a) does not hold. Since u' is not in L_1 , adding $u u'$ to F introduces a cycle that contains at least one of w_1, w_2 . Assume wlog that the cycle contains w_1 and that z is the neighbor of w_1 on the cycle (possibly $z \in \{u, u'\}$). We can now

obtain a forest F^* from F by adding uu' , adding u_iw_1 and removing w_1z . The obtained forest has $m + 1$ edges, has $m - 1$ edges of F , has maximum degree at most k , and at most $t + 2$ W -vertices as only u, u' can increase their degree in F^* to k . Figure 1 visualizes u_i, u, u', w_1, z, L_1 and the added and removed edges when going from F to F^* .

Case (c). In this case, T_i has at least $n/(k + 1)$ vertices. Let $w_j \in W_j$ be a G -neighbor of u_i in a different component T_j of F . Removing w_j from T_j splits $T_j \setminus w_j$ into a forest with k component trees L_1, \dots, L_k . So at least one of these components, say L_1 , has at most $(n - |V(T_i)| - 1)/k < n/(k + 1)$ vertices. Obtain a forest F^{**} from F by adding the edge u_iw_j and removing the unique edge of T_j connecting w_j to L_1 . The new forest also has m edges and has $m - 1$ edges of F . It also has at most $t + 1$ W -vertices as only u_i may become a new vertex of degree k . But in F^{**} , there is a component, namely L_1 , with fewer than $n/(k + 1)$ vertices. Hence, we arrive at either Case (a) or Case (b) for F^{**} . So, applying the proofs of these cases to F^{**} (and observing that the number of W -vertices in F^{**} is only $t + 1$ so (2) still holds because of the slack in the sharp inequality of (2)), we obtain a forest F^* with $m + 1$ edges, at least $m - 2$ edges of F , maximum degree at most k , and at most $t + 3$ W -vertices.

Case (d). Since G is connected, we still have an edge of G connecting some vertex $w_i \in W_i$ with some $w_j \in W_j$. Without loss of generality, $|V(T_j)| \leq n/2$. Removing w_j from T_j splits $T_j \setminus w_j$ into a forest with k component trees L_1, \dots, L_k . So at least one of these components, say L_1 , has at most $|V(T_j)|/k \leq n/(2k)$ vertices. Let u be a vertex of L_1 of degree 1 in F . So, u has at least $n/(k + 1) - n/(2k) > n/(6.8k) \geq t$ neighbors not in L_1 . It follows that u has a G -neighbor u' which is a U -vertex of F . Also notice that $u' \in T_j$ since we assume Case (a) does not hold. Now, let F^{**} be obtained from F by adding the edge uu' and removing the unique edge of T_j connecting w_j to L_1 . The new forest also has m edges and has $m - 1$ edges of F . It also has at most $t + 1$ W -vertices as only u' may become a new vertex of degree k . But observe that in F^{**} the degree of w_j is only $k - 1$. Since w_j has a G -neighbor (namely w_i) in a different component of F^{**} , we arrive in F^{**} at either Case (a) or Case (b) or Case (c). So, applying the proofs of these cases to F^{**} (and observing that the number of W -vertices in F^{**} is only $t + 1$ so (2) still holds because of the slack in the sharp inequality of (2)), we obtain a forest F^* with $m + 1$ edges, at least $m - 3$ edges of F , maximum degree at most k , and at most $t + 4$ W -vertices. \square

By repeated applications of Lemma 2.1 where we start with a large forest and repeatedly increase the number of edges until obtaining a spanning tree, we immediately obtain the following corollary.

Corollary 2.2. *Let $k \geq 3$ and let G be a connected graph with n vertices and minimum degree at least $n/(k + 1)$. Suppose that F is a spanning forest of G with $n - O(\ln n)$ edges and maximum degree at most k . Furthermore, assume that F has at most t vertices with degree k where $t \leq n/(7k)$. Then there exists a spanning tree of G with maximum degree at most k where all but at most $O(\ln n)$ of its edges are from F . \square*

3 From out-degree one orientations to bounded degree spanning trees

Let G be a graph with no isolated vertices. An *out-degree one orientation* of G is obtained by letting each vertex v of G choose precisely one of its neighbors, say u , and orient the edge vu as (v, u) (i.e from v to u). Observe that an out-degree one orientation may have cycles of length 2. Also note that an out-degree one orientation has the property that each component¹ contains precisely one directed cycle and that all cycles in the underlying graph of an out-degree one orientation are directed cycles. Furthermore, observe that the edges of the component that are not on its unique directed cycle (if there are any) are oriented “toward” the cycle. In particular, given the cycle, the orientation of each non-cycle edge of the component is uniquely determined. Let $\mathcal{H}(G)$ denote the set of all out-degree one orientations of G . Clearly, $|\mathcal{H}(G)| = d(G)$.

Most of our proofs use the probabilistic model of Alon [1]: Each $v \in V(G)$ chooses independently and uniformly at random a neighbor u and the edge vu is oriented (v, u) . In this way we obtain a uniform probability distribution over the sample space $\mathcal{H}(G)$. We let \vec{G} denote a randomly selected element of $\mathcal{H}(G)$ and let $\Gamma(v)$ denote the chosen out-neighbor of v .

We focus on certain parameterized subsets of $\mathcal{H}(G)$. Let $\mathcal{H}_{k,s}(G)$ be the subset of all elements of $\mathcal{H}(G)$ with maximum in-degree at most $k - 1$ and with at most s vertices of in-degree $k - 1$. If $s = n$ (i.e. we do not restrict the number of vertices with in-degree $k - 1$) then we simply denote the set by $\mathcal{H}_k(G)$. Let $\mathcal{H}_\ell^*(G)$ be the subset of all elements of $\mathcal{H}(G)$ with at most ℓ directed cycles (equivalently, at most ℓ components). Our proofs are mostly concerned with establishing lower bounds for the probability that $\vec{G} \in \mathcal{H}_{k,s}(G) \cap \mathcal{H}_\ell^*(G)$. Hence we denote

$$P_{k,s,\ell}(G) = \Pr[\vec{G} \in \mathcal{H}_{k,s}(G) \cap \mathcal{H}_\ell^*(G)] .$$

Lemma 3.1. *Let $k \geq 3$ be given. Suppose that G is a connected graph with minimum degree at least $n/(k + 1)$. Then:*

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n}P_{k,n/(7k),\ln n}(G)^{1/n} .$$

Proof. Let $p = P_{k,n/(7k),\ln n}(G)$. By the definition of p , we have that

$$|\mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)| \geq d(G)p .$$

Consider some $\vec{G} \in \mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$. As it has at most $\ln n$ directed cycles (and recall that these cycles are pairwise vertex-disjoint as each belongs to a distinct component), it has at most $\ln n$ edges that, once removed from \vec{G} , turn it into a forest F with at least $n - O(\ln n)$ edges. Viewed

¹A component of a directed graph is a component of its underlying undirected graph.

as an undirected graph, F has maximum degree at most k (since the in-degree of each vertex of \vec{G} is at most $k - 1$ and the out-degree of each vertex of \vec{G} is precisely 1). Thus, we have a mapping assigning each $\vec{G} \in \mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$ an undirected forest F . While this mapping is not injective, the fact that \vec{G} only has at most $\ln n$ components implies that each F is the image of at most $n^{O(\ln n)}$ distinct \vec{G} . Indeed, given an undirected F , suppose it has $t \leq \ln n$ components of sizes s_1, \dots, s_t . To turn it into an element of $\mathcal{H}(G)$, we must first add a single edge to each component to give a cycle, and then choose the orientation of each cycle in each component, which implies the orientation of non-cycle edges. Hence, the number of possible \vec{G} obtained from F is at most $\prod_{i=1}^t (2s_i^2) \leq n^{O(\ln n)}$. Furthermore, since \vec{G} has at most $n/(7k)$ vertices with in-degree $k - 1$, it follows that F has at most $n/(7k)$ vertices with degree k . By Corollary 2.2, there exists a spanning tree T of G with maximum degree at most k where all but at most $O(\ln n)$ of its edges are from F . Thus, we have a mapping assigning each $\vec{G} \in \mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$ a spanning tree T of G with maximum degree at most k . While this mapping is not injective, the fact that the edit distance between T and F is $O(\ln n)$ trivially implies that each T is the image of at most $n^{O(\ln n)}$ distinct F . Hence, we obtain that

$$c_k(G) \geq d(G)pn^{-O(\ln n)}.$$

Taking the n 'th root from both sides of the last inequality therefore concludes the lemma. \square

We also require an upper bound for the probability that \vec{G} has many components. The following lemma is proved by Kostochka [10] (see Lemma 2 in that paper, applied to the case where the minimum degree is at least $n/(k + 1)$, as we assume).

