Dense graphs are antimagic

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Abstract

An *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labeling. A conjecture of Ringel (see [4]) states that every connected graph, but K_2, is antimagic. Our main result validates this conjecture for graphs having minimum degree $\Omega(\log n)$. The proof combines probabilistic arguments with simple tools from analytic number theory and combinatorial techniques. We also prove that complete partite graphs (but K_2) and graphs with maximum degree at least $n - 2$ are antimagic.

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1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the notation and terminology of [2]. An *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labeling. The following is conjectured in [4]:

Conjecture 1.1 *Every connected graph, but K_2, is antimagic.*

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In this paper we prove that conjecture 1.1 holds for several classes of graphs. Our main result validates Conjecture 1.1 for all graphs with minimum degree $\Omega(\log n)$.

Theorem 1.2 *There exists an absolute constant C such that every graph with n vertices and minimum degree at least $C \log n$ is antimagic.*

In fact, our proof can be optimized to obtain that even a lower bound of $\Omega(\log n / \log \log n)$ for the minimum degree suffices. Since this improvement in negligible and complicates the computations significantly, we omit its proof. The proof of Theorem 1.2 requires several probabilistic tools and some simple techniques from analytic number theory, together with some combinatorial ideas, and is presented in the next two sections.

It is rather straightforward to prove that n-vertex graphs with a vertex of degree $n-1$ are antimagic. It is somewhat surprising that this ceases to be straightforward when the minimum degree is $n-2$. The following theorem is proved in Section 4.

Theorem 1.3 *If G has $n \geq 4$ vertices and $\Delta(G) \geq n-2$ then G is antimagic.*

It is still an open problem to decide whether connected graphs with $\Delta(G) \geq n-k$ and $n > n_0(k)$ are antimagic, for any fixed $k \geq 3$.

In the final section we prove the following.

Theorem 1.4 *All complete partite graphs, but K_2, are antimagic.*

2   A few probabilistic lemmas

In the next section we describe some lemmas that will be useful in the proof of Theorem 1.2.

Lemma 2.1 *There are absolute positive constants $c_1, c_2 > 0$ such that the following holds. Let t be a positive integer, let $A \subset \{1, 2, \ldots, t\}$ be a set of size $|A| \geq t - 2d$, where d is a positive integer, $d \leq t/30$, and define $p = \lfloor td^{1/2} \rfloor$. Let $w = e^{2\pi i/p}$ be a primitive root of 1 of order p, and let a_1, a_2 be two distinct randomly chosen elements of A (where all pairs are equally likely).*
(i) If x is an integer, and $0 < x < d^{1/2}$ or $p - d^{1/2} < x < p$, then with probability at least c_1,

$$|\frac{w^{a_1 x} + w^{a_2 x}}{2}| \leq 1 - c_2 \frac{min(x, p-x)^2}{d}. \tag{1}$$

(ii) If x is an integer and $d^{1/2} \leq x \leq p - d^{1/2}$ then with probability at least c_1,

$$|\frac{w^{a_1 x} + w^{a_2 x}}{2}| \leq 1 - c_2. \tag{2}$$

Proof Note that

$$|\frac{w^{a_1 x} + w^{a_2 x}}{2}| = |\frac{1 + w^{(a_2 - a_1)x}}{2}| = |cos(\frac{(a_2 - a_1)\pi x}{p})|.$$

If $0 < x < d^{1/2}$ then, since A contains most of the numbers in the interval $\{1, 2, \ldots, t\}$, it follows that with probability $\Omega(1)$, $(a_2 - a_1)$ is between, say, $t/4$ and $3t/4$, supplying the assertion of (1), since $|cos(y)| = 1 - \Theta(y^2)$ for every $0 < y < 3\pi/4$. As $|cos(y)| = |cos(\pi - y)|$ the result for $p - d^{1/2} < x < p$ follows as well.

