# DEFINABLE STRUCTURES IN O-MINIMAL THEORIES: ONE DIMENSIONAL TYPES

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Abstract. Let $\mathcal{N}$ be a structure *definable* in an o-minimal structure $\mathcal{M}$ and $p \in S_{\mathcal{N}}(N)$, a complete $\mathcal{N}$-1-type. If $\dim_{\mathcal{M}}(p) = 1$ then $p$ supports a combinatorial pre-geometry. We prove a Zilber type trichotomy: Either $p$ is trivial, or it is linear, in which case $p$ is non-orthogonal to a generic type in an $\mathcal{N}$-definable (possibly ordered) group whose structure is linear, or, if $p$ is *rich* then $p$ is non-orthogonal to a generic type of an $\mathcal{N}$-definable real closed field.

As a result we obtain a similar trichotomy for definable one-dimensional structures in o-minimal theories.

In this paper we prove a trichotomy theorem for one-dimensional types in structures definable in o-minimal theories. With this we conclude the work started in [4], of which this is a direct continuation.

Recall that a structure $\mathcal{N}$ is said to be *definable* in an o-minimal structure $\mathcal{M}$ if the universe $N$, of $\mathcal{N}$, as well as all its atomic relations, are definable sets (possibly of several variables, possibly using parameters) in the structure $\mathcal{M}$.

In [4] we proved a weak version of Zilber's trichotomy:

**Theorem 1.** *Let $\mathcal{N}$ be a stable structure definable in an o-minimal structure $\mathcal{M}$. If $\dim_{\mathcal{M}}(N) = 1$ then $\mathcal{N}$ is 1-based.*

The local nature of phenomena in o-minimal theories does not leave room for more precise global statements in the unstable case. The aim of this paper is to remedy this situation by applying the results of [4] and [2] to obtain a complete classification of 1-$\mathcal{M}$-dimensional types in $\mathcal{N}$ without any additional global assumptions on $\mathcal{N}$. Our main result can be summed up by (see definitions below):

**Theorem 2.** *Let $\mathcal{M}$ be an o-minimal structure and $\mathcal{N}$ definable in $\mathcal{M}$. Let $p \in S_{\mathcal{N}}(N)$ be one-$\mathcal{M}$-dimensional. Then exactly one of the following holds:*

(1) *$p$ is trivial.*
(2) *$p$ is linear, in which case $p$ is non-orthogonal to a generic type of an $\mathcal{N}$-definable (possibly locally ordered) group $G$. The structure which $\mathcal{N}$ induces on $G$ is linear, i.e., given by definable (possibly local) subgroups of $G^n$.*
(3) *$p$ is* rich*, in which case it is non-orthogonal to a generic type of an $\mathcal{N}$-definable real closed field $R$.*

In fact, our results will be more precise and give a stable/unstable dichotomy (see Theorem 2.1). As a corollary to the above we can complete the analysis of definable one dimensional structures which began in [4]:

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**Theorem 3.** *Let $\mathcal{N}$ be a definable structure in a sufficiently saturated o-minimal structure $\mathcal{M}$. If $\dim_{\mathcal{M}}(N) = 1$ then exactly one of the following holds:*

1. *$\mathcal{N}$ is degenerate.*
2. *$\mathcal{N}$ is linear: There exists $X \subseteq N^{eq}$ with $\dim_{\mathcal{M}}(X) = 1$ and such that the structure which $\mathcal{N}$ induces on $X$ is either a strongly minimal locally modular group or elementarily equivalent to a group-interval in an ordered vector space. Moreover, no field is interpretable in $\mathcal{N}$.*
3. *$\mathcal{N}$ is rich and interprets a real closed field.*

To get the work going we need some preliminaries that we explain in the next section.

## 1. Some notation, terminology and background

We start by fixing some conventions of notation and terminology. We fix $\mathcal{N}$ a structure definable in an o-minimal densely ordered structure $\mathcal{M}$, and assume that $\mathcal{M}$ and $\mathcal{N}$ are sufficiently saturated. We let $M, N$ denote the universes of $\mathcal{M}, \mathcal{N}$, respectively and use $A, B \subseteq N$ etc. to denote small sets.