Lemma 3.2. [10] *Let G be a graph with minimum degree at least $n/(k + 1)$. The expected number of components of \vec{G} is at most $(k + 1) \ln n$.* \square

For $\vec{G} \in \mathcal{H}(G)$, let $B_i^{\vec{G}}$ denote the set of vertices with in-degree i . We will omit the superscript and simply write B_i whenever \vec{G} is clear from context.

We define the following K -stage model for establishing a random element of $\mathcal{H}(G)$. This model is associated with a positive integer K and a convex sum of probabilities $p_1 + \dots + p_K = 1$. In the first part of the K -stage model, we select uniformly and independently (with replacement) K elements of $\mathcal{H}(G)$ as in the aforementioned model of Alon. Denote the selected elements by \vec{G}_c for $c = 1, \dots, K$. Let $\Gamma_c(v)$ denote the out-neighbor of v in \vec{G}_c . In the second part of the K -stage model, we let each vertex $v \in V(G)$ choose precisely one of $\Gamma_1(v), \dots, \Gamma_K(v)$ where $\Gamma_c(v)$ is chosen with probability p_c . Observe that the resulting final element \vec{G} consisting of all $n = |V(G)|$ selected edges is also a uniform random element of $\mathcal{H}(G)$. Also note that for any given partition of $V(G)$ into parts V_1, \dots, V_K , the probability that all out-edges of the vertices of V_c are taken from \vec{G}_c for all $c = 1, \dots, K$ is precisely $\prod p_c^{|V_c|}$.

As mentioned in the introduction, most of our proofs for lower-bounding $c_k(G)$ contain two major ingredients. The first ingredient consists of using the K -stage model for a suitable K in

Theorem or case thereof	K	nibble step	completion step	combination
1.1, $k \geq 5$	2	Lemma 4.4	Lemma 4.5	Lemma 4.6
1.1, $k = 4$	5	Lemma 4.8	Lemma 4.9	Lemma 4.10
1.1, $k = 3$	20	Lemma 4.8	Lemma 4.9	Lemma 4.10
1.3	2	Lemma 5.2	Lemma 5.3	Lemma 5.4

Table 2: A roadmap for the proofs of Theorems 1.1, 1.3, 1.4.

order to establish a lower bound for $P_{k,s,\ell}(G)$ (with $\ell = \ln n$). This first ingredient further splits into several steps:

a) The *nibble step* where we prove that with nonnegligible probability, there is a forest with a linear number of edges consisting of edges of $\vec{G}_1, \dots, \vec{G}_{K-1}$ and which satisfies certain desirable properties.

b) The *completion step* where we prove that given a forest with the properties of the nibble step we can, with nonnegligible probability, complete it into an out-degree one orientation with certain desirable properties using only the edges of \vec{G}_K .

c) A *combination lemma* which uses (a) and (b) above to prove a lower bound for $P_{k,s,\ell}(G)$.

The second ingredient uses Lemma 3.1 applied to the lower bound obtained in (c) to yield the final outcome of the desired proof. Table 2 gives a roadmap for the various lemmas used for establishing steps (a) (b) (c), and the value of K used.

4 Proof of Theorem 1.1

In this section we assume that G is r -regular with $r \geq n/(k+1)$. We will consistently be referring to the notation of Section 3. When $k \geq 5$ we will use the two-stage model ($K = 2$) and when $k \in \{3, 4\}$ (dealt with in the next subsection) we will need to use larger K (see Table 2).

4.1 The case $k \geq 5$

We first need to establish several lemmas (the first lemma being straightforward).

Lemma 4.1. *Let G be an r -regular graph with $r \geq n/(k+1)$. For $0 \leq i \leq n$, the probability that $v \in B_i$ (i.e., that v has in-degree i in \vec{G}) is*

$$\Pr[v \in B_i] = \binom{r}{i} \frac{1}{r^i} \left(1 - \frac{1}{r}\right)^{r-i} \leq (1 + o_n(1)) \frac{1}{i!e}.$$

Furthermore, the in-degree of v in \vec{G} is nearly Poisson as for all $0 \leq i \leq k$,

$$\Pr[v \in B_i] = (1 \pm O(n^{-1})) \frac{1}{i!e}. \quad \square$$

Lemma 4.2. *Let G be an r -regular graph with $r \geq n/(k+1)$. For all $0 \leq i \leq k$ and for any set X of vertices of G it holds that*

$$\Pr \left[\left| |X \cap B_i| - \frac{|X|}{i!e} \right| > n^{2/3} \right] < \frac{1}{n^2}.$$

Proof. Consider the random variable $|X \cap B_i|$. By Lemma 4.1, its expectation, denoted by X_0 , is $X_0 = (1 \pm O(\frac{1}{n})) \frac{|X|}{i!e} = \frac{|X|}{i!e} \pm O_n(1)$. Now, suppose we expose the edges of \vec{G} one by one in n steps (in each step we choose the out-neighbor of another vertex of G), and let X_j be the expectation of $|X \cap B_i|$ after j steps have been exposed (so after the final stage we have $X_n = |X \cap B_i|$). Then X_0, X_1, \dots, X_n is a martingale satisfying the Lipschitz condition (each exposure increases by one the in-degree of a single vertex), so by Azuma's inequality (see [2]), for all $\lambda > 0$,

$$\Pr [||X \cap B_i| - X_0| > \lambda\sqrt{n}] < 2e^{-\lambda^2/2}.$$

Using, say, $\lambda = n^{1/10}$ the lemma immediately follows. \square

Lemma 4.3. *Let G be an r -regular graph with $r \geq n/(k+1)$. For all $3 \leq t \leq k$ the following holds: With probability at least $\frac{1}{10}$, \vec{G} has a set of at most $\frac{5.9n}{3et!}$ edges, such that after their removal, the remaining subgraph has maximum in-degree at most $t-1$.*

Proof. Let

$$Q_{\vec{G},t} = \sum_{i=t}^n (i-t+1)|B_i|$$

be the smallest number of edges we may delete from \vec{G} in order to obtain a subgraph where all vertices have in-degree at most $t-1$. We upper-bound the expected value of $Q_{\vec{G},t}$. By Lemma 4.1 we have that

$$\mathbb{E}[Q_{\vec{G},t}] = \sum_{i=t}^n (i-t+1)\mathbb{E}[|B_i|] \leq (1 + o_n(1)) \frac{n}{e} \left(\sum_{i=t}^n (i-t+1) \frac{1}{i!} \right).$$

Now, for all $t \geq 4$, each term in the sum $\sum_{i=t}^n (i-t+1) \frac{1}{i!}$ is smaller than its predecessor by at least a factor of 2.5, which means that for all n sufficiently large

$$\mathbb{E}[Q_{\vec{G},t}] \leq \frac{5.3n}{3et!}.$$

It is easily verified that for $t = 3$, the last inequality also holds since $\sum_{i=3}^{\infty} \frac{i-2}{i!} < 5.3/18$. By Markov's inequality, we therefore have that with probability at least $\frac{1}{10}$, for $t \geq 3$ it holds that

$$Q_{\vec{G},t} \leq \frac{5.9n}{3et!}.$$

Thus, with probability at least $\frac{1}{10}$, we can pick a set of at most $\frac{5.9n}{3et!}$ edges of \vec{G} , such that after their removal, the remaining subgraph has maximum in-degree at most $t - 1$. \square

Lemma 4.4. *Let G be an r -regular graph with $r \geq n/(k+1)$. With probability at least $\frac{1}{20}$, \vec{G} has a spanning forest F such that:*

- (a) F has maximum in-degree at most $k - 2$.
- (b) F has at least $n - \frac{2n}{e^{(k-1)!}}$ edges.
- (c) The number of vertices of F with in-degree at most $k - 3$ is at least $(1 - o_n(1)) \frac{n}{e} \sum_{i=0}^{k-3} \frac{1}{i!}$.

