On the Density of a Graph and its Blowup

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Abstract

It is well-known that, of all graphs with edge-density p, the random graph $G(n,p)$ contains the smallest density of copies of $K_{t,t}$, the complete bipartite graph of size $2t$. Since $K_{t,t}$ is a t-blowup of an edge, the following intriguing open question arises: Is it true that of all graphs with triangle density p^3, the random graph $G(n,p)$ contains close to the smallest density of $K_{t,t,t}$, which is the t-blowup of a triangle?

Our main result gives an indication that the answer to the above question is positive by showing that for *some* blowup, the answer must be positive. More formally we prove that if G has triangle density p^3, then there is some $2 \leq t \leq T(p)$ for which the density of $K_{t,t,t}$ in G is at least $p^{(3+o(1))t^2}$, which (up to the $o(1)$ term) equals the density of $K_{t,t,t}$ in $G(n,p)$. We also raise several open problems related to these problems and discuss some applications to other areas.

1 Introduction

One of the main family of problems studied in extremal graph theory is how does the lack and/or number of copies of one graph H in a graph G affect the lack and/or number of copies of another graph H' in G. Perhaps the most well known problems of this type are Ramsey's Theorem and Turán's Theorem. Our investigation here is concerned with the relation between the densities of certain fixed graphs in a given graph. Some well known results of this type are Goodman's Theorem [15], the Chung-Graham-Wilson Theorem [10] as well as the well known conjectures of Sidorenko [23] and Simonovits [24]. Some recent results of this type have been obtained recently by Razborov [21] and Nikiforov [20], and an abstract investigation of problems of this type was taken recently by Lovász and Szegedy [19]. In this paper we introduce an extremal problem of this type, which is related to some of these well-studied problems, and to problems in other areas such as quasi-random graphs and Communication Complexity.

Let us start with some standard notation. Given a graph H on h vertices $v_1, \ldots, v_h$ and a sequence of h positive integers $a_1, \ldots, a_h$, we denote by $B = H(a_1, \ldots, a_h)$ the $(a_1, \ldots, a_h)$-blowup

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of H obtained by replacing every vertex $v_i \in H$ with an independent set I_i of a_i vertices, and by replacing every edge (v_i, v_j) of H with a complete bipartite graph connecting the independent sets I_i and I_j. For brevity, we will call $B = H(b, \ldots, b)$ the b-blowup of H, that is, the blowup in which all vertices are replaced with an independent set of size b.

For a fixed graph H and a graph G we denote by $c_H(G)$ the number of copies of H in G, or more formally the number of injective mappings from $V(H)$ to $V(G)$ which map edges of H to edges of G. For various reasons, it is usually more convenient to count *homomorphisms* from H to G, rather than count copies of H in G. Let us then denote this quantity by $Hom_H(G)$, that is, the number of (not necessarily injective) mappings from $V(H)$ to $V(G)$ which map edges of H to edges of G (allowing[1] two endpoints of an edge to be mapped to the same vertex of G). We now let $d_H(G) = Hom_H(G)/n^h$ denote the *H-density* of G (or the density of H in G). Note that $0 \leq d_H(G) \leq 1$ and we can think of $d_H(G)$ as the probability that a random map $\phi : V(H) \mapsto V(G)$ is a homomorphism. We will also say that a graph on n vertices has edge-density p if it has $p\binom{n}{2}$ edges.

The main motivation of our investigation comes from extremal graph theory. It is a well known fact that of all graphs with edge-density p, the random graph $G(n, p)$ contains the smallest asymptotic density of copies of C_4 (the 4-cycle) [2]. Let $K_{a,b}$ denote the complete bipartite graph on sets of vertices of sizes a and b and note that $K_{a,b}$ is the (a, b)-blowup of an edge and that C_4 is just $K_{2,2}$. It is actually known that for any $K_{a,b}$ the random graph $G(n, p)$ has the smallest density of $K_{a,b}$ over all graphs with edge density p. We also recall the famous conjectures of Sidorenko [23] and Simonovits [24] which state that the above phenomenon holds for all bipartite graphs, that is, that for any bipartite graph B, of all graphs with edge density p, the random graph $G(n, p)$ has the smallest B-density.

The question we raise in this paper can thus be thought of as an attempt to extend the above results/conjectures from blowups of an edge, to blowups of arbitrary graphs. Let us state it explicitly.

Problem 1 *Let H be a fixed graph and set $B = H(a_1, \ldots, a_h)$. Assuming that $d_H(G) = \gamma$, how small can $d_B(G)$ be?*

Motivated by the fact regarding blowups of an edge, it is natural to ask if it is the case that over all graphs G satisfying $d_H(G) = \gamma$, the density of B is minimized by a random graph of an appropriate density (where B is some blowup of H). This turns out to be false even when H is a triangle and B is the 2-blowup of H. This fact was noted by Conlon et al. [11] who observed that a blow-up of K_5 has triangle-density $12/25$ and B-density $0.941(12/25)^4$. On the other hand, a random graph with triangle-density $12/25$ has B-density $(12/25)^4$. Hence we get that blowups of triangles

[1]We note that the standard definition of homomorphism does not allow the end points of an edge to be mapped to the same vertex. However, this relaxed definition is easier to consider and will not make any difference when counting the asymptotic number of homomorphisms.

[2]This fact is implicit in some early works of Erdős

and blowups of edges behave quite differently. We also recall the Chung-Graham-Wilson Theorem [10] which says that if a graph has edge-density p and $K_{2,2}$-density p^4 then the graph is quasi-random. It is thus natural to ask the following; let B be the 2-blowup of K_3. Is it true that if G has the same K_3-density and B-density as $G(n,p)$ then G is quasi-random? As it turns out, the example of [11] shows that this is not the case. We refer the reader to the excellent survey on quasi-random graphs by Krivelevich and Sudakov [17] for the precise definitions related to quasi-random graphs.

As we will see shortly, Problem 1 seems challenging even for the first non-trivial case of H being the triangle (denoted K_3), so we will mainly consider this case. To simplify the notation, let us denote by $K_{a,b,c}$ the (a,b,c)-blowup of K_3. So $K_{2,2,2}$ is the 2-blowup of the triangle and the question we are interested in is the following: Suppose the triangle-density of G is γ. How small can the density of $B = K_{a,b,c}$ be in G? Let us denote by $f_B(\gamma)$ the infimum of this quantity over all graphs with triangle-density at least γ. That is:

Definition 1.1 *For a real $\gamma > 0$ and an integer n, let $\mathcal{G}_\gamma(n)$ denote the set of n vertex graphs with triangle-density at least γ. For a blowup $B = K_{a,b,c}$ we define*[3]

$$f_B(\gamma) = \liminf_{n\to\infty} \min_{G\in\mathcal{G}_\gamma(n)} d_B(G) \ . \tag{1}$$

So Problem 1 can be restated as asking for a bound on the function $f_B(\gamma)$. Let's start with some simple bounds one can obtain for $f_B(\gamma)$. A simple upper bound for $f_B(\gamma)$ can be obtained by considering the number of triangles and copies of $K_{a,b,c}$ in the random graph $G(n,\gamma^{1/3})$. In the other direction, a simple lower bound can be obtained from the Erdős-Simonovits Theorem [12] regarding the number of copies of complete 3-partite hypergraphs in dense 3-uniform hypergraphs. These two bounds give the following:

Proposition 1.2 *Let $B = K_{a,b,c}$. Then we have the following bounds*

$$\gamma^{abc} \le f_B(\gamma) \le \gamma^{(ab+bc+ac)/3} \ .$$

Our main results in this paper suggest that it should be possible to improve upon the simple bounds in the above proposition. But before turning to the technical part of the paper, let us mention two other problems that are related to the problem we address here. As it turns out, in the case of $B = K_{2,2,2}$, the question of bounding $f_B(\gamma)$ was also considered recently (and independently) due to a different motivation. Alon, Raz and Yehudayoff [3] observed that improving the lower bound of $B = K_{2,2,2}$ from $f_B(\gamma) \ge \gamma^8$ to $f_B(\gamma) \ge \gamma^{8-c}$ for some $c > 0$ would give a lower bound for the disjointedness problem in the number-on-the-forehead model. Although this lower bound was obtained recently by other means [9, 18] it would be very intriguing to obtain such results via results from extremal graph theory. See [9, 18] and their references for more details on this problem.

