2-connected graphs with small 2-connected dominating sets

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Abstract

Let G be a 2-connected graph. A subset D of $V(G)$ is a 2-*connected dominating set* if every vertex of G has a neighbor in D and D induces a 2-connected subgraph. Let $\gamma_2(G)$ denote the minimum size of a 2-connected dominating set of G. Let $\delta(G)$ be the minimum degree of G. For an n-vertex graph G, we prove that

$$\gamma_2(G) \le n\frac{\ln \delta(G)}{\delta(G)}(1+o_\delta(1))$$

where $o_\delta(1)$ denotes a function that tends to 0 as $\delta \to \infty$. The upper bound is asymptotically tight. This extends the results in [3,5,10,12].

Key words: Domination, Connectivity, Minimum degree

1 Introduction

All graphs considered here are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [4]. A subset D of vertices in a graph G is a *dominating set* if every vertex not in D has a neighbor in D. The *domination number*, denoted $\gamma(G)$, is the minimum size of a dominating set. A graph G with more than r vertices is r-*connected* if deleting any set of at most $r-1$ vertices results in a connected graph. If G is an r-connected graph, then G has a dominating set that induces an r-connected subgraph (simply take the whole graph as a dominating set). Such a dominating set is called an r-*connected dominating set*. Let $\gamma_r(G)$ denote the minimum size of an r-connected dominating set of G. The parameter $\gamma_1(G)$ is also called the *connected domination number* of G. Note that for $r \ge 1$, every r-connected

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dominating set is a *strong* dominating set, namely, every vertex of G (whether in the dominating set or not) is dominated.

The problem of finding small dominating sets is an active topic of research in the area of graph algorithms and combinatorics (see e.g. [8,9]). In particular, upper bounds as a function of the minimum degree of the graph are well studied. A classic result proved independently by Lovász [10] (see another proof in [2]), Arnautov [3], and Payan [12] states that $\gamma(G) \le n\frac{1+\ln(\delta+1)}{\delta+1}$ for every n-vertex graph G with minimum degree δ. This result is asymptotically optimal for general graphs. Alon [1] proved by probabilistic methods that when n is large there exists a δ-regular graph with no dominating set of size less than $(1+o(1))\frac{1+\ln(\delta+1)}{\delta+1}n$. (For $\delta \le 3$, exact results were obtained in [11,13]). For connected domination, Caro, West, and Yuster [5] showed by more complicated arguments that the bound obtained by Lovász, Payan and Arnautov also holds in a much more restricted case, namely $\gamma_1(G) \le n\frac{\ln\delta}{\delta}(1+o_\delta(1))$. They also supplied a sequential deterministic algorithm that produces a connected dominating set with (at most) this size, in polynomial time. Thus, it is interesting to determine whether the bound of Lovász, Payan, and Arnautov also holds for r-connected dominating sets, for every fixed r. Namely:

Conjecture 1 *Let r be a fixed positive integer. If G is an r-connected graph with n vertices and minimum degree δ then $\gamma_r(G) \le n\frac{\ln\delta}{\delta}(1+o_\delta(1))$.*

The result of Caro, West and Yuster shows that the conjecture holds for $r = 1$. In this paper we prove it for $r = 2$.

Theorem 2 *If G is a 2-connected graph with n vertices and minimum degree δ, then $\gamma_2(G) \le n\frac{\ln\delta}{\delta}(1+o_\delta(1))$.*

Notice that both Conjecture 1 and Theorem 2 are relevant (and interesting) only for δ sufficiently large. The proof of the 2-connected case turns out to be more complicated than the proof for the case $r = 1$. This is partly due to the fact that $\gamma_1(G)/\gamma(G)$ is bounded by the constant 3 (this is shown in [6] and also in [5]). However, one cannot bound $\gamma_2(G)/\gamma(G)$ by any constant (see example in Section 2). There are also other obstacles when considering the 2-connected case.

The upper bound in Theorem 2 is asymptotically sharp. This is due to the fact that $\gamma_r(G) \ge \gamma(G)$ and that the construction of Alon mentioned above for $\gamma(G)$ yields, for every fixed r, an r-connected graph (assuming that n and δ are large enough).

In Section 2 we introduce the required tools needed for the proof of Theorem 2. The proof itself, which uses the probabilistic method, appears in Section 3.