Proof. By Lemma 4.3, with probability at least $\frac{1}{10}$ we can remove at most $\frac{5.9n}{3e^{(k-1)!}}$ edges from \vec{G} , such that after their removal, the remaining subgraph has maximum in-degree at most $k - 2$.

By Lemma 3.2, with probability at most $\frac{1}{40}$ we have that \vec{G} has more than $40(k+1) \ln n$ components. Recalling that in \vec{G} each component can be made a tree by removing a single edge from its unique directed cycle, with probability at least $1 - \frac{1}{40}$ we have that \vec{G} can be made acyclic by removing at most $40(k+1) \ln n$ edges.

By Lemma 4.2 applied to $X = V(G)$, with probability at least $1 - (k-1)/n^2 > 1 - 1/40$ we have that for all $0 \leq i \leq k-3$, the number of vertices of \vec{G} with in-degree i is at least $n/(i!e) - n^{2/3} \geq (1 - o_n(1))n/(i!e)$. Thus, with probability at least $1 - 1/40$ there are at least $(1 - o_n(1)) \frac{n}{e} \sum_{i=0}^{k-3} \frac{1}{i!}$ vertices of \vec{G} with in-degree at most $k - 3$.

We therefore obtain that with probability at least $\frac{1}{10} - \frac{1}{40} - \frac{1}{40} = \frac{1}{20}$, the claimed forest exists and has at least $n - \frac{5.9n}{3e^{(k-1)!}} - 40(k+1) \ln n \geq n - \frac{2n}{e^{(k-1)!}}$ edges. \square

Using the two-stage model, consider \vec{G}_1 and \vec{G}_2 as denoted in Section 3. We say that \vec{G}_1 is *successful* if it has a spanning forest as guaranteed by Lemma 4.4. By that lemma, with probability at least $\frac{1}{20}$, we have that \vec{G}_1 is successful. Assuming \vec{G}_1 is successful, designate a spanning forest F_1 of it satisfying the properties of Lemma 4.4. Let $X_1 \subset V(G)$ be the set of vertices with out-degree 0 in F_1 . Thus, we have by Lemma 4.4 that $|X_1| \leq \frac{2n}{e^{(k-1)!}} = ng_k$.

Now, consider the set of edges of \vec{G}_2 emanating from X_1 , denoting them by $E_2 = \{(v, \Gamma_2(v)) \mid v \in X_1\}$. By adding E_2 to F_1 we therefore obtain an out-degree one orientation of G , which we denote (slightly abusing notation) by $E_2 \cup F_1$.

Lemma 4.5. *Suppose that $k \geq 5$. Given that \vec{G}_1 is successful, and given the corresponding forest F_1 , the probability that $(E_2 \cup F_1) \in \mathcal{H}_{k-1}(G) \cap \mathcal{H}_{\ln n}^*(G)$ is at least*

$$(1 - (k+1)(f_k + g_k) - o_n(1))^{ng_k}.$$

Proof. Fix an arbitrary ordering of the vertices of X_1 , say $v_1, \dots, v_{|X_1|}$. We consider the edges $(v_i, \Gamma_2(v_i))$ one by one, and let $E_{2,i} \cup F_1$ be the graph obtained after adding to F_1 the edges $(v_j, \Gamma_2(v_j))$ for $1 \leq j \leq i$. Also let $E_{2,0} = \emptyset$. We say that $E_{2,i} \cup F_1$ is *good* if it satisfies the following two properties:

- (i) The in-degree of each vertex in $E_{2,i} \cup F_1$ is at most $k - 2$.
- (ii) Every component of $E_{2,i} \cup F_1$ with fewer than $n/\ln n$ vertices is a tree.

Trivially, $E_{2,0} \cup F_1 = F_1$ is good, since F_1 is a forest where the in-degree of each vertex is at most $k - 2$. We estimate the probability that $E_{2,i+1} \cup F_1$ is good given that $E_{2,i} \cup F_1$ is good.

Consider vertex v_{i+1} . By Property (c) of Lemma 4.4, v_{i+1} has at most $(1 + o_n(1))nf_k$ neighbors with in-degree $k - 2$ in F_1 (recall that $f_k = 1 - \frac{1}{e} \sum_{i=0}^{k-3} \frac{1}{i!}$). Thus, there is a subset S of at least $r - f_k(1 + o_n(1))n - i$ neighbors of v_{i+1} in G which still have in-degree at most $k - 3$ in $E_{2,i} \cup F_1$. Now, if the component of v_{i+1} in $E_{2,i} \cup F_1$ has fewer than $n/\ln n$ vertices, then further remove from S all vertices of that component. In any case, $|S| \geq r - f_k(1 + o_n(1))n - i - n/\ln n$. The probability that $\Gamma_2(v_{i+1}) \in S$ is therefore at least

$$\begin{aligned} \frac{r - f_k(1 + o_n(1))n - i - \frac{n}{\ln n}}{r} &= 1 - \frac{f_k(1 + o_n(1))n + i + \frac{n}{\ln n}}{r} \\ &\geq 1 - \frac{f_k(1 + o_n(1))n + \frac{2n}{e^{(k-1)!}} + \frac{n}{\ln n}}{n/(k+1)} \\ &= 1 - (k+1)(f_k + g_k) - o_n(1). \end{aligned}$$

Now, to have $\Gamma_2(v_{i+1}) \in S$ means that we are not creating any new components of size smaller than $n/\ln n$, so all components of size at most $n/\ln n$ up until now are still trees. It further means that $E_{2,i+1} \cup F_1$ still has maximum in-degree at most $k - 2$. In other words, it means that $E_{2,i+1} \cup F_1$ is good. We have therefore proved that the final $E_2 \cup F_1$ is good with probability at least

$$(1 - (k+1)(f_k + g_k) - o_n(1))^{|X_1|} \geq (1 - (k+1)(f_k + g_k) - o_n(1))^{ng_k}.$$

Finally, observe that for $E_2 \cup F_1$ to be good simply means that it belongs to $\mathcal{H}_{k-1}(G) \cap \mathcal{H}_{\ln n}^*(G)$. \square

Lemma 4.6. *Let $k \geq 5$. Then,*

$$P_{k,0,\ln n}(G)^{1/n} \geq (1 - o_n(1))z_k.$$

Proof. Using the two-stage model, we have by Lemma 4.4 that \vec{G}_1 is successful with probability at least $\frac{1}{20}$. Thus, by Lemma 4.5, with probability at least

$$\frac{1}{20}(1 - (k+1)(f_k + g_k) - o_n(1))^{ng_k}$$

the following holds: There is an out-degree one orientation \vec{G} consisting of $x \geq n - \frac{2n}{e^{(k-1)!}}$ edges of \vec{G}_1 , and hence at most $n - x \leq ng_k$ edges of \vec{G}_2 , which is in $\mathcal{H}_{k-1}(G) \cap \mathcal{H}_{\ln n}^*(G)$ (observe that being in $\mathcal{H}_{k-1}(G)$ is the same as being in $\mathcal{H}_{k,0}$, i.e. there are zero vertices with in-degree $k-1$ since every vertex has maximum in-degree at most $k-2$).