If $d^{1/2} \le x \le p - d^{1/2}$ then $((a_2 - a_1)x)(mod\ p)$ is between, say, $p/4$ and $3p/4$ with probability $\Omega(1)$, implying the assertion of (2). ∎

Lemma 2.2 *There is an absolute positive constant C_1 such that the following holds. Let t, d be integers, $t/30 \ge d \ge \lfloor C_1 \log t \rfloor$, and let p and w be as in Lemma 2.1. Let a_{i1}, a_{i2}, $1 \le i \le d$, be $2d$ pairwise distinct elements of $\{1, 2, \ldots, t\}$, chosen randomly with all choices of d pairwise disjoint pairs being equally likely. Define*

$$T(x) = \prod_{i=1}^{d} \frac{w^{a_{i1}x} + w^{a_{i2}x}}{2}.$$

Then with probability at least $1 - 1/t^2$ the following holds:
(i) For every integer x satisfying $0 < x < d^{1/2}$ or $p - d^{1/2} < x < p$, $|T(x)| \le e^{-min(x, p-x)^2}$.
(ii) For every integer x satisfying $d^{1/2} \le x \le p - d^{1/2}$, $|T(x)| \le \frac{1}{t^2}$.

Proof Fix an integer x between 1 and $p - 1$. The random choice of the elements a_{i1}, a_{i2} can be obviously done as follows. Start with $A_1 = \{1, 2, \ldots, t\}$, choose a random pair a_{11}, a_{12} of distinct elements of A_1, and define $A_2 = A_1 - \{a_{11}, a_{12}\}$. Next choose a random pair a_{21}, a_{22} of distinct elements of A_2 and omit them from A_2 to get A_3, etc. Since in each step the remaining size of the set A_i is bigger than $t - 2d$, we can apply Lemma 2.1 and conclude that in each step, with probability at least c_1, the absolute value of the obtained term $\frac{w^{a_{i1}x} + w^{a_{i2}x}}{2}$ is at most $1 - c_2 \frac{min(x, p-x)^2}{d}$ in case $0 < x < d^{1/2}$ or $p - d^{1/2} < x < p$, and at most $1 - c_2$ in case $d^{1/2} \le x \le p - d^{1/2}$. Let us call the i-th chosen pair a_{i1}, a_{i2} *successful* if this inequality holds.

By the standard estimates for Binomial distributions (see, e.g., [1], Appendix A) and by our assumption that $d = \Omega(\log t)$ it follows that if C_1 is chosen appropriately, then with probability at least $1 - 1/t^2$, for every fixed admissible x there are sufficiently many successful pairs to insure the assertion of the lemma. ∎

Lemma 2.3 *There are absolute positive constants C_1, C_2 such that the following holds. Let t and $d \ge \lfloor C_1 \log t \rfloor$ be as in Lemma 2.2, and let $a_{i,1}, a_{i,2}$, $1 \le i \le d$ be $2d$ pairwise distinct elements of $\{1, 2, \ldots, t\}$, chosen randomly with all choices of d pairwise disjoint pairs being equally likely.*

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Then, with probability at least $1 - 1/t^2$ the following holds. For each i, $1 \leq i \leq d$, choose $j_i \in \{1, 2\}$ randomly, independently and uniformly, and consider the random sum $Q = \sum_{i=1}^{d} a_{i,j_i}$. Then, for every integer S, the probability that Q is equal S is at most $\frac{C_2}{td^{1/2}}$.

Proof Put $p = \lfloor td^{1/2} \rfloor$. It suffices to prove that the probability that Q is equal to S modulo p is at most $O(1/p)$. It is thus enough to show that this is the case if the conclusions of Lemma 2.2 hold. Note that the probability that Q is equal to S modulo p is precisely

$$\frac{1}{p} \sum_{x=0}^{p-1} \prod_{i=1}^{d} \frac{w^{a_{i1}x} + w^{a_{i2}x}}{2} w^{-Sx} = \frac{1}{p} \sum_{x=0}^{p-1} T(x) w^{-Sx}.$$

However, if the conclusions of Lemma 2.2 hold, then the term corresponding to $x = 0$ in the above sum is $1/p$, and the absolute value of the term corresponding to x is at most $e^{-min(x,p-x)^2}/p$ for each x satisfying $0 < x < d^{1/2}$ or $p - d^{1/2} < x < p$, and is at most $\frac{1}{pt^2}$ for each other x in the sum. As $\sum_{x>0} e^{-x^2} = O(1)$ and as $1/t^2 < 1/p$, the desired result follows. ∎

In the proof of Theorem 1.2 we need to use the symmetric Lovász Local Lemma [3]. Here it is, following the notations in [1]. Let $A_1, \ldots, A_s$ be events in an arbitrary probability space.