[3] It is actually not too hard to deduce from the regularity-lemma that the lim inf in (1) is actually a proper limit.

Finally, note that one can naturally consider the following variant of Problem 1: Let H be a fixed graph and let B' be any *subgraph* of $H(a_1, \ldots, a_h)$. How small can $f_{B'}(G)$ be if $f_H(G) = \gamma$? We note that while Problem 1 for the case of H being an edge is well understood, the above variant of Problem 1 is open even when H is an edge. This is the conjecture of Sidorenko [23] and Simonovits [24] (which we have mentioned earlier). We thus focus our attention on Problem 1.

1.1 Balanced blowups and the main results

When considering the case $B = K_{2,2,2}$, the bounds given by Proposition 1.2 become $\gamma^8 \leq f_B(\gamma) \leq \gamma^4$. Recall also the result of [11] which can be stated as saying that we can further improve this upper bound to $f_B(\gamma) < 0.941\gamma^4$. So we see that the $K_{2,2,2}$-density can be smaller than the $K_{2,2,2}$-density in a random graph with the same triangle-density. It is thus natural to ask if there are examples in which the $K_{2,2,2}$-density is polynomially smaller than in the random graph. By taking an appropriate graph power of the example of [11] we can show that this is indeed the case.

Proposition 1.3 *Set $B = K_{2,2,2}$. Then for all small enough γ we have*

$$f_B(\gamma) \leq \gamma^{4.08} \ .$$

The proof of Proposition 1.3 appears at the end of Subsection 2.3. The above proposition implies that one cannot hope to show that the random graph has the smallest $K_{2,2,2}$-density, even up to a small polynomial factor.

Let us now turn to consider the more general case in which $B = K_{t,t,t}$. In this case Proposition 1.2 gives the bounds $\gamma^{t^3} \leq f_B(\gamma) \leq \gamma^{t^2}$ and the question is finding the correct exponent of $f_B(\gamma)$. Given Proposition 1.3 it is thus natural to ask if we can obtain a similar polynomial improvement over the upper bound of Proposition 1.2 for other blowups $K_{t,t,t}$. Our first main result in this paper is the following general improved upper bound.

Theorem 1 *There are absolute constants $t_0,\ c > 0$, so that for all $t \geq t_0$ and all small enough γ,*

$$f_B(\gamma) \leq \gamma^{t^2(1+c)} \ ,$$

where $B = K_{t,t,t}$.

So the above theorem states that for every large enough t there are graphs whose $K_{t,t,t}$-density is far from the corresponding density in a random graph with the same triangle-density. Our second main result in this paper complements the above theorem by showing that if a graph G has triangle-density γ, then for at least *one* blowup $K_{t,t,t}$, the graph must have $K_{t,t,t}$-density asymptotically close to γ^{t^2}, namely, as the density expected in the random graph. More formally, we prove the following.

Theorem 2 *For every $0 < \gamma, \delta < 1$ there are $N = N(\gamma, \delta)$ and $T = T(\gamma, \delta)$ such that if G is a graph on $n \geq N$ vertices and its triangle-density is γ, then there is some $2 \leq t \leq T$ for which the $K_{t,t,t}$-density of G is at least $\gamma^{(1+\delta)t^2}$.*

So Theorem 2 states that in any graph G with K_3-density γ, there is *some* t for which the $K_{t,t,t}$-density in G is almost as large as the $K_{t,t,t}$-density in a random graph with the same triangle-density. A natural question is if the dependence on G in Theorem 2 can be removed, that is, if one can strengthen Theorem 2 by showing that it holds with T depending only on γ and t. Observe, however, that by Theorem 1 this is not the case for any small enough γ and $\delta = c$, where c is the constant in Theorem 1. So we see that the value of t must depend on the specific graph, although it is bounded by a quantity that depends only on γ and δ. We still believe though that the following is true.

Conjecture 1 *There is an absolute constant C such that*

$$f_B(\gamma) \geq \gamma^{Ct^2}$$

where $B = K_{t,t,t}$.

Recall that by Theorem 1 even if the above conjecture is true, we must have $C > 1$. See Section 4 for further discussion on this conjecture and some related results.

1.2 Organization

The rest of this paper is organized as follows. In section 2 we focus on large blowups and prove Theorem 2. Our first main tool for the proof of Theorem 2 is the quantitative version of the Erdős-Stone Theorem, the so called Bollobás-Erdős-Simonovits Theorem [6, 7], regarding the size of blowups of K_r in graphs whose density is larger than the Turán density of K_r. To the best of our knowledge, this is the first application of the Bollobás-Erdős-Simonovits Theorem in which the exact bound on the size of the blowup of K_r plays in important role. Our second main tool is a functional variant of Szemerédi's regularity lemma [26] due to Alon et al. [2]. We believe that this combination of the results of [6, 7] and [2] may be of independent interest. In Section 3 we prove Theorem 1. The main idea is to first prove Theorem 1 for γ close to 1 using a simple (yet hard to analyze) graph. We then extend the result to all small enough γ using tensor products and random graphs. In section 4 we consider some additional result. Specifically we consider the densities of small skewed blowups and prove that in some cases one can obtain nearly tight bounds of their densities. The proof of these results apply the so called Triangle Removal Lemma of Rusza-Szemerédi as well as the Rusza-Szemerédi graphs. We also mention some additional problems related to the study of $f_B(\gamma)$.

2 The Density of Large Symmetric Blowups

2.1 Background on the Regularity Lemma

We start with the basic notions of regularity, some of the basic applications of regular partitions and state the regularity lemmas that we use in the proof of Theorem 2. See [16] for a comprehensive survey on the Regularity Lemma. We start with some basic definitions. For every pair of nonempty disjoint vertex sets A and B of a graph G, we define $e(A, B)$ to be the number of edges of G between A and B. The *edge-density* of the pair is defined by $d(A, B) = e(A, B)/|A||B|$.

Definition 2.1 (γ-regular pair) *A pair (A, B) is γ-*regular*, if for any two subsets $A' \subseteq A$ and $B' \subseteq B$, satisfying $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$, the inequality $|d(A', B') - d(A, B)| \leq \gamma$ holds.*

Let G be a graph obtained by taking a copy of K_3, replacing every vertex with a sufficiently large independent set, and every edge with a random bipartite graph. The following well known lemma shows that if the bipartite graphs are "sufficiently" regular, then G contains the same number of triangles as the random graph does. For brevity, let us say that three vertex sets A, B, C are ϵ-regular if the three pairs (A, B), (B, C) and (A, C) are all ϵ-regular. Several versions of this lemma were previously proved in papers using the Regularity Lemma. See e.g. Lemma 4.2 in [14].

Lemma 2.2 *For every ζ there is an $\epsilon = \epsilon_{2.2}(\zeta)$ satisfying the following. Let A, B, C be pairwise disjoint independent sets of vertices of size m each that are ϵ-regular and satisfy $d(A, B) = \alpha_1$, $d(A, C) = \alpha_2$ and $d(B, C) = \alpha_3$. Then (A, B, C) contain at most $(\alpha_1\alpha_2\alpha_3 + \zeta)\, m^3$ triangles.*

Comment 2.3 *Although this is not stated explicitly, the function $\epsilon_{2.2}(\zeta)$ in Lemma 2.2 can be assumed to be monotone increasing in ζ. We will use this assumption for all similar functions throughout the paper.*

The following lemma also follows from Lemma 4.2 in [14].

Lemma 2.4 *For every t and ζ there is an $\epsilon = \epsilon_{2.4}(t, \zeta)$ such that if G is a $3t$-partite graph on disjoint vertex sets $A_1, \ldots, A_t$, $B_1, \ldots, B_t$, $C_1, \ldots, C_t$ of size m, and these sets satisfy:*

- *(A_i, B_j, C_k) are ϵ-regular for every $1 \leq i, j, k \leq t$.*
- *For every $1 \leq i, j, k \leq t$ we have $d(A_i, B_j) \geq \alpha_1$, $d(A_i, C_k) \geq \alpha_2$ and $d(B_j, C_k) \geq \alpha_3$.*

Then G contains at least $(\alpha_1\alpha_2\alpha_3 - \zeta)^{t^2} m^{3t}$ copies of $K_{t,t,t}$ each having precisely one vertex from each partite set.