For $a \in N$, and $A \subseteq N$, we let $\mathrm{tp}_{\mathcal{N}}(a/A)$ be the type of $a$ over $A$, in sense of the structure $\mathcal{N}$. Such types are called $\mathcal{N}$-types. We let $S_{\mathcal{N}}(A)$ be the set of complete $\mathcal{N}$-types over $A$.

A complete non-algebraic $\mathcal{N}$-type $p$ is called *one-$\mathcal{M}$-dimensional* if it contains an $\mathcal{N}$-formula whose $\mathcal{M}$-dimension is one. For simplicity, we will just call those types *one dimensional.*

**Remark** It is worth noting that the notion of $\mathcal{M}$-dimension is not intrinsic to $\mathcal{N}$, but depends on the particular interpretation of $\mathcal{N}$ in $\mathcal{M}$. E.g., if $\mathcal{N}$ is a dense linear ordering then it can be interpreted in $\mathcal{M}$ either as a subset of $M$, or as a subset of $M^2$, with the lexicographic ordering. In the first case, every nonalgebraic 1-type in $\mathcal{N}$ is one dimensional while in the second case, there are clearly two dimensional 1-types and also one-dimensional ones: if one fixes two points in the same fibre, the interval between them gives rise to one-dimensional types.

If $p$ is a one-dimensional $\mathcal{N}$-type then the restriction of $\mathrm{acl}_{\mathcal{N}}(\cdot)$ to the set of realizations of $p$ forms a pre-geometry (Exchange holds because it is true in $\mathcal{M}$, but note that this property of $p$ is independent of the interpretation. This can be also seen by noting that $p$ has $\mathrm{U}^{\text{þ}}$-rank 1, and this property is intrinsic to $\mathcal{N}$). Similarly, $\mathrm{acl}_{\mathcal{N}}$ induces a pre-geometry on any union of one-dimensional types.

For one-dimensional types (or more generally, for types of $\mathrm{U}^{\text{þ}}$-rank 1) the definition of non-orthogonality of minimal types (in the stable context) readily generalises:

**Definition 1.1.** Let $p, q \in S_{\mathcal{N}}(A)$ be one-dimensional. Then $p$ and $q$ are *nonorthogonal over $A$* if there is $B \supseteq A$ and $a \models p$, $b \models q$ such that $\mathrm{tp}_{\mathcal{N}}(a/B)$ and $\mathrm{tp}_{\mathcal{N}}(b/B)$ are non-algebraic and $a, b$ are inter-algebraic in $\mathcal{N}$ over $B$.

Thus, in order to make clear the statement of Theorem 2 it remains to explain the definition of *rich* types. For that purpose we first remind the notion of $\mathcal{N}$-curves.

**Definition 1.2.** An $\mathcal{N}$-curve is an $\mathcal{N}$-definable set $C$ with $\dim_{\mathcal{M}}(C) = 1$.

For simplicity, and as no confusion can arise, we will refer to $\mathcal{N}$-curves simply as *curves.*

1.1. **Pre-geometries and nontrivial types.** Our main interest in this paper is in extracting algebraic information from the $\mathrm{acl}_{\mathcal{N}}(.)$ pre-geometry of a 1-dimensional type $p$, along the line of Zilber's Trichotomy.

We recall that a definable family of curves $\mathcal{F} = \{C_q : q \in Q\}$ is called *almost normal* if for every $q_1 \in Q$, there are only finitely many $q_2 \in Q$ such that $C_{q_1} \cap C_{q_2}$ is infinite.

**Definition 1.3.** Let $p \in S_{\mathcal{N}}(A)$ be a 1-dimensional type.

- $p$ is called *trivial* if for every $a_1, a_2, a_3 \models p$ and $B \supseteq A$, if $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_1, a_2B)$ then $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_1B)$ or $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_2B)$.
- $p$ is called *rich* if for any $b \models p$ there exists a finite set $A$, independent from $b$ over $N$ (i.e. $\mathrm{tp}_{\mathcal{N}}(b/AN)$ is non-algebraic), and an almost normal family $\mathcal{F}$ of curves, $\mathcal{N}$-definable over the set $AN$, such that $\{f \in \mathcal{F} : \langle b, b\rangle \in f\}$ is infinite.
- $p$ is *linear* if $p$ is not trivial and not rich.