Assuming that this holds, let X be the set of vertices whose out-edge in \vec{G} is from \vec{G}_1 . Now let $p_1 + p_2 = 1$ be the probabilities associated with the two-stage model where we will use $p_2 < \frac{1}{2}$. The probability that in the second part of the two-stage model, each vertex $v \in X$ will indeed choose $\Gamma_1(v)$ and each vertex $v \in V(G) \setminus X$ will indeed choose $\Gamma_2(v)$ is precisely

$$p_1^x p_2^{n-x} \geq (1 - p_2)^{n-ng_k} p_2^{ng_k} .$$

Optimizing, we will choose $p_2 = g_k$. Recalling that the final outcome of the two-stage model is a completely random element of $\mathcal{H}(G)$, we have that

$$P_{k,0,\ln n}(G) \geq \frac{1}{20} (1 - (k+1)(f_k + g_k) - o_n(1))^{ng_k} (1 - g_k)^{n-ng_k} g_k^{ng_k} .$$

Taking the n 'th root from both sides and recalling that $z_k = (1 - (k+1)(f_k + g_k))^{g_k} (1 - g_k)^{1-g_k} g_k^{g_k}$ yields the lemma. \square

Proof of Theorem 1.1 for $k \geq 5$. By Lemma 4.6 we have that $P_{k,0,\ln n}(G)^{1/n} \geq (1 - o_n(1))z_k$. As trivially $P_{k,0,\ln n}(G) \leq P_{k,n/(7k),\ln n}(G)$ we have by Lemma 3.1 that

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n}(1 - o_n(1))z_k = (1 - o_n(1))r \cdot z_k .$$

\square

4.2 The cases $k = 3$ and $k = 4$

Lemma 4.5 doesn't quite work when $k \in \{3, 4\}$ as the constant $1 - (k+1)(f_k + g_k)$ is negative in this case ($f_4 = 1 - 2/e$ and $g_4 = 1/(3e)$). To overcome this, we need to make several considerable adjustments in our arguments. Among others, this will require using the K -stage model for K relatively large ($K = 20$ when $k = 3$ and $K = 5$ when $k = 4$ will suffice). Recall that in this model we have randomly chosen out-degree one orientations $\vec{G}_1, \dots, \vec{G}_K$. Define the following sequence:

$$q_i = \begin{cases} \frac{1}{e}, & \text{for } i = 1 \\ \frac{q_{i-1}}{e^{q_{i-1}}}, & \text{for } i > 1 . \end{cases}$$

Slightly abusing notation, for sets of edges F_1, \dots, F_i where $F_j \subset E(\vec{G}_j)$ we let $\cup_{j=1}^i F_j$ denote the graph whose edge set is the union of these edge sets.

Definition 4.7. For $1 \leq i \leq K - 1$ we say that \vec{G}_i is successful if \vec{G}_i has a subset of edges F_i such that all the following hold:

- (a) $i = 1$ or \vec{G}_{i-1} is successful (so the definition is recursive).
- (b) F_1, \dots, F_i are pairwise-disjoint and $\cup_{j=1}^i F_j$ is a forest.
- (c) The maximum in-degree and maximum out-degree of $\cup_{j=1}^i F_j$ is at most 1.
- (d) $\cup_{j=1}^i F_j$ has $(1 \pm o_n(1))nq_i$ vertices with in-degree 0.
- (e) For all $v \in V(G)$ the number of G -neighbors of v having in-degree 0 in $\cup_{j=1}^i F_j$ is $(1 \pm o_n(1))rq_i$.
- (f) For all $v \in V(G)$ the number of G -neighbors of v having out-degree 0 in $\cup_{j=1}^i F_j$ is $(1 \pm o_n(1))rq_i$.

Lemma 4.8. For all $1 \leq i \leq K - 1$, \vec{G}_i is successful with probability at least $\frac{1}{2^i}$.

Proof. We prove the lemma by induction. Observe that for $i \geq 2$, it suffices to prove that, given that \vec{G}_{i-1} is successful, then \vec{G}_i is also successful with probability $\frac{1}{2}$. For the base case $i = 1$ it just suffices to prove that items (b)-(f) in Definition 4.7 hold with probability $\frac{1}{2}$ without the preconditioning item (a), so this is easier than proving the induction step; thus we shall only prove the induction step. In other words, we assume that \vec{G}_{i-1} is successful and given this assumption, we prove that \vec{G}_i is successful with probability $\frac{1}{2}$.

For notational convenience, let $F = \cup_{j=1}^{i-1} F_j$. Let X_{i-1} be the set of vertices with out-degree 0 in F . Since \vec{G}_{i-1} is successful we have that $|X_{i-1}| = (1 \pm o_n(1))nq_{i-1}$ (in a digraph with maximum in-degree 1 and maximum out-degree 1, the number of vertices with in-degree 0 equals the number of vertices with out-degree 0). Consider the set of edges of \vec{G}_i emanating from X_{i-1} , denoting them by $E_i = \{(v, \Gamma_i(v)) \mid v \in X_{i-1}\}$. By adding E_i to F we therefore obtain an out-degree one orientation of G , which we denote by $E_i \cup F$. We would like to prove that by deleting just a small amount of edges from E_i , we have a subset $F_i \subset E_i$ such that $F_i \cup F$ satisfies items (b)-(f) of Definition 4.7. Fix some ordering of X_{i-1} , say $v_1, \dots, v_{|X_{i-1}|}$. Let $E_{i,h} \cup F$ be the graph obtained after adding to F the edges $(v_j, \Gamma_i(v_j))$ for $1 \leq j \leq h$. Also let $E_{i,0} = \emptyset$.

We start by taking care of Item (b). For $0 \leq h < |X_{i-1}|$, we call v_{h+1} *friendly* if the component of v_{h+1} in $E_{i,h} \cup F$ has at most \sqrt{n} vertices and $\Gamma_i(v_{h+1})$ belongs to that component. The probability of being friendly is therefore at most \sqrt{n}/r , so the expected number of friendly vertices is at most $|X_{i-1}|\sqrt{n}/r \leq (1 \pm o_n(1))nq_{i-1}\sqrt{n}/(n/5) < n^{2/3}$ (recall that we assume that $r \geq n/(k+1)$ and that $k \in \{3, 4\}$ so $r \geq n/5$). By Markov's inequality, with probability $p_{(b)} = 1 - o_n(1)$, there are at most $n^{3/4}$ friendly vertices. But observe that removing from $E_i \cup F$ the edges of E_i emanating from friendly vertices results in a digraph with maximum out-degree 1 in which every component with at most \sqrt{n} vertices is a tree. Thus, with probability $p_{(b)} = 1 - o_n(1)$ we can remove a set $E_i^* \subset E_i$ of at most $n^{3/4} + n/\sqrt{n} = n^{3/4} + \sqrt{n} < 2n^{3/4}$ edges from E_i such that $(E_i \setminus E_i^*) \cup F$ still constitutes a forest (recall that F is a forest since \vec{G}_{i-1} is successful).

We next consider Item (c). While trivially the maximum out-degree of $E_i \cup F$ is one (being an out-degree one orientation), this is not so for the in-degrees. It could be that a vertex whose

in-degree in F is 0 or 1 has significantly larger in-degree after adding E_i . So, we perform the following process for reducing the in-degrees. For each $v \in V(G)$ whose in-degree in $E_i \cup F$ is $t > 1$, we randomly delete precisely $t - 1$ edges of E_i entering v thereby reducing v 's in-degree to 1 (note: this means that if v 's in-degree in F is 1 we remove all edges of E_i entering it and if v 's in-degree in F is 0 we just keep one edge of E_i entering it, and the kept edge is chosen uniformly at random). Let E_i^{**} be the edges removed from E_i by that process. Then we have that $(E_i \setminus E_i^{**}) \cup F$ has maximum in-degree at most 1 and maximum out-degree at most 1.