The following lemma is an easy consequence of Lemma 2.4, obtained by taking t multiple copies of each partite set.

Lemma 2.5 *For every t and ζ there is an $\epsilon = \epsilon_{2.5}(t, \zeta)$ such that if G is a 3-partite graph on disjoint vertex sets A, B, C of size m and these sets satisfy:*

- *(A, B, C) is ϵ-regular.*
- *$d(A, B) \geq \alpha_1$, $d(A, C) \geq \alpha_2$ and $d(B, C) \geq \alpha_3$.*

Then G contains at least $(\alpha_1\alpha_2\alpha_3 - \zeta)^{t^2} m^{3t}$ distinct homomorphisms of $K_{t,t,t}$.

A partition $\mathcal{A} = \{V_i \mid 1 \leq i \leq k\}$ of the vertex set of a graph is called an *equipartition* if $|V_i|$ and $|V_j|$ differ by no more than 1 for all $1 \leq i < j \leq k$ (so in particular each V_i has one of two possible sizes). The *order* of an equipartition denotes the number of partition classes (k above). A *refinement* of an equipartition $\mathcal{A}$ is an equipartition of the form $\mathcal{B} = \{V_{i,j} \mid 1 \leq i \leq k,\ 1 \leq j \leq \ell\}$ such that $V_{i,j}$ is a subset of V_i for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$.

Definition 2.6 (γ-regular equipartition) *An equipartition $\mathcal{B} = \{V_i \mid 1 \leq i \leq k\}$ of the vertex set of a graph is called γ-regular if all but at most γk^2 of the pairs $(V_i, V_{i'})$ are γ-regular.*

The Regularity Lemma of Szemerédi can be formulated as follows.

Lemma 2.7 ([26]) *For every m and $\gamma > 0$ there exists $T = T_{2.7}(m, \gamma)$ with the following property: If G is a graph with $n \geq T$ vertices, and $\mathcal{A}$ is an equipartition of the vertex set of G of order at most m, then there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ of order k, where $m \leq k \leq T$ and $\mathcal{B}$ is γ-regular.*

Our main tool in the proof of Theorem 2 is Lemma 2.9 below, proved in [2]. This lemma can be considered a strengthening of Lemma 2.7, as it guarantees the existence of an equipartition and a refinement of this equipartition that have stronger properties compared to those of the standard γ-regular equipartition. This stronger notion is defined below.

Definition 2.8 ($\mathcal{E}$-regular equipartition) *For a function $\mathcal{E}(r) : \mathbb{N} \mapsto (0, 1)$, a pair of equipartitions $\mathcal{A} = \{V_i \mid 1 \leq i \leq k\}$ and its refinement $\mathcal{B} = \{V_{i,j} \mid 1 \leq i \leq k,\ 1 \leq j \leq \ell\}$, where $V_{i,j} \subset V_i$ for all i, j, are said to be $\mathcal{E}$-regular if*

1. *All but at most $\mathcal{E}(0)k^2$ of the pairs (V_i, V_j) are $\mathcal{E}(0)$-regular.*
2. *For all $1 \leq i < i' \leq k$, for all $1 \leq j, j' \leq \ell$ but at most $\mathcal{E}(k)\ell^2$ of them, the pair $(V_{i,j}, V_{i',j'})$ is $\mathcal{E}(k)$-regular.*
3. *All $1 \leq i < i' \leq k$ but at most $\mathcal{E}(0)k^2$ of them are such that for all $1 \leq j, j' \leq \ell$ but at most $\mathcal{E}(0)\ell^2$ of them $|d(V_i, V_{i'}) - d(V_{i,j}, V_{i',j'})| < \mathcal{E}(0)$ holds.*

It will be very important for what follows to observe that in Definition 2.8 we may use an arbitrary *function* rather than a fixed γ as in Definition 2.6 (such functions will be denoted by $\mathcal{E}$ throughout the paper). The following is one of the main results of [2].

Lemma 2.9 ([2]) *For any integer m and function $\mathcal{E}(r) : \mathbb{N} \mapsto (0,1)$ there is $S = S_{2.9}(m, \mathcal{E})$ such that any graph on at least S vertices has an $\mathcal{E}$-regular equipartition $\mathcal{A}$, $\mathcal{B}$ where $|\mathcal{A}| = k \geq m$ and $|\mathcal{B}| = k\ell \leq S$.*

2.2 Main Idea and Main Obstacle

Let us describe the main intuition behind the proof of Theorem 2, and where its naive implementation fails. Recall that Lemma 2.2 says that an ϵ-regular triple contains the "correct" number of triangles we expect to find in a "truly" random graph with the same density. So given a graph with triangle density γ, we can apply the Regularity Lemma with (say) $\epsilon = \gamma$. Suppose we get a partition into k sets, for (say) some $k \leq T_{2.7}(\gamma, 1/\gamma^2)$. So the situation now is that the number of triangles spanned by any triple V_i, V_j, V_k is more or less determined by the densities between the sets. Since G has triangle-density γ, we get (by averaging) that there must be some triple V_i, V_j, V_k whose triangle-density is also close to being at least γ. Suppose the densities between V_i, V_j, V_k are α_1, α_2 and α_3. Since the number of triangles between V_i, V_j, V_k is determined by the densities connecting them, we get that $\alpha_1\alpha_2\alpha_3$ is close to γ. Now, if ϵ is small enough, then we can also apply Lemma 2.4 on V_i, V_j, V_k in order to infer that they contain close to $(n/k)^{3t}\alpha_1^{t^2}\alpha_2^{t^2}\alpha_3^{t^2}$ copies of $K_{t,t,t}$. Hence, by the above consideration, this number is close to $(n/k)^{3t}\gamma^{t^2}$. Now, since for large enough $t \geq t(k)$ we have $(n/k)^{3t}\gamma^{t^2} = n^{3t}\gamma^{(1+o(1))t^2}$ we can choose a large enough $t = t(k)$ to get the desired result. Since k is bounded by a function of γ so is t.

The reason why the above argument fails is that in order to apply Lemma 2.4 with a given t, the value of ϵ in the ϵ-regular partition needs to depend on t. So we arrive at a circular situation in which ϵ needs to be small enough in terms of t (to allow us to apply Lemma 2.4), and t needs to be large enough in terms of ϵ (to allow us to infer that $(n/k)^{3t}\gamma^{t^2} = n^{3t}\gamma^{(1+o(1))t^2}$).

We overcome the above problem by applying Lemma 2.9 which more or less allows us to find a partition which is $f(k)$-regular where k is the number of partition classes. However, this is an over simplification of this result (as can be seen from Definition 2.8), and our proof requires several other ingredients that enable us to apply Lemma 2.9. Most notably, we need to use a classic result of Bollobás, Erdős and Simonovits [6, 7] and adjust it to our setting.

2.3 Some preliminary lemmas

We now turn to discuss two simple (yet crucial) lemmas that will be later used in the proof of Theorem 2. Let us recall that Turán's Theorem asserts that every graph with edge-density larger than $1 - \frac{1}{r-1}$ contains a copy of K_r, the complete graph on r vertices. The Erdős-Stone Theorem strengthens this result by asserting that if the edge-density is larger than $1 - \frac{1}{r-1}$, then the graph actually contains a blowup of K_r. More precisely, there is a function $f(n, \beta, r)$ such that every n-vertex graph with edge-density $1 - \frac{1}{r-1} + \beta$ contains an $f(n, \beta, r)$-blowup of K_r (and $f(n, \beta, r)$ goes to infinity with n).

The determination of the growth rate of $f(n,\beta,r)$ received a lot of attention until Bollobás, Erdős and Simonovits [6, 7] determined that for fixed β and r we have $f(n,\beta,r)=\Theta(\log n)$. See [20] for a short proof of this result and for related results and references. As it turns out, the bound $\Theta(\log n)$ will be crucial for our proof (a bound like $\log^{1-\epsilon} n$ would not be useful for us). Let us state an equivalent formulation of this result for the particular choice of $r=3$ and $\beta=1/24$.