We recall the following definitions from [4]:

**Definition 1.4.** A 1-dimensional structure $\mathcal{N}$ definable in $\mathcal{M}$ is called *degenerate* if for every $a_1, a_2, a_3 \in N$ and $B \subseteq N$, if $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_1, a_2B)$ then $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_1B)$ or $a_3 \in \mathrm{acl}_{\mathcal{N}}(a_2B)$. $\mathcal{N}$ is called *linear* if it is non-degenerate and every definable almost normal family of curves in $N^2$ has dimension at most 1. It is called *rich* if there exists a two-dimensional family of curves in $N^2$.

The goal is to associate to linear types a (type-)definable, possibly ordered, module and to rich types a definable infinite field. We first establish several facts:

It is easily checked that any extension of a rich type is rich (one verifies that a rich type has at least one global rich extension, and thus any global extension of a rich type is itself rich). However, showing that any extension of a non-trivial type is non-trivial requires a little more effort. Since we will have to work not only with the given type $p$ but also with several of its extensions at the same time, it will be convenient to clarify this technical point already at this stage.

**Definition 1.5.** For $A \subseteq N$ and one-dimensional types $p, q, r \in S_{\mathcal{N}}(A)$, we say that $p, q, r$ *form a nontrivial configuration over* $A$ if there are $a \models p, b \models q$ and $c \models r$ such that $c \in \mathrm{acl}_{\mathcal{N}}(abA)$ and each pair of $\{a, b, c\}$ is $\mathcal{N}$-independent over $A$ (we say that $a, b$ are $\mathcal{N}$-independent over $A$ if $\mathrm{tp}_{\mathcal{N}}(a/bA)$ and $\mathrm{tp}(b/aA)$ are non-algebraic). Such $\{a, b, c\}$ are said to realize a *strong nontrivial configuration over* $A$ if in addition each pair is $\mathcal{M}$-independent over $A$.

In this terminology, a one-dimensional complete $\mathcal{N}$-type over $A$ is nontrivial iff there is $B \supseteq A$ and extensions $q, r, s \in S_{\mathcal{N}}(B)$ of $p$ which form a nontrivial configuration over $B$. In particular, if some complete non-algebraic extension of $p$ is nontrivial then $p$ is nontrivial as well.

Note that if $\{a, b, c\}$ realise a nontrivial configuration over $A$ and $\dim_{\mathcal{M}}(abc/A) = 2$ then the configuration is strong over $A$. It follows that if $\{a, b, c\}$ realise a nontrivial configuration over $A$ and $a'b'c'$ is an $\mathcal{M}$-generic realisation of $\mathrm{tp}_{\mathcal{N}}(abc/A)$ then $\{a', b', c'\}$ realise a strong nontrivial configuration over $A$.

**Lemma 1.6.** *Let $p, q, r \in S_{\mathcal{N}}(A)$ be one-dimensional types over $A \subseteq \mathcal{N}$ forming a nontrivial configuration over $A$.*

(1) *Suppose that* $s \in S_{\mathcal{N}}(A)$ *is nonorthogonal to* $r$. *Then there is* $B \supseteq A$ *and* $p_1, q_1, s_1 \in S_{\mathcal{N}}(B)$ *extending* $p, q, s$, *respectively, such that* $p_1, q_1, s_1$ *form a nontrivial configuration over* $B$.
(2) *Each of the types* $p, q$ *and* $r$ *is nontrivial.*