2 Preliminary Lemmas

We start with three lemmas that are required for the proof of Theorem 2. In order to present the first lemma, we need to state several definitions. A *k-dominating set* of a graph G is a subset $X \subset V(G)$ such that every v outside X has at least k neighbors in X. A *block* of a graph H is a maximal subgraph B of H such that B has no cut-vertex. Every edge of H belongs to precisely one block. A block with three or more vertices is a 2-connected subgraph. It is an elementary exercise to see that if H is not a block then at least one block of H (in fact, at least two) has precisely one cut-vertex of H. Such a block is called a *leaf* block. For a subset X of vertices of G, let $G[X]$ denote the subgraph of G induced by X.

We begin with a lemma that shows that any 2-dominating set (with at least three vertices) of a 2-connected graph can be extended into a 2-connected dominating set by an iterative process that reduces the number of blocks until there is only one.

Lemma 3 *Let G be a 2-connected graph. If X is a 2-dominating set of G (with $|X| > 2$), and $G[X]$ has s blocks, then there exists a 2-connected dominating set of G, containing X, whose size is less than $|X| + 10(s-1)$.*

Proof: Let t denote the number of components of $G[X]$. Clearly, $1 \le t \le s$. We shall consider the weight function w defined by $w(X) = 4t+s$. We will show that, as long as $s > 1$, we can find at most two vertices outside X such that adding them to X decreases w. Hence, by adding at most $8t+2s-10 \le 10(s-1)$ vertices, we get $w = 5$ and $s = t = 1$. In particular, there exists a 2-connected dominating set of G.

Consider first the case $t > 1$. Thus, we have $X = A \cup B$ such that no edge of G connects a vertex in A with a vertex in B. Let $a \in A$ and $b \in B$ be two vertices whose distance in G is the smallest possible. This distance is at most 3, since otherwise there is a vertex in the middle of the shortest path between a and b that is not dominated by X. Consider first the case where a and b have a common neighbor, c, outside X. Adding c to X decreases the number of components by some $p \ge 1$, but it may increase the number of blocks (by adding cut-edges) by at most $p+1$. Hence, $w(X \cup \{c\}) \le 4(t-p) + (s+p+1) \le 4t+s-2$. Now consider the case where the distance in G between a and b is 3. Let (a,c,d,b) be a shortest path from a to b; note that $c, d \notin X$. Adding c and d to X decreases the number of components by some $p \ge 1$, but may increase the number of blocks by at most $p+2$. Hence, $w(X \cup \{c\ ,\ d\}) \le 4(t-p) + (s+p+2) \le 4t+s-1$.

Now consider the case $t = 1$. If $s = 1$ we are done. Otherwise, let U be the vertex set of a leaf block in $G[X]$, and let u be the unique cut-vertex of $G[X]$

belonging to U. Let $a \in U \setminus \{u\}$ and $b \in X \setminus U$ be two vertices whose distance in $G-u$ is the smallest possible. This distance is finite since $G-u$ is connected. We claim also that this distance is at most 3. Indeed, otherwise, there is a vertex in the middle of the shortest path between a and b in $G-u$ having no neighbor in $X \setminus \{u\}$, contradicting the fact that X is a 2-dominating set. Adding the one or two vertices on this shortest path between a and b to X decreases the number of blocks and therefore decreases w. □

Remark: Note that the requirement that X is a 2-dominating set is needed. There are examples showing that $\gamma_2(G)/\gamma(G)$ may be arbitrary large. Here is one: Take a cycle on the vertices $v_1, \ldots, v_n$, where $n \geq 7$. Connect v_3 to all the other vertices except for v_1 and v_5. The resulting graph is 2-connected (it is Hamiltonian). The domination number is 3, since v_1, v_3, v_5 is a dominating set, and two vertices cannot dominate everything. However, every 2-connected dominating set has at least $n-4$ vertices.

The second tool we need is a special case of a theorem of Kouider and Lonc [7].

Lemma 4 (Kouider, Lonc [7]) *The vertex set of an n-vertex graph with minimum degree d can be covered with at most n/d subgraphs such that each is a vertex, an edge, or a cycle.* □

The third lemma shows that if a graph has many vertices of (relatively) high degree, then it also has a large subgraph whose minimum degree is relatively high.

Lemma 5 *Fix ϵ and d with $1/2 > \epsilon > 0$ and $d \geq 1$, and let G be a graph with n vertices. If at most ϵn vertices have degree less than d, then G has a set of at least $(1-3\epsilon)n$ vertices inducing a subgraph with minimum degree greater than $d/4$.*