Lemma 2.4 (The Local Lemma, symmetric version) *If there are positive constants p and r such that $p(r + 1) < 1/3$, each A_i is mutually independent of all other events but at most r, and $\Pr[A_i] \leq p$ for all i, then with positive probability no event A_i holds.* ∎

3 Proof of Theorem 1.2

Let C be a sufficiently large absolute constant such that for all sufficiently large positive integer n, and for all $t \in [dn/2, dn]$ where $d = \lfloor C \log n \rfloor$ it holds that $t/30 - 1 \geq d \geq \lfloor C_1 \log t \rfloor + 1$ where C_1 is the constant from Lemma 2.3. In order to prove Theorem 1.2 it suffices to prove that for all sufficiently large n, if G is a graph with n vertices, m edges, and $\delta(G) \geq d$, then G is antimagic.

It is convenient to split the description of the proof into five phases.

Phase 1: As long as there are two adjacent vertices each having degree at least $d + 1$, we assign the edge connecting them the highest yet unused label and delete the edge. Let G' denote the spanning subgraph of G obtained at the end of this process. Denote the set of vertices with degree d in G' by A and the set of vertices with degree at least $d + 1$ in G' by B. Notice that it is possible that $B = \emptyset$ and also note that B induces an independent set in G'. Also, each vertex v has a partial sum denoted $r(v)$, which is the sum of the labels assigned to the edges incident with v that were deleted in this phase. Let $t \leq m$ denote the number of edges of G' and notice that $t \in [dn/2, dn]$. Hence, our goal is to assign the set $\{1, \ldots, t\}$ of labels to the edges of G' such that all vertex sums (including the partial sums contributed from the labels assigned already) are distinct.

Phase 2: We partition the edge-set of G' into $t/2$ pairs (clearly, we may assume, without loss of generality, that t is even) as follows. Let $d'(v)$ denote the degree of v in G'. For each $v \in B$ we arbitrarily choose a set $F(v)$ of edges incident with v, such that $|F(v)|$ is even and

$$\left| (d'(v) - |F(v)|) - d \right| \leq 1.$$

Notice that for any two distinct vertices v and u of B we have $F(v) \cap F(u) = \emptyset$. We partition each $F(v)$ into pairs, arbitrarily. Put $k = (t - |\cup_{v \in B} F(v)|)/2$. We partition the remaining edges of G' not in $\cup_{v \in B} F(v)$ into k pairs, with the property that the two edges in each pair do not share a common endpoint. This can be done by observing that the complement of the line graph of the remaining edges has a very high minimum degree and therefore has a perfect matching. Finally, for each edge $e \in G'$, let $p(e)$ denote the edge paired with e.

Phase 3: We randomly partition the set of t labels into $t/2$ pairs of labels. The randomly selected label pairs are then arbitrarily assigned to the edge pairs created in Phase 2. For $e \in G'$ let $L(e)$ denote the pair of labels assigned to the pair $\{e, p(e)\}$. Clearly $L(p(e)) = L(e)$.

Phase 4: For each $v \in B$ let $f(v)$ denote the sum of the labels assigned to the edges of $F(v)$. Notice that although we have yet to specify which edge gets which label (there are two choices), $f(v)$ is well defined. Now, for each $v \in B$, let $H(v)$ denote the set of edges incident with v in G', and not belonging to $F(v)$. For each $v \in A$, let $H(v)$ denote the set of edges incident with v in G'. Notice that $d + 1 \geq |H(v)| \geq d - 1$ for all $v \in V$ (in fact, $|H(v)| = d$ for $v \in A$). For each $v \in V$, consider the set of pairs of labels $\{L(e) \ : \ e \in H(v)\}$. There are $2^{|H(v)|}$ choices to select one label from each pair $L(e)$, and each choice yields a possible sum. Denote the set of possible sums by $Q(v)$, and notice that $|Q(v)| \leq 2^{|H(v)|}$. By Lemma 2.3, with probability at least $1 - 1/t^2$, no specific value in $Q(v)$ is obtained by more than a fraction of $C_2/td^{1/2}$ of the $2^{|H(v)|}$ choices for a suitable absolute constant C_2. We therefore *fix* an assignment of label pairs to edge pairs having the property that for *all* $v \in V$, no value from $Q(v)$ is obtained by more than a fraction of $C_2/td^{1/2}$ of the $2^{|H(v)|}$ possible selections.