Theorem 3 (Bollobás-Erdős-Simonovits [6, 7]) *There is an absolute constant c, such that every graph on at least c^t vertices and edge-density at least $13/24$ contains a copy of $K_{t,t,t}$.*

As a 3-partite graph with edge-density at least $7/8$ between any two parts has overall density greater than $13/24$ we have:

Corollary 1 *There is an absolute constant C, such that every 3-partite graph with parts of equal size C^t and edge-density at least $7/8$ between any two parts, contains a copy of $K_{t,t,t}$.*

We will need the following lemma guaranteeing many copies of a large blowup of K_3.

Lemma 2.10 *If G is a 3-partite graph on vertex sets X, Y and Z of equal size m, and the three densities $d(X,Y)$, $d(X,Z)$ and $d(Y,Z)$ are all at least $31/32$, then G contains at least $\lfloor m^{3t}/C^{3t^2}\rfloor$ copies of $K_{t,t,t}$, where C is an absolute constant.*

Proof: Let C be the constant of Corollary 1. If $m<C^t$ there is nothing to prove (as $\lfloor m^{3t}/C^{3t^2}\rfloor=0$) so let us assume that $m\geq C^t$. We first claim that at least $1/4$ of the graphs spanned by three sets of vertices $X'\subseteq X$, $Y'\subseteq Y$, $Z'\subseteq Z$, where $|X'|=|Y'|=|Z'|=C^t$, have edge-density at least $7/8$. Indeed, suppose we randomly pick the sets X', Y' and Z'. The expected density of non-edges between (X',Y'), (X',Z') and (Y',Z') is $1/32$ (for each pair separately), so by Markov's inequality, with probability at least $1/4$ this density is at most $1/8$ for all three pairs simultaneously.

By Corollary 1, every graph of size at least C^t, whose edge-density is at least $7/8$, contains a copy of $K_{t,t,t}$. So by the above consideration, at least $1/4$ of the $\binom{m}{C^t}^3$ choices of A',B',C' contain a $K_{t,t,t}$. Since each such $K_{t,t,t}$ is counted $\binom{m-t}{C^t-t}^3$ times, we have that the number of distinct copies of $K_{t,t,t}$ in G is at least

$$\frac{1}{4}\binom{m}{C^t}^3\Big/\binom{m-t}{C^t-t}^3\geq m^{3t}/C^{3t^2}\ .$$

■

Let us briefly mention that if one fixes an integer t, then it is always possible to choose an $\epsilon=\epsilon(t)$ such that if the densities of the three bipartite graphs in Lemma 2.10 are $1-\epsilon$ (rather than $31/32$) then one can find many copies of $K_{t,t,t}$. This follows from a simple counting argument, and in this case there is no need to use the Bollobás-Erdős-Simonovits Theorem. However, in our application we will not have the freedom of choosing the parameters this way. The reason is that ϵ in the above reasoning will be given by $\mathcal{E}(0)$ from Definition 2.8, while t will be roughly the integer S in Theorem 2.9, which is much larger than $1/\mathcal{E}(0)$. In particular, we will not have the freedom of choosing $\epsilon = \mathcal{E}(0)$ to be small enough as a function of t, for the reason that t itself will depend on $\epsilon = \mathcal{E}(0)$.

The proof of Theorem 2 that we give in the next subsection only covers the case of $\gamma \ll \delta$. As the following lemma shows, we can then "lift" this result to arbitrary $0 < \gamma, \delta < 1$. For the proof we will use the notion of graph tensor products, defined as follows: for an integer k let $G^{\otimes k}$ be the k^{th} tensor product of $G = (V, E)$, that is, the graph whose vertices are sequences of k (not necessarily distinct) vertices of G, and where vertex $v = (v_1, \ldots, v_k)$ is connected to vertex $u = (u_1, \ldots, u_k)$ if and only if for every $1 \le i \le k$ either $v_i = u_i$ or $(v_i, u_i) \in E$.

Lemma 2.11 *If Theorem 2 holds for every $\delta > 0$ and every small enough $\gamma < \gamma_0(\delta)$, then it also holds for every $0 < \gamma, \delta < 1$.*

Proof: Assume to the contrary that there exist a $\delta > 0$, a $\gamma \ge \gamma_0(\delta)$ and arbitrarily large graphs with triangle-density γ in which the $K_{t,t,t}$-density is smaller than $\gamma^{(1+\delta)t^2}$ for every $2 \le t \le T(\gamma_0^2(\delta), \delta)$. Let G be one such graph on at least $N(\gamma_0^2(\delta), \delta)$ vertices. The key observation is that for every graph H, if the H-density of G is γ then the H-density of $G^{\otimes k}$ is γ^k. Let then k be the smallest integer satisfying $\gamma^k < \gamma_0(\delta)$ and note that in this case we have $\gamma_0^2(\delta) \le \gamma^k < \gamma_0(\delta)$. We thus get that $G^{\otimes k}$ is a graph on at least $N(\gamma_0^2(\delta), \delta) \ge N(\gamma^k, \delta)$ vertices, with triangle-density γ^k and for all $2 \le t \le T(\gamma^k, \delta) \le T(\gamma_0^2(\delta), \delta)$ its $K_{t,t,t}$-density is smaller than $\gamma^{k(1+\delta)t^2}$, which contradicts the assumption of the lemma. ■

We end this subsection with the proof of Proposition 1.3.

Proof (of Proposition 1.3): Let K be an $m/5$ blowup of K_5. Recall that [11] noted that for large m, the triangle-density of K is $12/25$, while its $K_{2,2,2}$-density is $156/5^5 < 0.941(12/25)^4$. For any integer k, let $G = K^{\otimes k}$ be the k^{th} tensor-product of K. So the triangle-density of G is $(12/25)^k$ and its $K_{2,2,2}$-density is at most $(0.941)^k(12/25)^{4k} = (12/25)^{k(4+\zeta)}$, where $\zeta = \frac{\log 0.941}{\log 12/25} > 0.08$. For any positive integer k, setting $\gamma = (12/25)^k$ we thus get a graph whose triangle-density is γ while its $K_{2,2,2}$-density is at most $\gamma^{4.08}$. This proves the bound stated in Proposition 1.3 for a sequence of γ's approaching 0. One can extend this to arbitrary small γ using the same idea used in the proof of Theorem 1. ■

2.4 Proof of Theorem 2

We prove the theorem for every $0 < \delta < 1$ and for every $0 < \gamma < 1$ which is small enough so that

$$\gamma < \left(\frac{1}{128C^3}\right)^{2/\delta} , \tag{2}$$

where C is the absolute constant from Lemma 2.10. By Lemma 2.11 this will establish the theorem for all $0 < \delta, \gamma < 1$. We note that the main idea of the proof given below basically implements the "naive" idea described in Subsection 2.2, while utilizing Lemma 2.9 in order to actually make it work.

For a given positive integer r, let $t = t(r, \delta, \gamma)$ be a large enough integer such that

$$\frac{1}{r^{3t}} \left(\frac{\gamma}{64}\right)^{t^2} \geq 2C^{3t^2} \gamma^{(1+\delta)t^2} \tag{3}$$

holds. Since we assume that γ and δ satisfy (2), it is enough to make sure that t satisfies

$$\frac{1}{r^{3t}} \geq \gamma^{\frac{1}{2}\delta t^2} ,$$

hence we can take

$$t(r, \delta, \gamma) = \max\{2, \ \frac{6 \log r}{\delta \log \frac{1}{\gamma}}\} . \tag{4}$$

We now define a function $\mathcal{E}(r)$ as follows:

$$\mathcal{E}(r) = \begin{cases} \min\{\frac{1}{32}, \ \gamma/30, \ \epsilon_{2.2}(\gamma/4)\}, & r = 0 \\ \\ \min\{\frac{1}{32}, \ \epsilon_{2.5}(t(r,\delta,\gamma), \gamma/64), \ \epsilon_{2.4}(t(r,\delta,\gamma), \gamma/64)\}, & r \geq 1 . \end{cases} \tag{5}$$

Given γ and δ let $\mathcal{E}(r)$ be the function defined above. Set $m = 30/\gamma$ and let $S = S_{2.9}(m, \mathcal{E})$ be the constant from Lemma 2.9. Given a graph G on $n \geq S$ vertices and parameters γ and δ, we apply Lemma 2.9 with $m = 30/\gamma$ and with the function $\mathcal{E}(r)$ defined above. The lemma returns an $\mathcal{E}$-regular partition consisting of an equipartition $\mathcal{A} = \{V_i \mid 1 \leq i \leq k\}$ and a refinement $\mathcal{B} = \{V_{i,j} \mid 1 \leq i \leq k, \ 1 \leq j \leq \ell\}$, where $k\ell \leq S(m, \mathcal{E})$ and $k \geq m$. Note that S depends only on δ and γ.