Proof: Since $\epsilon < 0.5$, we have that G has more than $nd/4$ edges. It is well known that every graph with n vertices and αn edges has a subgraph with minimum degree at least α (see, e.g. [4] p. xvii). Hence, G has a subgraph with minimum degree greater than $d/4$. Let Q be the largest set of vertices of a subgraph of G with minimum degree greater than $d/4$. Let $q = |Q|$ and let X be the vertices outside Q. By the maximality of Q, every vertex of X has at most $d/4$ neighbors in Q, and every subgraph of $G[X]$ has a vertex of degree at most $d/4$. Hence, $G[X]$ has at most $(n-q)d/4$ edges. Furthermore, X contains at least $n-q-\epsilon n$ vertices whose degree in G is at least d. Thus, the sum of the degrees (in $G[X]$) is at least $(n-q-\epsilon n)3d/4$. Hence, we must have

$$(n-q)d/4 \geq (n-q-\epsilon n)3d/8.$$

Thus, $q \geq (1-3\epsilon)n$. □

3 Proof of the main result

The basic idea of the proof of Theorem 2 is the following. We choose a random subset X of vertices of the graph. Given X, we (deterministically) add to it the subset Y of all vertices that have only a "few" (or no) neighbors in X. Clearly, $X \cup Y$ is a dominating set. If necessary, we add to $X \cup Y$ a few more vertices in order to make it a 2-connected dominating set. The crucial argument is that with high probability, this process results in a 2-connected dominating set that is not too large. The detailed proof follows.

Proof of Theorem 2: Fix $\epsilon \in (0, .5)$. We shall prove that, for sufficiently large δ, every 2-connected n-vertex graph G with minimum degree δ has a 2-connected dominating set of size at most $(1 + 100\epsilon)n\frac{\ln\delta}{\delta}$.

Let $p = (1 + \epsilon)\frac{\ln\delta}{\delta}$, and let X be a random set of vertices in G, where each vertex is chosen independently with probability p. Let Y be the set of vertices in G that have fewer than k neighbors in X, where $k = \lfloor\sqrt{\ln\delta}\rfloor$. Note that $X \cup Y$ is a k-dominating set of G. Let $H = G[X \cup Y]$, and let s be the number of blocks of H. According to Lemma 3, we can add at most $10(s - 1)$ vertices to $X \cup Y$ and obtain a 2-connected dominating set of G. With $|X| = x$ and $|Y| = y$, it follows that

$$\gamma_2(G) < x + y + 10s. \tag{1}$$

To obtain the desired upper bound on $\gamma_2(G)$, we first prove an upper bound on s in terms of other parameters. In the degree sequence of $G[X]$, listed in nondecreasing order, let $d - 1$ be the term at position $\lfloor\epsilon x\rfloor$. Thus $G[X]$ has at most ϵx vertices with degree less than d. By Lemma 5, X has a subset Q of size at least $(1 - 3\epsilon)x$ such that $\delta(G[Q]) > d/4$. By Lemma 4, Q can be covered using at most $4|Q|/d$ subgraphs of $G[Q]$ that are cycles, edges, or vertices. Adding the $x - |Q| + y$ vertices of $(X - Q) \cup Y$ as 1-vertex subgraphs yields a covering of $V(H)$ using r such subgraphs of H, where $r \leq 4x/d + 3\epsilon x + y$.

We claim that s, the number of blocks of H, is at most $2r - 1$. Enlarge the subgraphs in the covering to become blocks of H. This may combine some subgraphs, but in any case we obtain a covering of $V(H)$ using at most r blocks of H. This implies that H has at most $2r - 1$ blocks, using the fact that if the vertices of a graph G are covered by at most r blocks in G, then G has at most $2r - 1$ blocks. (Since blocks are connected, the union of the initial covering blocks has at most r components. The vertices of an omitted block lie in distinct components, so adding the block reduces the number of components, and this can happen at most $r - 1$ times.) We have shown that

$$s \leq 8\frac{x}{d} + 6\epsilon x + 2y.$$

It now follows from (1) that

$$\gamma_2(G) < (1+60\epsilon)x + 21y + 80\frac{x}{d}. \tag{2}$$

We shall bound the expectations of the summands in (2). Obviously,

$$E[x] = pn = (1+\epsilon)\, n\frac{\ln\delta}{\delta}. \tag{3}$$

Examining any δ neighbors of a vertex v in G yields an upper bound on the probability that v has fewer than k neighbors in X. Thus,

$$\Pr[x \in Y] \le \sum_{i=0}^{k-1} \binom{\delta}{i} p^i (1-p)^{\delta-i} < \sum_{i=0}^{k-1} (\delta p)^i e^{-p(\delta-k)} = \tag{4}$$

$$O\left(k(2\ln\delta)^k \delta^{-(1+\epsilon/2)}\right),$$

which is at most $o(\delta^{-1})$, so

$$E[y] = o\left(\frac{n}{\delta}\right). \tag{5}$$

The estimation of $E[\frac{x}{d}]$ is somewhat more delicate. Using conditional expectation, we split the computation into three parts. Let A be the event that $x > 3np$. Let B be the event that $x \le 3np$ and $d < k$. Let C be the event that $x \le 3np$ and $d \ge k$. Hence,

$$E\left[\frac{x}{d}\right] = E\left[\frac{x}{d} \mid A\right]\Pr[A] + E\left[\frac{x}{d} \mid B\right]\Pr[B] + E\left[\frac{x}{d} \mid C\right]\Pr[C]$$