*Proof.* (1) Assume that $B \supseteq A$ and $c, d$ are such that $\mathrm{tp}_{\mathcal{N}}(c/B)$, $\mathrm{tp}_{\mathcal{N}}(d/B)$ are non-algebraic extensions of $r$ and $s$ respectively, and such that $c, d$ are inter-algebraic in $\mathcal{N}$ over $B$. By an automorphism argument, there are $a, b$ such that

$$\mathrm{tp}_{\mathcal{N}}(a/B), \mathrm{tp}_{\mathcal{N}}(b/B), \mathrm{tp}_{\mathcal{N}}(c/B)$$

form a nontrivial configuration over $B$ and $\mathrm{tp}_{\mathcal{N}}(a/B)$, $\mathrm{tp}_{\mathcal{N}}(b/B)$ extend $p, q$, respectively. It follows from exchange that $\{a, b, d\}$ realize a nontrivial configuration of $p_1, q_1, s$ over $B$.

(2) Because $p, q, r$ form a nontrivial configuration each of the two types are nonorthogonal to each other. In particular, we may apply (1) to $p$, $q$, $r$ and take $p$ again for $s$. It follows that there are extensions $p_1, q_1, p_2 \in S_{\mathcal{N}}(B)$, $B \supseteq A$, of $p, q, p$, respectively, which form a nontrivial configuration over $B$. Applying (1) again (replacing now $q_1$ by an extension of $p$, we obtain a nontrivial configuration $p_1, p_2, p_3$ where all types extend $p$. It follows that $p$ is nontrivial. □

**Corollary 1.7.** *If* $p \in S_{\mathcal{N}}(A)$ *is 1-dimensional and non-trivial then every complete non-algebraic extension of* $p$ *is non-trivial.*

*Proof.* Assume that $B \supseteq A$, and $q \in S_{\mathcal{N}}(B)$ is a non-algebraic extension of a non-trivial type $p \in S_{\mathcal{N}}(A)$. Let $a \models q$.
Because $p$ is nontrivial there are $a_1, b_1, c_1 \models p$ and $B' \supseteq A$ such that

$$\mathrm{tp}_{\mathcal{N}}(a_1/B'), \mathrm{tp}_{\mathcal{N}}(b_1/B'), \mathrm{tp}_{\mathcal{N}}(c_1/B')$$

form a nontrivial configuration over $B'$. As we observed above, we may assume that this is a strong configuration (i.e. every two are also $\mathcal{M}$-independent).

Because $a_1 \equiv_A a$ in $\mathcal{N}$, we may assume that $a_1 = a$. Next, we may also assume that $b_1 c_1 B'$ is $\mathcal{M}$-independent from $B$ over $aA$. By that we mean that $\dim_{\mathcal{M}}(b_1 c_1 B'/aB) = \dim_{\mathcal{M}}(b_1 c_1 B'/aA)$. Indeed, this can be done by finding an $\mathcal{M}$-generic realisation of $\mathrm{tp}_{\mathcal{N}}(b_1 c_1 B'/aA)$ over $aB$ and calling it $b_1 c_1 B'$ again.

It follows that $a, b_1, c_1$ realize a nontrivial strong configuration over $BB'$. By Lemma 1.6 (2), the type $\mathrm{tp}_{\mathcal{N}}(a/BB')$ is nontrivial and therefore $q = \mathrm{tp}_{\mathcal{N}}(a/B)$ is nontrivial. □

**Remark** Instead of using $\mathcal{M}$-dimension in the above proof, we could use $\mathrm{U}^{\text{þ}}$-rank considerations. Indeed, the definitions of nonorthogonality and nontriviality make perfect sense for types of $\mathrm{U}^{\text{þ}}$-rank 1, and the proof of the last lemma goes through unaltered to the rosy context. However, in order to avoid introducing these concepts, which will not be used herein, we preferred sticking with o-minimal considerations.

1.2. **Stable types.** When investigating structures interpretable in o-minimal theories, the stable/unstable dichotomy arises naturally as a main division line in the analysis. Below we reformulate this division line on the level of type-definable sets, recalling a variation on a definition of Lascar and Poizat, [8]:

**Definition 1.8.** Let $\mathcal{N} \models T$ sufficiently saturated, for a theory $T$. A partial type $\Phi(x)$ is *unstable in* $\mathcal{N}$ if there exists an $\mathcal{N}$-formula $\varphi(x, y)$ defined over $N$ and

$\langle a_i, b_i\rangle_{i\in\omega}$, $a_i \models \Phi$, such that $N \models \varphi(a_i, b_j)$ iff $i < j$. If there is no such formula $\varphi$ then $\Phi$ is said to be *stable in* $\mathcal{N}$.