We now remove from G any edge whose endpoints belong to the same set V_i. We thus remove at most $n^2/(2m) < \frac{\gamma}{60}n^2$ edges. We also remove any edge connecting pairs (V_i, V_j) that are not $\mathcal{E}(0)$-regular. The first property of an $\mathcal{E}$-regular partition guarantees that we thus remove at most $\mathcal{E}(0)n^2 \leq \frac{\gamma}{30}n^2$ edges. We also remove any edge connecting a pair (V_i, V_j) for which there are more that $\mathcal{E}(0)\ell^2$ pairs i', j' which do not satisfy $|d(V_i, V_j) - d(V_{i,i'}, V_{j,j'})| < \mathcal{E}(0)$. By the third property of an $\mathcal{E}$-partition we infer that we thus remove at most $\frac{\gamma}{30}n^2$ edges. All together we have removed less than $\frac{\gamma}{12}n^2$ edges and so we have destroyed at most $\frac{\gamma}{2}n^3$ triangles in G (recall that we are counting homomorphisms so each triangle is counted 6 times). And so the new graph we obtain has triangle-density at least $\gamma/2$. Let us call this new graph G'.

As G' has triangle-density at least $\gamma/2$, we get (by averaging) that there must be three sets (V_i, V_j, V_p) that contain at least $\frac{1}{2}\gamma(n/k)^3$ triangles with one vertex in each of the sets V_i, V_j, V_p For what follows, let us set $\alpha_1 = d(V_i, V_j)$, $\alpha_2 = d(V_i, V_p)$ and $\alpha_3 = d(V_j, V_p)$. Because we have removed edges between non-$\mathcal{E}(0)$-regular pairs, we get that (V_i, V_j, V_p) must be $\mathcal{E}(0)$-regular. Letting Δ denote the number of triangles spanned by (V_i, V_j, V_p) we see that as $\mathcal{E}(0) \leq \epsilon_{2.2}(\gamma/4)$, we can apply Lemma 2.2 on (V_i, V_j, V_p) to conclude that

$$\frac{1}{2}\gamma\left(\frac{n}{k}\right)^3 \leq \Delta \leq (\alpha_1\alpha_2\alpha_3 + \frac{1}{4}\gamma)\left(\frac{n}{k}\right)^3 ,$$

implying that

$$\alpha_1\alpha_2\alpha_3 \geq \frac{1}{4}\gamma . \tag{6}$$

Let us say that a $3s$-tuple (where s is any positive integer) $1 \leq i_1 < \cdots < i_s \leq \ell$, $1 \leq j_1 < \cdots < j_s \leq \ell$, $1 \leq p_1 \cdots < p_s \leq \ell$ is *good* if it satisfies the following four properties:

1. For every i_a, j_b, p_c we have that $(V_{i,i_a}, V_{j,j_b}, V_{p,p_c})$ are $\mathcal{E}(k)$-regular.
2. For every i_a, j_b we have $d(V_{i,i_a}, V_{j,j_b}) \geq \alpha_1 - \mathcal{E}(0) \geq \alpha_1 - \frac{1}{8}\gamma \geq \frac{1}{2}\alpha_1$.
3. For every i_a, p_c we have $d(V_{i,i_a}, V_{p,p_c}) \geq \alpha_2 - \mathcal{E}(0) \geq \alpha_2 - \frac{1}{8}\gamma \geq \frac{1}{2}\alpha_2$.
4. For every j_b, p_c we have $d(V_{j,j_b}, V_{p,p_c}) \geq \alpha_3 - \mathcal{E}(0) \geq \alpha_3 - \frac{1}{8}\gamma \geq \frac{1}{2}\alpha_3$.

Suppose $i_1, \ldots, i_t,\ j_1, \ldots, j_t,\ p_1, \ldots, p_t$ is a good $3t$-tuple. Then the definition of $\mathcal{E}$ (via the function $\epsilon_{2.4}(t, \zeta)$ from Lemma 2.4) and the first property of a good $3t$-tuple, guarantee that we can apply Lemma 2.4 on $V_{i,i_1}, \ldots, V_{i,i_t}$, $V_{j,j_1}, \ldots, V_{j,j_t}$, $V_{p,p_1}, \ldots, V_{p,p_t}$, to conclude that they have at least

$$\left(\frac{n}{k\ell}\right)^{3t}\left(\frac{1}{8}\alpha_1\alpha_2\alpha_3 - \frac{1}{64}\gamma\right)^{t^2} \geq \left(\frac{n}{k\ell}\right)^{3t}\left(\frac{\gamma}{64}\right)^{t^2}$$

copies of $K_{t,t,t}$, where we have also used (6). Our choice of $t = t(k, \delta, \gamma)$ in (4) guarantees (via (3)) that the number of copies of $K_{t,t,t}$ in a good $3t$-tuple is at least

$$\left(\frac{n}{k\ell}\right)^{3t}\left(\frac{\gamma}{64}\right)^{t^2} \geq 2C^{3t^2}\left(\frac{n}{\ell}\right)^{3t}\gamma^{(1+\delta)t^2} . \tag{7}$$

But how can we be certain that a good $3t$-tuple exists? And if they do, how many are there? We first consider the case $\ell \geq C^t$. Let us now recall that $\mathcal{E}(r) \leq \frac{1}{32}$ for every $r \geq 0$ and so the second and third properties of an $\mathcal{E}$-regular partition guarantee that at least $\frac{31}{32}\ell^2$ of the choices $1 \leq i', j' \leq \ell$ are such that $(V_{i,i'}, V_{j,j'})$ is $\mathcal{E}(k)$-regular and satisfies $|d(V_i, V_j) - d(V_{i,i'}, V_{j,j'})| \leq \mathcal{E}(0)$. The same holds with respect to the other two pairs (V_j, V_p) and (V_i, V_p). Therefore, by Lemma 2.10, the sets V_i, V_j, V_p contain at least $\lfloor \ell^{3t}/C^{3t^2} \rfloor \geq 0.5\ell^{3t}/C^{3t^2}$ choices of good $3t$-tuples. Hence, combining this with (7) we infer that the number of copies of $K_{t,t,t}$ spanned by (V_i, V_j, V_p) is at least

$$\frac{\ell^{3t}}{2C^{3t^2}} \cdot 2C^{3t^2}\left(\frac{n}{\ell}\right)^{3t}\gamma^{(1+\delta)t^2} = n^{3t}\gamma^{(1+\delta)t^2} ,$$

implying that the $K_{t,t,t}$-density of G' (and so also in G) is at least $\gamma^{(1+\delta)t^2}$.