Phase 5: For each pair $\{e, p(e)\}$ we flip a coin to decide which edge gets which label from $L(e)$. All $t/2$ decisions are independent. Notice that the final weight of each $v \in B$ is a random variable given by adding to $r(v) + f(v)$ a number of $Q(v)$ which corresponds to the random coin flip. Similarly, the final weight of each $v \in A$ is a random variable given by adding to $r(v)$ a number of $Q(v)$. We claim that with positive probability, no two vertices of v will end up with the same final weight. For a pair of vertices u, v, let $B(u, v)$ denote the event that both u and v end up with the same final weight. We need to show that with positive probability no $B(u, v)$ holds. By our arguments from Phase 4,

$$\Pr[B(u, v)] \leq \frac{C_2}{td^{1/2}} \leq \frac{2C_2}{nd^{3/2}}.$$

We show that $B(u, v)$ is independent of all other events but at most $O(nd)$. Indeed, let Z denote the set of vertices that are endpoints of edges in $H(v) \cup H(u)$ and their matched edges. Clearly

$|Z| \leq 6(d+1) + 2$. Now any combination of the events $B(x,y)$ where neither x nor y are in Z is independent of $B(u,v)$. Thus, $B(u,v)$ is independent of all but at most $(6(d+1)+2)n$ other events. Since

$$\frac{2C_2}{nd^{3/2}} \cdot ((6(d+1)+2)n+1) << \frac{1}{3}$$

for n sufficiently large, we get, using Lemma 2.4, that with positive probability no $B(u,v)$ holds. This completes the proof of Theorem 1.2. $\blacksquare$

4 Proof of Theorem 1.3

The proof of Theorem 1.3 is divided into four lemmas. The first lemma handles the case of maximum degree $n-1$.

Lemma 4.1 *If G has n vertices and $\Delta(G) = n-1$ then G is antimagic.*

Proof Assume G has m edges, and let v be a vertex of degree $n-1$. Assign the distinct labels $1, \ldots, m-n+1$ arbitrarily to all the $m-n+1$ edges not incident with v. Denote the $n-1$ neighbors of v by $v_1, \ldots, v_{n-1}$ where $w'(v_i) \leq w'(v_{i+1})$ for $i = 1, \ldots, n-2$, and where w' is the sum of the labels given to edges incident with v_i. Now assign the label $m-n+1+i$ to the edge (v, v_i) for $i = 1, \ldots, n-1$. Notice that in the final labeling we have $w(v_i) = w'(v_i) + m - n + 1 + i$ and hence all the weights of $v_1, \ldots, v_{n-1}$ are distinct. Also $w(v) = (n-1)(m-n+1) + n(n-1)/2$ and hence $w(v)$ is larger than any $w(v_i)$. Thus, G is antimagic. $\blacksquare$

An *S-partial labeling* of $G = (V, E)$ is an assignment of distinct labels to some of the edges of G from a set of positive integers S (we allow $|S| > |E|$ in the definition of partial labeling). Given a partial labeling, the weight of a vertex v, denoted $w(v)$, is the sum of the labels of the edges incident with v, that received a label. A *completion* of an S-partial labeling is an assignment of distinct unused labels from S to all the remaining non-labeled edges of G. We need the following simple lemma:

Lemma 4.2 *Let G be any graph with r vertices and m edges. Let S be a set of positive integers with $|S| = m+2$. Then, any S-partial labeling that satisfies the property that no more than $\lceil r/2 \rceil$ vertices have the same positive weight, has a completion that also satisfies this property.*

Proof Assume that $t \leq m$ edges are labeled. We need to label the remaining $m-t$ edges. We use induction on $m-t$. If $m-t = 0$ there is nothing to prove. Assuming the lemma holds when $m-t = k-1$, we prove it for k. Assume, therefore, that $m-t = k > 0$. Pick an arbitrary non-labeled edge, $e = (x, y)$. Since we have $m+2$ possible labels and at most $m-1$ have been used, there are at least three possible labels that we may assign to e. Denote the three non-used labels by