The following are immediate from the definition (and compactness).

(1) Every extension of a stable partial $\mathcal{N}$-type is stable. (However, there are complete stable types all of whose formulas are unstable, see Example 2.4 below).
(2) Every partial unstable $\mathcal{N}$-type over a set $A$ can be extended to a complete unstable $\mathcal{N}$-type over $A$.
(3) By compactness, if $p \in S_{\mathcal{N}}(A)$ is stable, then for any $\phi(x, y)$ over $A$, there exists $\theta(x) \in p$ such that $\phi(x, y) \wedge \theta(x)$ does not have the order property. For formulas, this notion of stability may be ambiguous, because of the earlier definition (see for example [13]) of a stable formula (with a given partition of variables) $\delta(x, y)$ as one which does not have the order property, namely, that for no $\langle a_i, b_i\rangle_{i\in\omega}$ can it be that $\models \phi(a_i, b_i) \iff i < j$.

   In any case, by definition we have:
(4) An $\mathcal{N}$-formula $\varphi(x)$ is stable if and only if for every $\delta(x, y)$, the formula $\varphi(x) \wedge \delta(x, y)$ does not have the order property.

Complete stable types are studied in more detail in [3] where it is shown that such types share many of the good properties of types in stable theories. We give here a partial list (all results can be found in [2], some can actually be found in [8]).

**Fact 1.9.** *Let $\mathcal{N}$ be a sufficiently saturated model of an arbitrary theory.*

(1)
- *If $p \in S_{\mathcal{N}}(\mathrm{acl}_{\mathcal{N}}(A))$ is stable then it is stationary, namely has a unique non-forking extension (in particular, $p$ does not fork over $A$).*
- *If $p \in S_{\mathcal{N}}(A)$ is stable and stationary then it is definable, namely for all $\phi(x, y)$ over $A$ there is an $\mathcal{N}$-formula $\psi(y)$ over $A$ such that for all $b \in A$, $\psi(b)$ iff $\phi(x, b) \in p$.*
- *If $p \in S_{\mathcal{N}}(A)$ is stable then it satisfies the open mapping theorem, namely, if $q \supseteq p$ is a non-forking extension and $\phi(x, b) \in q$ then there is $\psi(x) \in p$ which is a finite positive boolean combination of $A$-conjugates of $\phi(x, b)$.*

(2) *If $\Phi$ is a stable partial type then it defines a stably embedded set, namely every relatively $\mathcal{N}$-definable subset of $\Phi(N)^k$ is relatively definable over parameters realising $\Phi$.*
(3) *If $\mathcal{N}$ is interpretable in an o-minimal theory and $p$ is a one-dimensional stable type, then* $\mathrm{U(p)} = 1$.

## 2. Statement of the main result

We are now ready to state the finer version of Theorem 2 referred to in the introduction.

**Theorem 2.1.** *Assume that $\mathcal{N}$ is a definable structure in a sufficiently saturated o-minimal $\mathcal{M}$ and that $p$ is a one-dimensional $\mathcal{N}$-type over an $\aleph_0$-saturated model $\mathcal{N}_0 \prec \mathcal{N}$. Then one, and only one, of the following holds:*

(1) *$p$ is trivial (with respect to $\mathrm{acl}_{\mathcal{N}}$).*
(2) *$p$ is non-trivial and stable: There is an $\mathcal{N}$-definable group $G$ over $N_0$, whose $\mathcal{N}$-induced structure is locally modular and strongly minimal, such that $p$ is*

*non-orthogonal the generic type of $G$. In particular, $p$ is strongly minimal and locally modular.*