We now consider the case $\ell < C^t$. Assume that in this case we can find just *one* good 3-tuple. Then the definition of $\mathcal{E}$ (via the function $\epsilon_{2.5}(t,\zeta)$ from Lemma 2.5) and the first property of a good 3-tuple, together guarantee that we can apply Lemma 2.5 on this 3-tuple in order to conclude that it contains at least

$$\left(\frac{n}{k\ell}\right)^{3t}\left(\frac{1}{8}\alpha_1\alpha_2\alpha_3 - \frac{\gamma}{64}\right)^{t^2} \geq \left(\frac{n}{k\ell}\right)^{3t}\left(\frac{\gamma}{64}\right)^{t^2}$$

distinct homomorphisms of $K_{t,t,t}$. Our choice of $t = t(k,\delta,\gamma)$ in (4) guarantees (via (3)) that the number of homomorphisms of $K_{t,t,t}$ in a good 3-tuple is at least

$$\left(\frac{n}{k\ell}\right)^{3t}\left(\frac{\gamma}{64}\right)^{t^2} \geq 2C^{3t^2}\left(\frac{n}{\ell}\right)^{3t}\gamma^{(1+\delta)t^2} \geq n^{3t}\gamma^{(1+\delta)t^2} \ ,$$

implying that the $K_{t,t,t}$-density in G' (and so also in G) is at least $\gamma^{(1+\delta)t^2}$. To see that a single good 3-tuple i_1, j_1, p_1 exists, consider picking i_1, j_1 and p_1 randomly and uniformly from $[\ell]$. Since $\mathcal{E}(k), \mathcal{E}(0) \leq \frac{1}{32}$ we infer that with positive probability i_1, j_1 and p_1 will satisfy the four properties of a good 3-tuple, so a good 3-tuple exists.

Finally, note that since $k \leq S$ we see that k is upper bounded by some function of γ and δ. As $t = t(k,\delta,\gamma)$ is chosen in (4) we see that $2 \leq t \leq T(\gamma,\delta)$ for some function $T(\gamma,\delta)$ and so the proof is complete. ■

3 Proof of Theorem 1

We will prove Theorem 1 by first proving it for some large γ (see Lemma 3.1 below), and then, by applying tensor products and taking random subgraphs, obtain a similar result for all small enough γ. Actually, the proof of Theorem 1 will cover all γ in the range $(0, 1-\frac{2}{t})$ so we actually need a very mild assumption on how small γ is.

Let $G(n,6t)$ denote the complete $6t$-partite graph with n vertices in each part. The main part of the proof of Theorem 1 is the following lemma.

Lemma 3.1 *There exists an absolute constant $c > 0$ and an integer t_0, so that for all $t \geq t_0$ and for all n sufficiently large, if γ_t denotes the triangle-density of $G(n,6t)$, then the $K_{t,t,t}$-density of $G(n,6t)$ is at most $\gamma_t^{t^2(1+c)}$.*

Proof: Throughout the proof n is assumed to be sufficiently large depending on t. Also t is assumed to be sufficiently larger than some absolute constant.

Let us first compute a lower bound on the triangle-density of $G(n,6t)$. Clearly,

$$\gamma_t > (1-1/6t)(1-2/6t) > 1-1/2t.$$

Since

$$\gamma_t^{t^2} > 0.9e^{-t/2} \ , \tag{8}$$

we can prove the lemma by showing that the $K_{t,t,t}$-density of $G(n,6t)$ is $C^{-t/2}$ for some $C > e$. Computing the $K_{t,t,t}$-density of $G(n,6t)$ requires, however, much more care. We start with the following combinatorial lemma.

Claim 3.2 *Let r and s be positive integers where $s \le r$. Suppose r labeled elements are placed at random into s labeled bins. The probability that no empty bin will remain is at most*

$$\frac{\binom{r-1}{s-1} r!}{s^r 2^{r-s}} .$$

Proof: There are precisely s^r ways to place the balls in the bins. Let's provide an upper bound for the number of such placements in which no empty bin remains. A vector of positive integers $(b_1, \ldots, b_s)$ with $\sum_{i=1}^{s} b_i = r$ corresponds to a configuration in which bin i receives b_i balls. There are precisely $\binom{r-1}{s-1}$ configurations. For a given configuration, the number of assignments corresponding to it is precisely

$$\frac{r!}{\Pi_{i=1}^{s} b_i!} \le \frac{r!}{2^{r-s}} .$$

It follows that the probability in the statement of the lemma is at most as claimed. ■

It will be convenient to name the vertex classes of $K_{t,t,t}$ as A_1, A_2, A_3, and to name the vertex classes of $G(n,6t)$ as $V_1, \ldots, V_{6t}$. Let $\mathcal{Q}$ be the set of ordered partitions of $[6t]$ into four nonempty parts. Thus, an element $\mathcal{Q}$ is a 4-tuple (Q_0, Q_1, Q_2, Q_3). A copy of $K_{t,t,t}$ in G is of *configuration* (Q_0, Q_1, Q_2, Q_3) if all the vertices of A_i are in $\cup_{j \in Q_i} V_j$, and precisely the vertex classes V_j with $j \in Q_0$ are those that do not contain any vertex of the copy.

Let $z = (z_0, z_1, z_2, z_3)$ be a vector of positive integers with $z_0 + z_1 + z_2 + z_3 = 6t$ and with $z_0 \ge 3t$. A tuple (Q_0, Q_1, Q_2, Q_3) is of *type* z if $|Q_i| = z_i$ for $i = 0, \ldots, 3$. Similarly, a copy of $K_{t,t,t}$ of configuration (Q_0, Q_1, Q_2, Q_3) is of *type* $z = (|Q_0|, |Q_1|, |Q_2|, |Q_3|)$.

There are $O(t^3)$ possible types. Thus, it suffices to prove that for any fixed type, the $K_{t,t,t}$-density of that given type is at most $\gamma_t^{(1+c')t^2}$ for some absolute constant c'. We therefore fix a type $z = ((6-3\beta)t, \beta_1 t, \beta_2 t, \beta_3 t)$ where $\beta = (\beta_1 + \beta_2 + \beta_3)/3$ and focus at proving an upper bound for its density.

We first observe that the number of configurations corresponding to z is

$$\frac{(6t)!}{(\beta_1 t)!(\beta_2 t)!(\beta_3 t)!((6-3\beta)t)!} .$$

For a given configuration (Q_0, Q_1, Q_2, Q_3) of type z, we compute the density of $K_{t,t,t}$s having this configuration using the following equivalent combinatorial procedure. We have $3t$ elements, with three colors $1, 2, 3$, where there are t elements of each color. We have $6t$ bins, and we ask for the probability of a random assignment of elements to the bins that has the following property: All the elements of color i are placed in bins j with $j \in Q_i$, and each bin $j \in Q_i$ has at least one element placed in it. This probability is a product of two probabilities $p_a p_b$. The first one, p_a, is that each element falls in a bin belonging to the correct class, and, conditioned on that, the second one, p_b, is that no bin in Q_i is empty for each $i = 1, 2, 3$. Thus, $p_b = p_{c,1} p_{c,2} p_{c,3}$ where $p_{c,i}$ is the probability that no bin in Q_i is empty, given that all elements of color i are placed only in these bins.

The first probability is trivial to compute. It is precisely just

$$p_a = (\beta_1/6)^t (\beta_2/6)^t (\beta_3/6)^t \ .$$

Instead of computing p_{c_i}, we shall provide an upper bound for it. We will use the upper bound provided by Claim 3.2 with $r = t$ and $s = \beta_i t$. It yields:

$$p_{c_i} \le \frac{\binom{t-1}{\beta_i t - 1} t!}{(\beta_i t)^t 2^{(1-\beta_i)t}} \ .$$

The $K_{t,t,t}$-density of our configuration is, therefore, bounded by

$$\frac{(6t)!}{(\beta_1 t)!(\beta_2 t)!(\beta_3 t)!((6-3\beta)t)!} \prod_{i=1}^{3} \left(\frac{\binom{t-1}{\beta_i t - 1} t!}{(\beta_i t)^t 2^{(1-\beta_i)t}} \cdot \left(\frac{\beta_i}{6}\right)^t \right) \ .$$

Using Stirling's formula asserting that $x! = \sqrt{2\pi x}(x/e)^x(1 + O(1/x))$, the last expression is at most

$$\left(\frac{6^6}{\beta_1^{\beta_1} \beta_2^{\beta_2} \beta_3^{\beta_3} (6-3\beta)^{6-3\beta}} + o_t(1) \right)^t \prod_{i=1}^{3} \left(\frac{\beta_i}{\beta_i^{\beta_i} (1-\beta_i)^{1-\beta_i} \beta_i 2^{1-\beta_i} 6e} + o_t(1) \right)^t \ .$$