a_1, a_2, a_3. We will show that at least one of the three possible assignments maintains the property in the statement of the lemma. If we assign a_i to e then, after the assignment, we have that the total weight of x is $w(x) + a_i$ and the total weight of y is $w(y) + a_i$. The other vertices did not change their total weight. Assume, for the sake of contradiction, that each of the three assignments fails. This means that for each $i = 1, 2, 3$, one of $w(x) + a_i$ or $w(y) + a_i$ appears at least $\lceil r/2 \rceil + 1$ times as a total weight. Hence, in the original partial labeling, we have that for each $i = 1, 2, 3$, one of $w(x) + a_i$ or $w(y) + a_i$ appears at least $\lceil r/2 \rceil$ times as a total weight. Assume, without loss of generality, that $w(x) + a_1$ and $w(x) + a_2$ appear at least $\lceil r/2 \rceil$ times as a total weight in the original labeling. The original labeling also has the weight $w(x)$ appearing in x. Thus, the overall number of vertices of G is at least $2\lceil r/2 \rceil + 1 > r$, a contradiction. We have proved that we can label e and maintain the desired property. Now, we remain with only $k - 1$ unlabeled edges, and we can complete the labeling using the induction hypotheses. ∎

Recall that an *even graph* is a graph whose vertices all have even degree. It is easy and well-known that any even graph can be decomposed into edge-disjoint simple cycles. It is also easy to see that any graph has a subforest such that the deletion of the edges of this subforest from the graph results in an even graph. We use these facts in the following two lemmas.

Lemma 4.3 *Let $G = (V, E)$ have n vertices and $\Delta(G) = n - 2$. If $|E| = m \geq 2n - 4$ then G is antimagic.*

Proof The statement implies $n \geq 4$. We can assume $n \geq 5$ since for $n = 4$ the only possible graph satisfying the assumption is C_4, which is trivially antimagic. Let v_n denote a vertex of maximum degree $n-2$. Let v_{n-1} be its unique non-neighbor and let $v_1, \ldots, v_{n-2}$ be the other vertices. Consider the induced subgraph G^* on all vertices except v_n. G^* has $n - 1$ vertices. Let F be a set of edges that induce a subforest in G^* such that the deletion of F from G^* results in an even subgraph G' of G^*. Clearly, $|F| \leq (n - 1) - 1 = n - 2$. Assign the edges of F the smallest even weights $2, 4, \ldots, 2|F|$ arbitrarily. Notice that this can be done since $2|F| \leq 2n - 4 \leq m$. Now, partition the edges of G' into edge-disjoint simple cycles $C_1, \ldots, C_p$. We label the edges of C_i sequentially from $i = 1$ until $i = p$. We first use all the remaining even weights, and when we exhaust them, we turn to using odd weights, starting from the smallest odd weights and continuing sequentially. Now, either all cycles have all their edges labeled with the same parity, or else at most one C_i contains both odd and even weights, such that the odd weights form one path in C_i and the even weights form the remaining path in C_i. In the latter case, exactly two vertices of G^* end up with a total odd weight, and we can assume neither of them is v_{n-1} since C_i has at least three vertices, while in the former case, all vertices of G^* end up with total even weight. We are now left with the largest $n - 2$ odd labels for the use in the edges incident with v_n. Denote the remaining labels by $t, t+2, \ldots, t+2n-6$ where $t + 2n - 6$ is either m or $m - 1$ (depending on the parity of m). Assume, without loss of generality, that after labeling G^*, we have $w_{G^*}(v_1) \leq w_{G^*}(v_2) \leq \cdots \leq w_{G^*}(v_{n-2})$.

If all the $w_{G^*}(v_i)$ are even then after labeling (v_i, v_n) with the label $t + 2i - 2$ for $i = 1, \ldots, n - 2$ we have that all the total weights of $v_1, \ldots, v_{n-2}$ are distinct and odd, while the total weight of v_{n-1} stays even. The total weight of v_n is clearly the largest of all total weights. Hence, the labeling is antimagic. We may therefore assume that for precisely two indices j and k, with $1 \le j < k \le n - 2$ have $w_{G^*}(v_j)$ and $w_{G^*}(v_k)$ odd. Since $n - 2 \ge 3$, there are at least three odd labels $t, t + 2, t + 4$. We may assign two of them to (v_n, v_j) and (v_n, v_k) so as to guarantee that the total weights of v_j and v_k after the assignment is even, but distinct from that of v_{n-1} which is also even, and distinct from each other. The third odd label not used from $t, t + 2, t + 4$, together with the other $n - 5$ odd labels, are assigned, sequentially, to the other vertices from $\{v_1, \ldots, v_{n-2}\} \setminus \{v_j, v_k\}$. Hence, the total weights of these vertices is odd and distinct, and the total weight of v_n is the largest. Hence, G is antimagic. ∎