(3) *$p$ is nontrivial and unstable: for every $a \models p$ there exist a parameter set $A \supseteq N_0$ with $\mathrm{tp}_{\mathcal{N}}(a/A)$ non-algebraic, a formula $\varphi(x) \in \mathrm{tp}(a/A)$, and an equivalence relation $E$ on $\varphi(N)$ with finite classes, which is $\mathcal{N}$-definable over $A$, such that either*

   (a) *$p$ is linear, in which case the structure which $\mathcal{N}$ induces on $\varphi(N)/E$ is that of an interval in an ordered vector space over an ordered division ring. Or,*

   (b) *$p$ is rich, in which case the structure which $\mathcal{N}$ induces on $\varphi(N)/E$ is an o-minimal expansion of a real closed field.*

It is easy to verify that the above theorem implies Theorem 2. The only additional observation is that, in the unstable case, if $\phi(N/E)$ has the structure of an interval $\langle I, 0, <, +\rangle$ in an ordered vector space then, fixing any $\alpha > 0$ in $I$ we obtain a definable group $G$ on $[0, \alpha)$ with addition modulo $\alpha$. The structure induced on $G$ is clearly linear.

To see that the above theorem also implies 3 for a 1-dimensional definable structure $\mathcal{N}$, it is sufficient to show that $\mathcal{N}$ is non-trivial if and only if there is a non-trivial 1-type in $\mathcal{N}$ and that it is rich if and only if there is a rich 1-type. Obviously, if $\mathcal{N}$ has a non-trivial 1-type it is non-trivial, the converse is given by Lemma 1.5(2). Now, if such a rich 1-type exists then, by the above, $\mathcal{N}$ admits a definable real closed field $R$ on a set of the form $X/E$, for $X \subseteq N$ and $E$ of finite classes. The field implies the existence of a 2-dimensional normal family of curves (graphs of polynomial function of fixed degree $n > 0$) on $R^2$ which translates to such a family on $N^2$, hence $\mathcal{N}$ is rich. For the opposite implication, assume that $\mathcal{N}$ is rich as witnessed by $\mathcal{F} = \{C_q : q \in Q\}$ an $\mathcal{N}$-definable 2-dimensional almost normal family of curves in $N^2$. Fix $q_0$ generic in $Q$ and $p_0 = \langle a, b\rangle$ generic in $C_{q_0}$. We claim that $\mathrm{tp}(b/q_0)$ is rich. Indeed, if $b' \equiv_{q_0} b$ then the family $\{C_q \circ C_{q_0}^{-1} : q \in Q\}$ is an almost normal family with infinitely many curves through $\langle b', b'\rangle$.

**Remark 2.2.** (1) The theorem also implies that, in our setting, complete one-dimensional nontrivial stable $\mathcal{N}$-types must contain a stable (in fact strongly minimal) formula (compare with Example 2.4 below).

(2) Theorem 2.1 does not say much about trivial types, so a few words on the subject are in place:
The analysis of unstable trivial types could be reduced *locally*, via Theorem 3.1 below, to that of 1-types in trivial o-minimal structures. Note however that a trivial one-dimensional type $p$ might not contain any formula whose structure is trivial, as the second part of Example 2.4 shows. A reasonable classification of trivial o-minimal structures is given in [9].

In the stable case, note that by a theorem of Lachlan ([7], Theorem 4.6), all trivial structures which are totally categorical are definable in the o-minimal structure $(\mathbb{Q}, \leq)$. It is thus left to treat those trivial $\mathcal{N}$-types which don't contain a stable trivial formula $\varphi$ or those for which the structure induced on $\varphi$ is not totally categorical.

2.1. **Some examples.** At the suggestion of the referee we add a few examples, showing among others that the statement of the theorem is in some sense the strongest possible. First, observe that trivial stable structures interpretable in o-minimal theories need not have any strongly minimal subsets:

**Example 2.3.** Let $\mathcal{L}_1 := \{P_i\}_{i\in\mathbb{N}}$ be a language with infinitely many unary predicates, and $T_1$ the $\mathcal{L}_1$-theory stating that they are all infinite and pairwise disjoint. Let $\mathcal{L} := \{P_{i,j} : i, j \in \mathbb{N}\}$ be the expansion of $\mathcal{L}_1$ with the theory $T$ stating that:

(1) $P_{i,0} \equiv P_i$ for all $i$.
(2) For a fixed $i_0$ the set $P_{i_0,j+1}$ splits $P_{i_0,j}$ to two infinite sets.