Following (8), it therefore remains to prove that

$$\frac{6^3}{(6-3\beta)^{6-3\beta} e^3 2^{3-3\beta}} \prod_{i=1}^{3} \frac{1}{\beta_i^{2\beta_i} (1-\beta_i)^{1-\beta_i}} < e^{-1/2}.$$

Now, the function

$$f(\beta_1, \beta_2, \beta_3) = \prod_{i=1}^{3} \beta_i^{2\beta_i} (1-\beta_i)^{1-\beta_i}$$

subject to $0 \le \beta_i \le 1$ and to the fact that $\beta_1 + \beta_2 + \beta_3 = 3\beta$, satisfies

$$f(\beta_1, \beta_2, \beta_3) \ge (\beta^{2\beta}(1-\beta)^{1-\beta})^3 \ . \tag{9}$$

To see this, observe that $\ln f = \sum_{i=1}^{3} g(\beta_i)$ where $g(x) = 2x \ln x + (1-x)\ln(1-x)$. Since $g(x)$ is convex in $(0, 1)$ we get by Jensen's inequality that

$$\sum_{i=1}^{3} g(\beta_i) \ge 3g(\sum_{i=1}^{3} \beta_i/3) = 3g(\beta)$$

yielding (9).

It therefore remains to prove that

$$\frac{6^3}{(6-3\beta)^{6-3\beta}e^3 2^{3-3\beta}}\left(\frac{1}{\beta^{2\beta}(1-\beta)^{1-\beta}}\right)^3 < e^{-1/2}.$$

Rearranging the terms, this is equivalent to showing that

$$(3e6^{-\beta}\beta^{2\beta}(2-\beta)^{2-\beta}(1-\beta)^{1-\beta})^6 > e \tag{10}$$

in $[0,1]$. We first prove that $h(x)=6^{-x}x^{2x}(2-x)^{2-x}(1-x)^{1-x}$ is log-convex in $(0,1)$. Indeed,

$$\ln h(x) = -x\ln 6 + 2x\ln x + (2-x)\ln(2-x) + (1-x)\ln(1-x)\ .$$

The second derivative of $\ln h(x)$ is thus $2/x + 1/(2-x) + 1/(1-x)$ which is positive in $(0,1)$. Hence, the minimum of $h(x)$ in $(0,1)$ is unique and is attained at the minimum of $\ln h(x)$. Equating the first derivative of $\ln h(x)$ to zero we obtain the equation

$$-\ln 6 + 2\ln x - \ln(2-x) - \ln(1-x) = 0\ .$$

This amounts to solving the quadratic equation $5x^2 - 18x + 12 = 0$ whose unique solution in $(0,1)$ is $x = 9/5 - \sqrt{21}/5$. At this point, the value of the left hand side of (10) is *greater than* $2.76 > e$. ■

Proof of Theorem 1: Let c and t_0 be the absolute constants from Lemma 3.1, and let $t \geq t_0$. We first prove that $f_B(\gamma) \leq \gamma^{t^2(1+c)}$ for all values γ of the form γ_t^k where $k \geq 1$ is an integer. Indeed, by Lemma 3.1, for all n sufficiently large, $G(n,6t)$ has triangle-density γ_t and $K_{t,t,t}$-density less than $\gamma_t^{(1+c)t^2}$. For any integer k, let $G_n = (G(n,6t))^{\otimes k}$ be the k^{th} tensor-product of $G(n,6t)$. So the triangle-density of G_n is $\gamma = \gamma_t^k$ and its $K_{t,t,t}$-density is at most $\gamma_t^{k(1+c)t^2} = \gamma^{(1+c)t^2}$. It follows that $f_B(\gamma) \leq \gamma^{t^2(1+c)}$.

To handle values of γ that are not of the form $\gamma = \gamma_t^k$, suppose $\gamma_t^{k+1} < \gamma < \gamma_t^k$ for some integer $k \geq 1$, and set, as above, $G_n = (G(n,6t))^{\otimes k}$. Then G_n has triangle-density γ_t^k and $K_{t,t,t}$-density at most $\gamma_t^{k(1+c)t^2}$. Suppose we now randomly remove every edge of G_n with probability $1-(\gamma/\gamma_t^k)^{1/3}$. Let's call the new graph we obtain G'_n. Then every triangle in G_n remains a triangle in G'_n with probability γ/γ_t^k, so the expected triangle-density of G'_n is γ. Similarly, every copy of $K_{t,t,t}$ remains a copy in G'_n with probability $(\gamma/\gamma_t^k)^{t^2}$, hence the expected $K_{t,t,t}$-density of G'_n is at most

$$(\gamma/\gamma_t^k)^{t^2}\cdot\gamma_t^{k(1+c)t^2} = \gamma^{t^2}\gamma_t^{kct^2} \leq \gamma^{(1+c/2)t^2}\ ,$$

where the inequality follows since we assume that $\gamma \geq \gamma_t^{k+1} \geq \gamma_t^{2k}$. So the expected triangle and $K_{t,t,t}$ densities in G'_n satisfy the same relation we obtained above for γ of the form γ_t^k, with a slightly smaller constant $c/2$. But to show that there is actually a subgraph of G_n with *both* of these densities as their expected values, we need to show that both densities attain a value close to their expectation with high probability. We can think of the process of obtaining G'_n from G_n as a Doob martingale (see [4]). In this case, after exposing every edge the triangle-density can change by $O(1/n^2)$. Since there are $O(n^2)$ such events, we get from Azuma's Inequality (see [4]) that the probability that the triangle-density of G'_n deviates from its expectation by an additive $O(1/\sqrt{n})$ term is bounded from above by $2^{-O(n)}$. A similar bound holds for the $K_{t,t,t}$-density of G'_n. We infer that with high probability G'_n has both triangle-density $\gamma \pm o(1)$ and $K_{t,t,t}$-density $\gamma^{(1+c/2)t^2} \pm o(1)$ thus completing the proof. ■

4 Additional Results

In this section we describe some additional results related to the study of the function $f_B(\gamma)$. We start with considering the case of B being a skewed blowup. Proposition 1.2 implies that when $B = K_{1,1,2}$ we have $\gamma^2 \leq f_B(\gamma) \leq \gamma^{5/3}$. In an independent recent investigation, motivated by an attempt to improve the bounds in the well-known Triangle Removal Lemma (see Theorem 5), Trevisan (see [27] page 239) observed that the γ^2 lower bound for the case $B = K_{1,1,2}$ can be (slightly) improved. This is a special case of the following theorem which extends the result of Trevisan to any $K_{1,1,t}$.

Theorem 4 *Set $B = K_{1,1,t}$. Then we have the following bound*

$$f_B(\gamma) \geq \omega(\gamma^t) \ .$$

The proof of Theorem 4 is a simple adaptation of the proof of Trevisan for the case $B = K_{1,1,2}$. We will need the Triangle Removal Lemma of [22]:

Theorem 5 ([22]) *If G is an n vertex graph from which one should remove at least ϵn^2 edges in order to destroy all triangles, then G contains at least $f(\epsilon)n^3$ triangles.*

Proof of Theorem 4: Suppose G has γn^3 triangles. Then by Theorem 5 we know that G contains a set of edges F of size at most $f(\gamma)n^2$, the removal of which makes G triangle-free, where $f(\gamma) = o(1)$, that is $\lim_{\gamma \to 0} f(\gamma) = 0$. For each edge $e \in E(G)$ let $c(e)$ be the number of triangles in G containing e as one of their edges. Observe that a copy of $K_{1,1,t}$ is obtained by taking t triangles sharing an edge. Also, as the removal of edges in F makes G triangle-free, every triangle in G has an edge of F as one if its edges. Therefore, we have that the number of copies of $K_{1,1,t}$ in G is

$$\sum_{e \in F} \binom{c(e)}{t} \geq \frac{1}{t^t} \sum_{e \in F} c(e)^t \geq \frac{1}{t^t |F|^{t-1}} \left(\sum_{e \in F} c(e) \right)^t \geq \frac{1}{t^t |F|^{t-1}} \gamma^t n^{3t} \geq \frac{1}{t^t f(\gamma)^{t-1}} \gamma^t n^{t+2} \ ,$$

implying the desired result with $1/(t^t f(\gamma)^{t-1})$ being the $\omega(1)$ term in the statement of the theorem. ■

We note that since the proof of Theorem 4 applies the Triangle Removal Lemma, which, in turn, applies Semerédi's Regularity Lemma, already for $t = 2$, the $\omega(\gamma^2)$ bound in Theorem 4 "just barely" beats the simple γ^2 bound of Proposition 1.2. The bound which the proof gives is roughly of order $\log^*(1/\gamma)\gamma^2$, and Tao [27] asked if it is possible to improve this bound to something like $\log\log(1/\gamma)\gamma^2$. While we can not rule out such a bound, we can still rule out a polynomially better bound by improving the upper bound of Proposition 1.2. This is a special case of the following theorem whose proof uses a variant of the Ruzsa and Szemerédi Theorem [22].