Lemma 4.4 *Let $G = (V, E)$ have $n \ge 4$ vertices and $\Delta(G) = n - 2$. If $|E| = m \le 2n - 5$ then G is antimagic.*

Proof Again, we will assume $n \ge 5$ as the case $n = 4$ is trivial to check. As in the proof of Lemma 4.3, let v_n denote a vertex of degree $n - 2$ and let v_{n-1} denote the unique non-neighbor of v_n. If v_{n-1} is an isolated vertex then we can use Lemma 4.1 for the subgraph induced by all vertices except v_{n-1} and obtain that G is antimagic (v_{n-1} will be the unique vertex with total weight 0 in this case). Thus, we assume v_{n-1} is not isolated. As in Lemma 4.3, let G^* be the subgraph induced by all vertices except v_n. Let s denote the number of edges of G^*. Hence, $s = m - (n - 2) \le (2n - 5) - (n - 2) = n - 3$. Notice that $s < m/2$. Thus, we can assign all edges of G^* only even weights. We consider three cases.

If $m = 2n - 5$ then $s = n - 3$ and we use all the even weights for labeling G^*. Then, we use the $n - 2$ odd weights to label the edges incident with v_n. As in Lemma 4.3, we assign the labels so as to guarantee that all vertices $v_1, \ldots, v_{n-2}$ have distinct total odd weight. v_{n-1} has total even weight, and v_n has maximum weight. Hence, G is antimagic.

If $m = 2n - 6$ or $m = 2n - 7$ we have only $n - 3$ odd weights and $m - n + 3 = s + 1$ even weights. Assign arbitrarily the first $s - 1$ even weights to all but one edge of G^*, denoted $e = (x, y)$. We may assume $v_1 \notin \{x, y\}$ since $n \ge 5$ so $n - 2 \ge 3$. Let r_1 and r_2 denote the largest two even weights. We may choose one of them for the label of e. Let a_1 denote the total weight of v_1 at this point and let a_2 denote the total weight of v_{n-1} at this point. If $v_{n-1} \in \{x, y\}$ we select r_1 for the label of e if and only if $r_2 + a_1 \ne a_2 + r_1$. Otherwise we select r_2 for the label of e and notice that in this case we must have $r_1 + a_1 \ne a_2 + r_2$. If $v_{n-1} \notin \{x, y\}$ we select r_1 for the label of e if and only if $r_2 + a_1 \ne a_2$. Otherwise we select r_2 for the label of e and notice that in this case we must have $r_1 + a_1 \ne a_2$. In any case we have shown that we can select a label for e such that if we select the other label for (v_n, v_1) we have that the total weight of v_1 is even and distinct from

the total weight of v_{n-1} which is also even. The remaining $n-3$ edges incident with v_n receive the odd labels so as to guarantee that the final weights of $v_2, \ldots, v_{n-2}$ are odd and distinct. Finally, v_n has the maximum total weight, so G is antimagic.