We next consider Item (d). For $u \in V(G)$, let W_u denote the G -neighbors of u in X_{i-1} . Since \vec{G}_{i-1} is successful, we have that $|W_u| = (1 \pm o_n(1))rq_{i-1}$. Let Z be the number of vertices with in-degree 0 in $E_i \cup F$. Suppose u has in-degree 0 in F . In order for u to remain with in-degree 0 in $E_i \cup F$ it must be that each vertex $v \in W_u$ has $\Gamma_i(v) \neq u$. The probability of this happening is precisely $(1 - 1/r)^{|W_u|} = (1 - 1/r)^{(1 \pm o_n(1))rq_{i-1}}$. Since \vec{G}_{i-1} is successful, there are $(1 \pm o_n(1))nq_{i-1}$ vertices u with in-degree 0 in F . We obtain that

$$\mathbb{E}[Z] = (1 \pm o_n(1))nq_{i-1} \left(1 - \frac{1}{r}\right)^{(1 \pm o_n(1))rq_{i-1}} = (1 \pm o_n(1))nq_i.$$

We can prove that Z is tightly concentrated around its expectation as we have done in Lemma 4.2 using martingales. Let $Z_0 = \mathbb{E}[Z]$ and let Z_h be the conditional expectation of Z after the edge $(v_h, \Gamma_i(v_h))$ of E_i has been exposed, so that we have $Z_{|X_{i-1}|} = Z$. Then, $Z_0, Z_1, \dots, Z_{|X_{i-1}|}$ is a martingale satisfying the Lipschitz condition (since the exposure of an edge can change the amount of vertices with in-degree 0 by at most one), so by Azuma's inequality, for every $\lambda > 0$,

$$\Pr[|Z - E[Z]| > \lambda | X_{i-1}] < 2e^{-\lambda^2/2}.$$

In particular, $Z = (1 \pm o_n(1))nq_i$ with probability $p_{(d)} = 1 - o_n(1)$ (the $o_n(1)$ term in the probability can even be taken to be exponentially small in n).

We next consider Item (e) whose proof is quite similar to the proof of Item (d) above. Let Z_v denote the number of G -neighbors of v with in-degree 0 in $E_i \cup F$. Since \vec{G}_{i-1} is successful, there are $(1 \pm o_n(1))rq_{i-1}$ G -neighbors of v with in-degree 0 in F so the expected value of Z_v is

$$\mathbb{E}[Z_v] = (1 \pm o_n(1))rq_{i-1} \left(1 - \frac{1}{r}\right)^{(1 \pm o_n(1))rq_{i-1}} = (1 \pm o_n(1))rq_i.$$

As in the previous paragraph, we apply Azuma's inequality to show that $Z_v = (1 \pm o_n(1))rq_i$ with probability $1 - o_n(1/n)$, so for all $v \in V(G)$ this holds with probability $p_{(e)} = 1 - o_n(1)$.

We finally consider Item (f) which is somewhat more delicate as we have to make sure that after removal of E_i^{**} , the vertices of X_{i-1} that remain with out-degree 0 are distributed roughly equally among all neighborhoods of vertices of G . Fix some $u \in V(G)$, and consider again W_u ,

the G -neighbors of u in X_{i-1} , recalling that $|W_u| = (1 \pm o_n(1))rq_{i-1}$. Suppose $v \in W_u$. We would like to estimate the probability that $(v, \Gamma_i(v)) \notin E_i^{**}$. For this to happen, a necessary condition is that $\Gamma_i(v)$ has in-degree 0 in F . As there are $(1 \pm o_n(1))rq_{i-1}$ G -neighbors of v with in-degree 0 in F , this occurs with probability $q_{i-1}(1 \pm o_n(1))$. Now, given that $\Gamma_i(v)$ has in-degree 0 in F , suppose that $\Gamma_i(v)$ has t in-neighbors in E_i . Then, the probability that $(v, \Gamma_i(v)) \notin E_i^{**}$ is $1/t$. As the probability that $\Gamma_i(v)$ has t in-neighbors in E_i (including v) is

$$\binom{s-1}{t-1} \frac{1}{r^{t-1}} \left(1 - \frac{1}{r}\right)^{s-t} = (1 \pm o_n(1)) \frac{q_{i-1}^{t-1}}{(t-1)!e^{q_{i-1}}}$$

where $s = (1 \pm o_n(1))rq_{i-1}$ is the number of G -neighbors of $\Gamma_i(v)$ in $X_{i-1} \setminus \{v\}$. We therefore have that

$$\Pr[(v, \Gamma_i(v)) \notin E_i^{**}] = (1 \pm o_n(1))q_{i-1} \sum_{t=1}^{\infty} \frac{1}{t} \cdot \frac{q_{i-1}^{t-1}}{(t-1)!e^{q_{i-1}}} = (1 \pm o_n(1)) (1 - e^{-q_{i-1}}) .$$

Since $|W_u| = (1 \pm o_n(1))rq_{i-1}$, we have from the last equation that the expected number of neighbors of u with out-degree 0 in $F \cup (E_i \setminus E_i^{**})$ is

$$(1 \pm o_n(1))rq_{i-1}e^{-q_{i-1}} = (1 \pm o_n(1))rq_i .$$

Once again, using Azuma's inequality as in the previous cases, we have that the number of neighbors of u with out-degree 0 in $F \cup (E_i \setminus E_i^{**})$ is $(1 \pm o_n(1))rq_i$ with probability $1 - o(1/n)$, so this holds with probability $p_{(f)} = 1 - o_n(1)$ for all $u \in V(G)$.

Finally, we define $F_i = E_i \setminus (E_i^* \cup E_i^{**})$ so items (b)-(f) hold for $F_i \cup F$ with probability at least $1 - (1 - p_{(b)}) - (1 - p_{(d)}) - (1 - p_{(e)}) - (1 - p_{(f)}) > \frac{1}{2}$ (recall that $|E_i^*| = o(n)$ so its removal does not change the asymptotic linear quantities stated in items (d),(e),(f)). \square

By Lemma 4.8, with probability at least $\frac{1}{2^i}$, we have that \vec{G}_i is successful. Assuming that \vec{G}_i is successful, let F_1, \dots, F_i satisfy Definition 4.7. Let X_i be the set of vertices with out-degree 0 in $\cup_{j=1}^i F_j$. Since \vec{G}_i is successful we have that $|X_i| = (1 \pm o_n(1))nq_i$. Consider the set of edges of \vec{G}_{i+1} emanating from X_i , denoting them by $E_{i+1} = \{(v, \Gamma_{i+1}(v)) \mid v \in X_i\}$. By adding E_{i+1} to $\cup_{j=1}^i F_j$ we therefore obtain an out-degree one orientation of G , which we denote by $E_{i+1} \cup (\cup_{j=1}^i F_j)$.

Lemma 4.9. *Let $i \geq 4$. Given that \vec{G}_i is successful, and given the corresponding forest $\cup_{j=1}^i F_j$, the probability that $(E_{i+1} \cup (\cup_{j=1}^i F_j)) \in \mathcal{H}_{3, nq_i(1 \pm o_n(1))}(G) \cap \mathcal{H}_{\ln n}^*(G)$ is at least*

$$(1 - 5q_i - o_n(1))^{nq_i} .$$

²We require this assumption so that the value $1 - 5q_i$ used in the lemma, is positive. Indeed, already $q_4 = 0.162038\dots$ satisfies this (observe also that $q_i = q_{i-1}/e^{q_{i-1}}$ is monotone decreasing).

Proof. Fix an arbitrary ordering of the vertices of X_i , say $v_1, \dots, v_{|X_i|}$. We consider the edges $(v_h, \Gamma_{i+1}(v_h))$ one by one, and let $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ be the graph obtained after adding to $\cup_{j=1}^i F_j$ the edges $(v_\ell, \Gamma_{i+1}(v_\ell))$ for $1 \leq \ell \leq h$. Also let $E_{i+1,0} = \emptyset$. We say that $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ is *good* if it satisfies the following properties:

- (i) The in-degree of each vertex of $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ is at most 2.
- (ii) Every component of $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ with fewer than $n/\ln n$ vertices is a tree.
- (iii) The number of vertices with in-degree 2 in $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ is at most h .