Thus,

$$E\left[\frac{x}{d}\right] \le n\Pr[x > 3np] + 3np\Pr[d < k] + \frac{3np}{k}. \tag{6}$$

We need to bound the two probabilities $\Pr[x > 3np]$ and $\Pr[d < k]$. Since x has binomial distribution $B(n,p)$, we can use large deviation inequalities to bound $\Pr[x > 3np]$. We use the inequality of Chernoff (cf. [2] Appendix A) which states that for every $\beta \ge 1$,

$$\Pr[x > \beta pn] < \left(e^{\beta-1}\beta^{-\beta}\right)^{pn}.$$

Putting $\beta = 3$ and using the inequality $\ln 27 - 2(1+\epsilon) > 1$ yields

$$\Pr[x > 3np] < \left(\frac{e^2}{27}\right)^{pn} < \left(\frac{e^2}{27}\right)^{(1+\epsilon)\ln\delta} < \frac{1}{\delta}. \tag{7}$$

By (4) the probability that a vertex v of G has fewer than k neighbors in X is at most $o(\delta^{-1})$. The event that v is chosen for X is independent of the number of neighbors it has in X (since there are no loops in the graph). Thus, for δ sufficiently large,

$$\Pr[v \in X \text{ and } d_{G[x]}(v) < k] < p\frac{1}{\delta}.$$

Let z denote the number of vertices in $G[X]$ with degree less than k. By the last inequality $E[z] < np/\delta$. By Markov's inequality,

$$\Pr[z > 0.5\epsilon np] = \Pr[z > 0.5\delta\epsilon(np/\delta)] < \frac{2}{\delta\epsilon}.$$

Let $q = 1 - 2/(\delta\epsilon) - \Pr[x < 0.6np]$. Hence, $z \le 0.5\epsilon np$ and $x \ge 0.6np$ with probability at least q. In this situation, $z \le \lfloor \epsilon x \rfloor - 1$. Thus, with probability at least q, the element at position $\lfloor \epsilon x \rfloor$ in the degree sequence of $G[X]$ has value at least k. It follows that $\Pr[d \ge k+1] \ge q$ and therefore

$$\Pr[d \le k] \le 1 - q = \frac{2}{\delta\epsilon} + \Pr[x < 0.6np]. \tag{8}$$

As before, we can bound $\Pr[x < 0.6np]$ using large deviation inequalities. We shall use the inequality of Chernoff (cf. [2]) appendix A) which states that for every $a > 0$,

$$\Pr[x - np < -a] < \exp\left(-\frac{a^2}{2pn}\right).$$

In particular, for $a = 0.4np$ we get

$$\Pr[x < 0.6np] < \exp(-0.08pn) < \exp(-0.08\ln\delta) = \frac{1}{\delta^{0.08}}.$$

Plugging the last inequality into (8) we get

$$\Pr[d \le k] < \frac{2}{\delta\epsilon} + \frac{1}{\delta^{0.08}}. \tag{9}$$

By (6), (7), and (9),

$$E\left[\frac{x}{d}\right] \le n\frac{1}{\delta} + 3np\left(\frac{2}{\delta\epsilon} + \frac{1}{\delta^{0.08}}\right) + \frac{3np}{k}. \tag{10}$$

Notice that (10) yields $E\left[\frac{x}{d}\right] = o(n\ln\delta/\delta)$. By (2), (3), (5), it holds for sufficiently large δ that

$$\gamma_2(G) = E[\gamma_2(G)] < (1 + 60\epsilon)E[x] + 21E[y] + 80E\left[\frac{x}{d}\right] \le$$

$$(1+60\epsilon)\,(1+\epsilon)\,n\frac{\ln\delta}{\delta}+21o\left(\frac{n}{\delta}\right)+80o\left(n\frac{\ln\delta}{\delta}\right)<(1+100\epsilon)n\frac{\ln\delta}{\delta}. \qquad \square$$

Remark: The proof of theorem 2 in fact gives a 2-connected k-dominating set of size at most $n\frac{\ln\delta}{\delta}(1+o_\delta(1))$ whenever $k \leq \lfloor\sqrt{\ln\delta}\rfloor$.

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