If $m \leq 2n - 8$ we have at least $s + 2$ even weights for the labeling. Let S denote the set of $s + 2$ largest even labels. Label one of the edges of G^* incident with v_{n-1} (recall that v_{n-1} is non-isolated) with the largest even label (this is either m or $m - 1$, depending on the parity of m). By Lemma 4.2 we can complete this labeling to a full labeling of G^* which uses only elements of S, such that no set of $\lceil (n-1)/2 \rceil + 1$ vertices in G^* have the same positive total weight. Now, consider the remaining $n - 2$ weights. We know that x of them are even and $n - 2 - x$ of them are odd, and, trivially, $s + x \leq n - 2 - x \leq s + x + 1$. Thus, $x \leq (n-3)/2$ in any case. Assume, without loss of generality, that $w_{G^*}(v_1) \leq w_{G^*}(v_2) \leq \cdots \leq w_{G^*}(v_{n-1})$. Denote the remaining x even labels by $\{r_1, \ldots, r_x\}$. The odd labels are $1, 3, 5, \ldots, 2n - 5 - 2x$. We assign the edge (v_i, v_n) the even label r_i, for $i = 1, \ldots, x$. We assign the edge (v_i, v_n) the odd label $2i - 2x - 1$ for $i = x + 1, \ldots, n - 2$. There are two cases. If for all $i = 1, \ldots, x$ we have $w_{G^*}(v_i) + r_i \neq w_{G^*}(v_{n-1})$ then, the final weights of $v_1, \ldots, v_x, v_{n-1}$ are even and distinct. The final weights of $v_{x+1}, \ldots, v_{n-2}$ are odd and distinct. The final weight of v_n is the largest of all, so G is antimagic. Hence, we may assume that for some i we have $w_{G^*}(v_i) + r_i = w_{G^*}(v_{n-1})$. Assume i is minimal with this property. We claim that $w_{G^*}(v_i) > 0$. Indeed, otherwise we would have $w_{G^*}(v_{n-1}) = r_i$ but this is impossible since we have labeled one of the edges incident with v_{n-1} with the largest even label, which is larger than r_i. Let $Z = \{j \ : \ j \geq i \ , \ w_{G^*}(v_i) = w_{G^*}(v_j)\}$. Recall that no set of $\lceil (n-1)/2 \rceil + 1$ vertices in G^* have the same positive total weight. Thus, $|Z| \leq \lceil (n-1)/2 \rceil$. Let k denote the largest element in Z. Notice that $k + x - i + 1 = |Z| + x \leq n - 2$. We modify the labeling of the edges (v_j, v_n) for $j = i, \ldots, k + (x - i + 1)$. We label (v_{k+j}, v_n) with the even label r_{i+j-1} for $j = 1, \ldots, x - i + 1$. We label (v_j, v_n) with the odd label $2(j - i) + 1$ for $j = i, \ldots, k$. Notice that now all the vertices $v_1, \ldots, v_{i-1}, v_{k+1}, \ldots, v_{k+x-i+1}$ receive distinct total even weights which are also distinct from the total weight of v_{n-1} which is the even number $w_{G^*}(v_{n-1})$. All the vertices $v_i, \ldots, v_k$ and $v_{k+x-i+2}, \ldots, v_{n-2}$ receive distinct total odd weights. Finally, v_n receives the maximal weight. Hence, G is antimagic. ∎

Proof of Theorem 1.3: The proof follows immediately from Lemma 4.1, Lemma 4.3 and Lemma 4.4. ∎

5   Proof of Theorem 1.4

It is convenient to separate the proof of Theorem 1.4 into two cases. The bipartite case and the k-partite case where $k \geq 3$, as the proofs of these cases are different. It is also convenient to

prove the bipartite case using the equivalent matrix formulation. The following two lemmas yield Theorem 1.4.

Lemma 5.1 *For all $m + n > 1$, the cells of an $m \times n$ matrix can be assigned the distinct integers $1, \ldots, mn$ such that all $m + n$ rows and columns receive distinct sums.*

Proof Assume, without loss of generality, that $m \leq n$. The case $m = 1$ is trivial so we assume $2 \leq m \leq n$. Furthermore, we may assume $n \geq 4$ since $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$ are easily verified as antimagic. We show how to assign the distinct numbers $1, \ldots, mn$ to the cells of a matrix $A_{m \times n}$. We assign the numbers $(i-1)n + 1, \ldots, in$ to the cells in row i, $i = 1, \ldots, m$. The assignment within each row is always monotone increasing if i is odd or if i is the last row. Otherwise, it is monotone decreasing. Let $R(i)$ denote the sum of the elements in row i and let $C(j)$ denote the sum of the elements in column j. Clearly, $R(i) - R(i-1) = n^2$ for $i = 2, \ldots, m$. Hence, all row sums are distinct and form an arithmetic progression with difference n^2. Also, for all even $i < m$, the sum of the first i rows is a constant vector. Thus, if m is odd we have $C(j) - C(j-1) = 1$ for $j = 2, \ldots, n$ and if m is even then $C(j) - C(j-1) = 2$. In any case, $C(n) - C(1) \leq 2(n-1)$. Since $2(n-1) < n^2$ we have that at most one column sum is equal to a row sum. Hence, we may assume that for one specific pair, $C(j) = R(i)$. Clearly $i < m$ since the last row contains the n largest elements and $n \geq m$. Assume first that $i > 1$. If i is even then $A(i, 1) - A(i-1, 1) = 2n - 1$. If i is odd then $A(i, n) - A(i-1, n) = 2n - 1$. In any case we can replace the values of two adjacent cells in rows i and $i - 1$ whose difference is precisely $2n - 1$. Notice that the sums of the columns do not change, and the sums of the rows, except i and $i - 1$, do not change. The new sum of row i is $R(i) - 2n + 1$ and the new sum of row $i - 1$ is $R(i-1) + 2n - 1$. However, $(R(i) - 2n + 1) - (R(i-1) + 2n - 1) = n^2 - 4n + 2 > 0$ for $n \geq 4$. Hence, all row sums are still distinct, but now $R(i) - 2n + 1$ is smaller than $C(1)$, so all row and column sums are distinct. Assume next that $i = 1$. If $m \geq 3$ we have $A(2, 1) - A(1, 1) = 2n - 1$ and when replacing them we have, as in the previous case, that all row sums remain distinct while the new sum of the first row is $R(1) + 2n - 1$, which is greater than $C(n)$. Finally, if $i = 1$ and $m = 2$ we can simply assign all odd numbers sequentially to the first row and all even numbers sequentially to the second row. The largest column sum is $4n - 1$ and the smallest row sum is n^2 and $n^2 > 4n - 1$. ∎