Trivially, $E_{i+1,0} \cup (\cup_{j=1}^i F_j) = \cup_{j=1}^i F_j$ is good, since by Definition 4.7, $\cup_{j=1}^i F_j$ is a forest where the in-degree of each vertex is at most 1. We estimate the probability that $E_{i+1,h+1} \cup (\cup_{j=1}^i F_j)$ is good given that $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ is good.

So, consider now the vertex v_{h+1} . Since $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ is assumed good, v_{h+1} has at most h neighbors with in-degree 2 in $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$. Thus, there is a subset S of at least $r - h$ G -neighbors of v_{h+1} which still have in-degree at most 1 in $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$. If the component of v_{h+1} in $E_{i+1,h} \cup (\cup_{j=1}^i F_j)$ has fewer than $n/\log n$ vertices, then also remove all vertices of this component from S . In any case we have that $|S| \geq r - h - n/\ln n$. The probability that $\Gamma_{i+1}(v_{h+1}) \in S$ is therefore at least

$$\frac{r - h - \frac{n}{\ln n}}{r} = 1 - \frac{h + \frac{n}{\ln n}}{r} \geq 1 - \frac{nq_i(1 \pm o_n(1)) + \frac{n}{\ln n}}{n/5} = 1 - 5q_i - o_n(1).$$

Now, to have $\Gamma_{i+1}(v_{h+1}) \in S$ means that we are not creating any new components of size smaller than $n/\ln n$, so all components of size at most $n/\ln n$ up until now are still trees and furthermore, $E_{i+1,h+1} \cup (\cup_{j=1}^i F_j)$ still has maximum in-degree at most 2 and at most one additional vertex, namely $\Gamma_{i+1}(v_{h+1})$, which may become now of in-degree 2, so it has at most $h + 1$ vertices with in-degree 2. Consequently, $E_{i+1,h+1} \cup (\cup_{j=1}^i F_j)$ is good. We have therefore proved that the final $E_{i+1} \cup (\cup_{j=1}^i F_j)$ is good with probability at least

$$(1 - 5q_i - o_n(1))^{|X_i|} \geq (1 - 5q_i - o_n(1))^{nq_i(1 \pm o_n(1))} = (1 - 5q_i - o_n(1))^{nq_i}.$$

Finally, notice that the definition of goodness means that $(E_{i+1} \cup (\cup_{j=1}^i F_j)) \in \mathcal{H}_{3,nq_i(1 \pm o_n(1))}(G) \cap \mathcal{H}_{\ln n}^*(G)$. \square

Lemma 4.10. *Let $K \geq 5$ be given.*

$$\begin{aligned} P_{4,0,\ln n}(G)^{1/n} &\geq P_{3,nq_{K-1}(1 \pm o_n(1)),\ln n}(G)^{1/n} \\ &\geq (1 - o_n(1)) \frac{(1 - 5q_{K-1})^{q_{K-1}}}{K}. \end{aligned}$$

Proof. The first inequality is trivial since an out-degree one orientation with maximum-in degree at most 3 has zero vertices with in-degree 4 or larger. So, we only prove the second inequality.

Consider the K -stage model, and let $i = K - 1 \geq 4$. By Lemma 4.8, with probability at least $\frac{1}{2^i}$, \vec{G}_i is successful. Thus, by Lemma 4.9 and Definition 4.7, with probability at least

$$\frac{1}{2^i}(1 - 5q_i - o_n(1))^{nq_i}$$

the following holds: There is an out-degree one orientation \vec{G} consisting of edges of \vec{G}_j for $j = 1, \dots, K$, at most $nq_i(1 \pm o_n(1))$ of which are edges of \vec{G}_K , which is in $\mathcal{H}_{3,nq_i(1 \pm o_n(1))}(G) \cap \mathcal{H}_{\ln n}^*(G)$.

Assuming that this holds, for $j = 1, \dots, K$ let X_j be the set of vertices whose out-edge in \vec{G} is from \vec{G}_j . Now, let p_1, \dots, p_K with $p_1 + \dots + p_K = 1$ be the probabilities associated with the K -stage model. The probability that in the second part of the K -stage model, each vertex $v \in X_j$ will indeed choose $\Gamma_j(v)$ is precisely

$$\prod_{j=1}^K p_j^{|X_j|}.$$

Using $p_j = \frac{1}{K}$ for all p_j 's and recalling that the final outcome of the K -stage model is a completely random element of $\mathcal{H}(G)$, we have that

$$P_{3,nq_i(1 \pm o_n(1)), \ln n}(G)^{1/n} \geq \frac{1}{2^i}(1 - 5q_i - o_n(1))^{nq_i} \left(\frac{1}{K}\right)^n.$$

Taking the n 'th root from both sides yields the lemma. \square

Proof of Theorem 1.1 for $k \in \{3, 4\}$. Consider first the case $k = 4$ where we will use $K = 5$. A simple computation gives that $q_{K-1} = q_4 = 0.162038\dots$ so we have by Lemma 4.10 that

$$P_{4,0, \ln n}(G)^{1/n} \geq (1 - o_n(1)) \frac{(1 - 5q_4)^{q_4}}{5} = (1 - o_n(1))0.1527\dots$$

As trivially $P_{4,0, \ln n}(G) \leq P_{4,n/28, \ln n}(G)$ we have by Lemma 3.1 that

$$c_4(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n}(1 - o_n(1))0.1527\dots = (1 - o_n(1))d \cdot 0.1527\dots$$

Consider now the case $k = 3$ where we will use $K = 20$. A simple computation gives that $q_{K-1} = q_{19} = 0.045821\dots$ so we have by Lemma 4.10 that

$$\begin{aligned} P_{3,n/21, \ln n}(G)^{1/n} &\geq P_{3,nq_{19}(1 \pm o_n(1)), \ln n}(G)^{1/n} \\ &\geq (1 - o_n(1)) \frac{(1 - 5q_{19})^{q_{19}}}{20} = (1 - o_n(1))0.0494\dots \end{aligned}$$

We now have by Lemma 3.1 that

$$c_3(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n}(1 - o_n(1))0.0494\dots = (1 - o_n(1))d \cdot 0.0494\dots$$

□

It is not too difficult to prove that if we additively increase the minimum degree requirement in Lemma 2.1 by a small constant, then we can allow for many more vertices of degree k in that lemma. This translates to an increase in the constants z_3 and z_4 . For example, in the case $k = 4$ a minimum degree of $n/5 + 2$ already increases z_4 to about 0.4 (instead of $z_4 = 0.1527\dots$ above) and in the case $k = 3$ a minimum degree of $n/4 + 17$ increases z_3 to about 0.2 (instead of $z_3 = 0.0494\dots$ above). However, we prefer to state Theorem 1.1 in the cleaner form of minimum degree exactly $n/(k + 1)$ for all $k \geq 3$.

4.3 Regular connected graphs with high minimum degree and $c_k(G) = 0$

In this subsection we show that the requirement on the minimum degree in Theorem 1.1 is essentially tight. For every $k \geq 2$ and for infinitely many n , there are connected r -regular graphs G with $r = \lfloor n/(k + 1) \rfloor - 2$ for which $c_k(G) = 0$. We mention that a construction for the case $k = 2$ is proved in [3].

Let $t \geq k + 4$ be odd. Let G_0, \dots, G_k be pairwise vertex-disjoint copies of K_t . Designate three vertices of each G_i for $1 \leq i \leq k$ where the designated vertices of G_i are $v_{i,0}, v_{i,1}, v_{i,2}$. Also designate $k + 2$ vertices of G_0 denoting them by $v_{0,0}, \dots, v_{0,k+1}$. We now remove a few edges inside the G_i 's and add a few edges between them as follows. For $1 \leq i \leq k$, remove the edges $v_{i,0}v_{i,1}$ and $v_{i,0}v_{i,2}$ and remove a perfect matching on the $t - 3$ undesignated vertices of G_i . Notice that after removal, each vertex of G_i has degree $t - 2$, except $v_{i,0}$ which has degree $t - 3$. Now consider G_0 and remove from it all edges of the form $v_{0,0}v_{0,j}$ for $1 \leq j \leq k + 1$. Also remove a perfect matching on the remaining $t - k - 2$ undesignated vertices of G_0 . Notice that after removal, each vertex of G_0 has degree $t - 2$, except $v_{0,0}$ which has degree $t - k - 2$. Finally, add the edges $v_{0,0}v_{i,0}$ for $1 \leq i \leq k$. After addition, each vertex has degree precisely $t - 2$. So the obtained graph G is connected, has $n = (k + 1)t$ vertices, and is r -regular for $r = n/(k + 1) - 2$. However, notice that any spanning tree of G must contain all edges $v_{0,0}v_{i,0}$ for $1 \leq i \leq k$ and must also contain at least one edge connecting $v_{0,0}$ to another vertex in G_0 . Thus, $v_{0,0}$ has degree at least $k + 1$ in every spanning tree.