Theorem 6 *Set $B = K_{1,1,t}$. Then we have the following bound*

$$f_B(\gamma) \leq \gamma^{t-o(1)}$$

where the $o(1)$ term goes to 0 with γ.

Proof: Suppose $S \subseteq [n]$ is a set of integers containing no 3-term arithmetic progression. We claim that in this case there is a graph $G = (V, E)$ with $|V| = 6n$ and $|E| = 3n|S|$, whose edges can be (uniquely) partitioned into $n|S|$ edge disjoint triangles. Furthermore, G contains no other triangles. To do this we define a 3-partite graph G on vertex sets A, B and C, of sizes n, $2n$ and $3n$ respectively, where we think of the vertices of A, B and C as representing the sets of integers $[n]$, $[2n]$ and $[3n]$. For every $1 \leq i \leq n$ and $s \in S$ we put a triangle $T_{i,s}$ in G containing the vertices $i \in A$, $i + s \in B$ and $i + 2s \in C$. It is easy to see that the above $n|S|$ triangles are edge disjoint, because every edge determines i and s. To see that G does not contain any more triangles, let us observe that G can only contain a triangle with one vertex in each set. If the vertices of this triangle are $a \in A$, $b \in B$ and $c \in C$, then we must have $b = a + s_1$ for some $s_1 \in S$, $c = b + s_2 = a + s_1 + s_2$ for some $s_2 \in S$, and $a = c - 2s_3 = a + s_1 + s_2 - 2s_3$ for some s_3. This means that $s_1, s_2, s_3 \in S$ form an arithmetic progression, but because S is free of 3-term arithmetic progressions it must be the case that $s_1 = s_2 = s_3$, implying that this triangle is one of the triangles $T_{i,s}$ defined above.

We now recall the well known construction of Behrend [5], which guarantees that for any integer m, there is a subset $S \subseteq [m]$ containing no 3-term arithmetic progression, satisfying $|S| \geq m/8^{\sqrt{\log m}}$. Let G' be the graph described above when using $[m]$ and the subset S. Finally, let G be an $n/6m$ blowup of G', that is, the graph obtained by replacing every vertex v of G' with an independent set I_v of size $n/6m$, and replacing every edge (u, v) of G' with a complete bipartite graph connecting I_u and I_v. Observe that G has n vertices, and that each triangle in G' gives rise to $(n/6m)^3$ triangles in G. Hence, the number of ways to map a triangle into G is

$$6m|S|\left(\frac{n}{6m}\right)^3 = \frac{n^3}{6^2 m 8^{\sqrt{\log m}}}$$

(recall that there are six ways to map a labeled triangle into a triangle of G). The crucial observation is that because all the triangles in G' are edge disjoint, the only copies of $K_{1,1,t}$ in G are those that are formed by picking t vertices from a set I_a, one vertex from a set I_b and one vertex from a set I_c for which a, b and c formed a triangle in G'. This means that the number of ways to map a $K_{1,1,t}$ into G is

$$m|S|(t+2)! \cdot 3\left(\frac{n}{6m}\right)^2 \binom{\frac{n}{6m}}{t} \leq \frac{3(t+1)(t+2)n^{t+2}}{6^{t+2}m^t 8^{\sqrt{\log m}}} \ .$$

Now setting

$$\gamma = \frac{1}{6^2 m 8^{\sqrt{\log m}}}$$

we see that the triangle-density of G is γ, while the density of $K_{1,1,t}$ in G is at most

$$\frac{3(t+1)(t+2)}{6^{t+2}m^t 8^{\sqrt{\log m}}} = \gamma^t \cdot (3(t+1)(t+2)6^{t-2}8^{(t-1)\sqrt{\log m}}) = \gamma^{t-o(1)} \ ,$$

thus completing the proof. ■

Note that Theorems 4 and 6 together determine the correct exponent of $f_B(\gamma)$ for $B = K_{1,1,t}$. That is, we get the following corollary.

Corollary 2 *Set $B = K_{1,1,t}$. Then we have*

$$\omega(\gamma^t) < f_B(\gamma) < \gamma^{t-o(1)} \ . \tag{11}$$

The problem of determining the correct order of the $o(1)$ terms in (11) remains open and seems challenging.

Comment 4.1 *Both Theorems 4 and 6 were also obtained independently by N. Alon [1].*

If we consider $B = K_{1,2,2}$, then Proposition 1.2 gives $\gamma^4 \leq f_B(\gamma) \leq \gamma^{8/3}$. The same proof as that of Theorem 4, and the same construction used for the proof of Theorem 6, give the following improved bounds $\omega(\gamma^4) \leq f_B(\gamma) \leq \gamma^{3-o(1)}$. Note that as opposed to the case of $B = K_{1,1,2}$ in which our bounds determined the correct exponent of $f_B(\gamma)$, in the case of $B = K_{1,2,2}$ we only know that the correct exponent of $f_B(\gamma)$ is between 3 and 4.

Alon [1] has recently proved the following result, related to Theorem 2.

Theorem 7 (Alon [1]) *Set $B = K_{t,t,t}$. Then we have*

$$f_B(\gamma) \geq \gamma^{t^2/\gamma^2} \ .$$

Alon's result implies that for any $t \geq 1/\gamma^2$ one can improve upon the lower bound of Proposition 1.2. Alon's argument is based on an idea used by Nikiforov [20] to tackle an Erdős-Stone [13] type question. We now show that a slightly weaker bound can be derived directly from a recent result of Nikiforov [20].

Theorem 8 *If a graph has triangle-density γ, then its $K_{t,t,t}$-density is at least $2^{-O(t^2/\gamma^3)}$.*

Proof (sketch): By a result of Nikiforov [20], a graph with triangle-density γ has a $K_{t,t,t}$ with $t = \gamma^3 \log n$. Or in other words, every graph on at least $2^{t/\gamma^3}$ vertices, whose triangle-density is γ, has a copy of $K_{t,t,t}$. As in the proof of Lemma 2.10, if a graph has triangle-density γ, then most subsets of vertices of size $2^{t/\gamma^3}$ have (roughly) the same density, so they contain a $K_{t,t,t}$. We thus get that G has $\frac{1}{4}\binom{n}{2^{t/\gamma^3}}$ sets which contain a $K_{t,t,t}$ and since each $K_{t,t,t}$ is counted $\binom{n-3t}{2^{t/\gamma^3}-3t}$ times we get that G has $n^{3t}/2^{O(t^2/\gamma^3)}$ distinct copies of $K_{t,t,t}$. ■

Observe that in a random graph $G(n, \gamma^{1/3})$, whose triangle density is γ, we expect to find a $K_{t,t,t}$ with $t = c \log_{1/\gamma} n$ for some absolute constant c. It seems very interesting to try and improve Nikiforov's result [20] mentioned above by showing the following:

Problem 2 *Is there an absolute constant $c > 0$, such that if a graph G has triangle-density γ, then G has a $K_{t,t,t}$ of size $t = c \log_{1/\gamma} n$?*

Besides being an interesting problem on its own, we note that such an improved bound, together with the argument we gave in the proof of Theorem 8, would imply that if the triangle-density of a graph is γ, then its $K_{t,t,t}$-density is at least $\gamma^{O(t^2)}$, which would establish Conjecture 1.

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