Lemma 5.2 *If G is a k-partite graph with $k \geq 3$ then G is antimagic.*

Proof Let the sizes of the vertex classes be $n_1, \ldots, n_k$ where $n_i \leq n_{i+1}$ for $i = 1, \ldots, k-1$. We may assume $n_1 \geq 2$ since otherwise Lemma 4.1 applies. Let A denote a vertex class of size n_1 and let $B = V(G) \backslash A$ denote the remaining vertices. Put $|B| = m$ and notice that $m \geq n_2 + n_3 \geq 2n_2 \geq 2n_1 \geq 4$. Let q denote the number of edges with both endpoints in B. Our initial labeling assigns arbitrary distinct labels from $1, \ldots, q$ to the edges with both endpoints in B. We denote $B = \{u_1, \ldots, u_m\}$

where $w'(u_i) \leq w'(u_{i+1})$ for $i = 1, \ldots, m-1$ and $w'(u_i)$ is the weight of u_i after the initial labeling. We now show how to complete the labeling by assigning the labels $q+1, \ldots, q+mn_1$ to the edges incident with $A = \{v_1, \ldots, v_{n_1}\}$. Let $c(v_i, u_j)$ denote the label received by (v_i, u_j). We define:

$$c(v_i, u_j) = \begin{cases} (i-1)m + j + q & j \text{ odd} \\ (n_1 - i)m + j + q & j \text{ even} \end{cases}$$

except for the case when m is even, when we define $c(v_i, u_m) = im + q$. The additional contribution to the weights of the vertices of B is:

$$\frac{n_1}{2} \left(2q + 2j + m(n_1 - 1) \right).$$

Since the last expression is an increasing function of j we obtain that the final labeling satisfies

$$w(u_1) < w(u_2) < \cdots < w(u_m),$$

as required. Observe that,

$$w(u_m) \leq \frac{n_1}{2} \left(2q + m(n_1 + 1) \right) + q(m - n_2) - \frac{(m - n_2)(m - n_2 - 1)}{2}.$$

On the other hand it is easily verified that the values of $w(v_i)$ are:

$$w(v_i) = \frac{m}{2} \left(2i + 2q + n_1(m-1) \right), \ for \ m \ odd$$

and

$$w(v_i) = \frac{m}{2} \left(4i + 2q + n_1(m-2) - 1 \right), \ for \ m \ even.$$

In both cases we have $w(v_i) < w(v_{i+1})$. Thus, it suffices to verify that $w(v_1) > w(u_m)$. Following some simple calculations we obtain that we have to verify the inequality:

$$\frac{m}{2}(m + mn_1 + 2 - n_1 - 2n_2) + \frac{n_2}{2}(n_2 + 1) + q(n_2 - n_1) > \frac{mn_1}{2}(n_1 + 2).$$

Indeed, this inequality is correct since already

$$\frac{m}{2}(m + mn_1 + 2 - n_1 - 2n_2) > \frac{mn_1}{2}(n_1 + 2)$$

as $m \geq 2n_2 \geq 2n_1 \geq 4$. ∎

Acknowledgment

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