Suppose next that $k \geq 2$ is even and suppose that $t \geq k + 5$ be odd. We slightly modify the construction above. First, we now take G_0 to be K_{t+1} . Now, there are $k + 3$ designated vertices in G_0 , denoted by $v_{0,0}, \dots, v_{0,k+2}$. The removed edges from the G_i for $1 \leq i \leq k$ stay the same. The removed edges from G_0 are as follows. We remove all edges of the form $v_{0,0}v_{0,j}$ for $1 \leq j \leq k + 2$. We remove a perfect matching on the vertices $v_{0,1}, \dots, v_{0,k+2}$. We also remove a Hamilton cycle on the $t - k - 2$ undesignated vertices of G_0 . Finally, as before, add the edges $v_{0,0}v_{i,0}$ for $1 \leq i \leq k$. After addition, each vertex has degree precisely $t - 2$. So the obtained graph G is connected, has $n = (k + 1)t + 1$ vertices, and is r -regular for $r = (n - 1)/(k + 1) - 2 = \lfloor n/(k + 1) \rfloor - 2$. However, notice that as before, $v_{0,0}$ has degree at least $k + 1$ in every spanning tree.

5 Proofs of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. As their proofs are essentially identical, we prove them together. We assume that $k \geq k_0$ where k_0 is a sufficiently large absolute constant satisfying the claimed inequalities. Although we do not try to optimize k_0 , it is not difficult to see from the computations that it is a moderate value.

Consider some $\vec{G} \in \mathcal{H}(G)$. An ordered pair of distinct vertices $u, v \in V(G)$ is a *removable edge* if $\Gamma(u) = v$ (so in particular $uv \in E(G)$) and the in-degree of v in \vec{G} is at least $k - 1$.

Lemma 5.1. *Suppose that $k \geq k_0$. Let G be a graph with minimum degree at least $\delta = n/(k + 1)$ and maximum degree at most $\Delta = n(1 - 3\sqrt{\ln k/k})$. Then with probability at least $\frac{1}{2}$, \vec{G} has at most $n/(14k)$ removable edges. The same holds if G has minimum degree at least $\delta^* = \frac{n}{k}(1 + 3\sqrt{\ln k/k})$ and unrestricted maximum degree.*

Proof. Consider some ordered pair of distinct vertices $u, v \in V(G)$ such that $uv \in E(G)$. For that pair to be a removable edge, it must hold that: (i) $\Gamma(u) = v$, and (ii) v has at least $k - 2$ in-neighbors in $N(v) \setminus u$. As (i) and (ii) are independent, and since $\Pr[\Gamma(u) = v] = 1/d(u)$, we need to estimate the number of in-neighbors of v in $N(v) \setminus u$, which is clearly at most v 's in-degree in \vec{G} . So let D_v be the random variable corresponding to v 's in-degree in \vec{G} . Observe that D_v is the sum of independent random variables $D_v = \sum_{w \in N(v)} D_{v,w}$ where $D_{v,w}$ is the indicator variable for the event $\Gamma(w) = v$.

Consider first the case where G has minimum degree at least δ and maximum degree at most Δ . In particular, $D_v \leq X$ where $X \sim \text{Bin}(\Delta, 1/\delta)$.

$$\mathbb{E}[X] = \frac{\Delta}{\delta} = (k + 1)(1 - 3\sqrt{\ln k/k}) = k(1 - o_k(1)).$$

Now let $a = k - 2 - \mathbb{E}[X] = 3\sqrt{k \ln k}(1 - o_k(1))$. Then by Chernoff's inequality (see [2] Appendix A) we have that for sufficiently large k ,

$$\begin{aligned} \Pr[D_v \geq k - 2] &\leq \Pr[X \geq k - 2] = \Pr[X - \mathbb{E}[X] \geq a] \\ &\leq e^{-a^2/(2\mathbb{E}[X]) + a^3/(2(\mathbb{E}[X])^2)} \\ &\leq e^{-(1 - o_k(1))9k \ln k/(2k) + (1 + o_k(1))27k^{3/2} \ln^{3/2} k/(2k^2)} \\ &\leq \frac{1}{k^4} \end{aligned} \tag{3}$$

where the last inequality holds for $k \geq k_0$. It follows that the probability that u, v is a removable edge is at most $(1/d(u))/k^4 \leq 1/(\delta k^4) \leq 1/(nk^2)$.

Consider next the case where G has minimum degree at least δ^* . In particular, $D_v \leq X$ where

$X \sim \text{Bin}(n, 1/\delta^*)$.

$$\mathbb{E}[X] = \frac{n}{\delta^*} = \frac{k}{1 + 3\sqrt{\ln k/k}} = k(1 - o_k(1)).$$

Now let $a = k - 2 - \mathbb{E}[X] = 3\sqrt{k \ln k}(1 - o_k(1))$. So as in (3), we obtain that $\Pr[D_v \geq k - 2] \leq 1/k^4$. It follows that the probability that u, v is a removable edge is at most $1/(\delta^* k^4) \leq 1/(nk^2)$.

As there are fewer than n^2 ordered pairs to consider, the expected number of removable edges is in both cases is at most n/k^2 . By Markov's inequality, with probability at least $\frac{1}{2}$, \vec{G} has at most $2n/k^2 \leq n/(14k)$ removable edges. \square

Lemma 5.2. *Suppose that $k \geq k_0$. Let G be a graph with minimum degree at least $\delta = n/(k + 1)$ and maximum degree at most $\Delta = n(1 - 3\sqrt{\ln k/k})$. With probability at least $\frac{1}{4}$, \vec{G} has a spanning forest F such that:*

- (a) F has maximum in-degree at most $k - 2$.
- (b) F has at least $n - n/(7k)$ edges.

The same holds if G has minimum degree at least $\delta^ = \frac{n}{k}(1 + 3\sqrt{\ln k/k})$ and unrestricted maximum degree.*

Proof. By Lemma 3.2, with probability at most $\frac{1}{4}$ we have that \vec{G} has more than $4(k + 1) \ln n$ components. Recalling that in \vec{G} each component can be made a tree by removing a single edge from its unique directed cycle, with probability at least $\frac{3}{4}$ we have that \vec{G} can be made acyclic by removing at most $4(k + 1) \ln n$ edges. By Lemma 5.1, with probability at least $\frac{1}{2}$, \vec{G} has at most $n/(14k)$ removable edges. So, with probability at least $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ we have a forest subgraph of \vec{G} with at least $n - 4(k + 1) \ln n - n/(14k) \geq n - n/(7k)$ edges in which all removable edges have been removed. But observe that after removing the removable edges, each vertex has in-degree at most $k - 2$. \square

Using the two-stage model, consider the graphs \vec{G}_1, \vec{G}_2 as denoted in Section 3. For a given $k \geq k_0$, we say that \vec{G}_1 is *successful* if it has a spanning forest as guaranteed by Lemma 5.2. By that lemma, with probability at least $\frac{1}{4}$, we have that \vec{G}_1 is successful. Assuming it is successful, designate a spanning forest F_1 of it satisfying the properties of Lemma 5.2. Let $X_1 \subset V(G)$ be the set of vertices with out-degree 0 in F_1 . Thus, we have by Lemma 5.2 that $|X_1| \leq n/(7k)$. Consider the set of edges of the \vec{G}_2 emanating from X_1 , denoting them by $E_2 = \{(v, \Gamma_2(v)) \mid v \in X_1\}$. By adding E_2 to F_1 we therefore obtain an out-degree one orientation of G , which we denote by $E_2 \cup F_1$. The following lemma is analogous to Lemma 4.5.

Lemma 5.3. *Given that \vec{G}_1 is successful, and given the corresponding forest F_1 , the probability that $(E_2 \cup F_1) \in \mathcal{H}_{k, n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$ is at least $(\frac{5}{8})^{n/(7k)}$.*

Proof. Fix an arbitrary ordering of the vertices of X_1 , say $v_1, \dots, v_{|X_1|}$. We consider the edges $(v_i, \Gamma_2(v_i))$ one by one, and let $E_{2,i} \cup F_1$ be the graph obtained after adding to F_1 the edges

$(v_j, \Gamma_2(v_j))$ for $1 \leq j \leq i$. Also let $E_{2,0} = \emptyset$. We say that $E_{2,i} \cup F_1$ is *good* if it satisfies the following two properties:

- (i) The in-degree of each vertex in $E_{2,i} \cup F_1$ is at most $k - 1$.
- (ii) Every component of $E_{2,i} \cup F_1$ with fewer than $n/\ln n$ vertices is a tree.
- (iii) The number of vertices in $E_{2,i} \cup F_1$ with in-degree $k - 1$ is at most i .

Note that $E_{2,0} \cup F_1 = F_1$ is good, since F_1 is a forest where the in-degree of each vertex is at most $k - 2$. We estimate the probability that $E_{2,i+1} \cup F_1$ is good given that $E_{2,i} \cup F_1$ is good. By our assumption, v_{i+1} has at most i neighbors with in-degree $k - 1$ in $E_{2,i} \cup F_1$. Thus, there is a subset S of at least $d(v_{i+1}) - i$ neighbors of v_{i+1} in G which still have in-degree at most $k - 2$ in $E_{2,i} \cup F_1$. As in Lemma 4.5, we may further delete at most $n/\ln n$ vertices from S in case the component of v_{i+1} in $E_{2,i} \cup F_1$ has fewer than $n/\ln n$ vertices so that in any case we have that $|S| \geq d(v_{i+1}) - i - n/\ln n$. The probability that $\Gamma_2(v_{i+1}) \in S$ is therefore at least

$$\frac{d(v_{i+1}) - i - \frac{n}{\ln n}}{d(v_{i+1})} \geq 1 - \frac{i + \frac{n}{\ln n}}{\frac{n}{k+1}} \geq 1 - \frac{\frac{n}{7k} + \frac{n}{\ln n}}{\frac{n}{k+1}} \geq \frac{5}{6}$$

(note that $d(v_{i+1}) \geq n/(k+1)$ trivially holds also in the assumption of Theorem 1.4). To have $\Gamma_2(v_{i+1}) \in S$ means that we are not creating any new components of size smaller than $n/\ln n$ and that $E_{2,i+1} \cup F_1$ has at most $i + 1$ vertices with in-degree $k - 1$. In other words, it means that $E_{2,i+1} \cup F_1$ is good. We have therefore proved that the final $E_2 \cup F_1$ is good with probability at least

$$\left(\frac{5}{6}\right)^{|X_1|} \geq \left(\frac{5}{6}\right)^{n/(7k)}.$$

Finally, note that the goodness of $E_2 \cup F_1$ means that it is in $\mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$. □

Lemma 5.4.

$$P_{k,n/(7k),\ln n}(G)^{1/n} \geq (1 - o_n(1)) \left(1 - \frac{1}{7k}\right)^{1 - \frac{1}{7k}} \left(\frac{1}{9k}\right)^{\frac{1}{7k}} = (1 - o_n(1)) z_k^*.$$

Proof. Considering the two-stage model, we have by Lemma 5.2 that with probability at least $\frac{1}{4}$, \vec{G}_1 is successful. Thus, by Lemma 5.3, with probability at least

$$\frac{1}{4} \left(\frac{5}{6}\right)^{n/(7k)}$$

the following holds: There is an out-degree one orientation \vec{G} consisting of $x \geq n - n/(7k)$ edges of \vec{G}_1 and hence at most $n/(7k)$ edges of \vec{G}_2 , which is in $\mathcal{H}_{k,n/(7k)}(G) \cap \mathcal{H}_{\ln n}^*(G)$. Assuming that this holds, let X be the set of vertices whose out-edge in \vec{G} is from \vec{G}_1 . Now, let $p_1 + p_2 = 1$ be the probabilities associated with the two-stage model where we assume $p_2 < \frac{1}{2}$. The probability that

in the second part of the two-stage model, each vertex $v \in X$ will indeed choose $\Gamma_1(v)$ and each vertex $v \in V(G) \setminus X$ will indeed choose $\Gamma_2(v)$ is precisely

$$p_1^x p_2^{n-x} \geq (1 - p_2)^{n-n/(7k)} p_2^{n/(7k)} .$$

Optimizing, we will choose $p_2 = 1/(7k)$. Recalling that the final outcome of the two-stage model is a completely random element of $\mathcal{H}(G)$, we have that

$$P_{k,n/(7k),\ln n}(G) \geq \frac{1}{4} \left(\frac{5}{6}\right)^{\frac{n}{7k}} \left(1 - \frac{1}{7k}\right)^{n-\frac{n}{7k}} \left(\frac{1}{7k}\right)^{\frac{n}{7k}} \geq \frac{1}{4} \left(1 - \frac{1}{7k}\right)^{n-\frac{n}{7k}} \left(\frac{1}{9k}\right)^{\frac{n}{7k}} .$$

Taking the n 'th root from both sides yields the lemma. □

Proof of Theorems 1.3 and 1.4. Combining Lemma 5.4 and Lemma 3.1 we have that

$$c_k(G)^{1/n} \geq (1 - o_n(1))d(G)^{1/n} z_k^* .$$

□

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References

- [1] N. Alon. The number of spanning trees in regular graphs. *Random Structures & Algorithms*, 1(2):175–181, 1990.
- [2] N. Alon and J. Spencer. *The probabilistic method*. John Wiley & Sons, 2004.
- [3] D. W. Cranston and O. Suil. Hamiltonicity in connected regular graphs. *Information Processing Letters*, 113(22-24):858–860, 2013.
- [4] B. Cuckler and J. Kahn. Hamiltonian cycles in dirac graphs. *Combinatorica*, 29(3):299–326, 2009.
- [5] A. Czygrinow, G. Fan, G. Hurlbert, H. A. Kierstead, and W. T. Trotter. Spanning trees of bounded degree. *The Electronic Journal of Combinatorics*, pages R33–R33, 2001.
- [6] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.

- [7] A. Ferber, M. Krivelevich, and B. Sudakov. Counting and packing hamilton cycles in dense graphs and oriented graphs. *Journal of Combinatorial Theory, Series B*, 122:196–220, 2017.
- [8] C. Greenhill, M. Isaev, M. Kwan, and B. D. McKay. The average number of spanning trees in sparse graphs with given degrees. *European Journal of Combinatorics*, 63:6–25, 2017.
- [9] R. Grone and R. Merris. A bound for the complexity of a simple graph. *Discrete mathematics*, 69(1):97–99, 1988.
- [10] A. V. Kostochka. The number of spanning trees in graphs with a given degree sequence. *Random Structures & Algorithms*, 6(2-3):269–274, 1995.
- [11] B. D. McKay. Spanning trees in regular graphs. *European Journal of Combinatorics*, 4(2):149–160, 1983.
- [12] G. N. Sárközy, S. M. Selkow, and E. Szemerédi. On the number of hamiltonian cycles in dirac graphs. *Discrete Mathematics*, 265(1-3):237–250, 2003.
- [13] S. Win. Existenz von gerüsten mit vorgeschriebenem maximalgrad in graphen. *Mathematischen Seminar der Universität Hamburg*, 43(1):263–267, 1975.