Clearly, $T$ is stable has a one-dimensional interpretation in $(\mathbb{R}, <)$ but no formula in $T$ is strongly minimal.

As we observed earlier, Theorem 2 implies that every one-dimensional stable non-trivial $\mathcal{N}$-type contains a stable formula. However, stable trivial types might not contain any such stable formula:

**Example 2.4.** Let $\mathcal{L}_2$ be the expansion of $\mathcal{L}_1$ (from 2.3 ) by $\{\leq_i\}_{i\in\omega}$, a countable set of binary relations, and let $T_2$ be the expansion of $T_1$ stating that each $\leq_i$ is a dense linear ordering on $P_i$ with no endpoints. Clearly, $T_2$ has quantifier elimination, so it is easy to check that the type $p := \{x \notin P_i : i \in \omega\}$ is stable and complete, but every formula in $p$ is unstable. Indeed, the only stable formulas, even over parameters, are algebraic.

We can further expand each $P_i$ by an ordered group operation $+_i : P_i^2 \to P_i$, making each $G_i = \langle P_i, <_i, +_i\rangle$ into an ordered divisible abelian group. The whole structure can be interpreted in $\langle \mathbb{R}, <, +, \cdot\rangle$, with each $G_i$ living on $(i, i + 1)$ and $+_i$ the image of $+$ under some definable bijection between $\mathbb{R}$ and $(i, i+1)$. In this case, the above type $p$ is also trivial but does not contain any trivial formula.

Returning to the statement of Theorem 2.1, let us see that the equivalence relation $E$ (with finite classes) appearing in the statement of the theorem cannot be avoided:

**Example 2.5.** Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, \leq)$ and $N := \{0\}\times\mathbb{R}\cup\{1\}\times\mathbb{R}$. Let $S := \mathbb{R}$ and consider the two sorted structure $\mathcal{N} := (N, S, \pi, \cdots)$ where $\pi : N \to S$ is the natural projection, $S$ is taken with all its $\mathcal{R}$-induced structure, and there is no further structure in $\mathcal{N}$. In particular $S$ is o-minimal. Then a type extending $N$ is trivial (resp. linear, resp. rich) if an only if its image under $\pi$ is. But clearly, no element of $N$ is contained in an $\mathcal{N}$-definable o-minimal structure.

Note that in the unstable case Theorem 2.1 does not imply, when $p$ is linear or rich, that $p/E$ itself is a generic type of a definable group or real closed field. Instead, only extensions of $p$ satisfy this. This is unavoidable as the following variant of an example from [2] shows.

**Example 2.6.** Consider $\mathcal{N} := (\mathbb{R}, +, *, I)$, where $I$ is a unary predicate interpreted as the unit interval and $* \subseteq I^3$ is that part of the graph of multiplication contained in the unit cube. By the trichotomy theorem for o-minimal theories, a real closed field is definable near each point of $\mathcal{N}$. It is not hard to check that in $\mathcal{N}$ every bounded interval is linearly ordered, but no unbounded interval is. In particular, no real closed field is definable on an unbounded interval (see [10] for more details). Now let $(\mathbb{R}, +, \cdot, \leq) \prec \mathcal{R}^*$ and let $\mathcal{N}^* \succeq \mathcal{N}$ be the reduct of $\mathcal{R}^*$ to $(+, *, I)$. Let $p := \mathrm{tp}_{\mathcal{N}}(a/N)$ for some $a \in R^*$ which satisfies $a > b$ for all $b \in \mathbb{R}$. It is not hard to check that $p$ itself is not a generic type in any definable field (for such a field would have to have an unbounded domain), but every realization of $p$ is contained in a definable real closed field on a bounded interval.

We conclude this set of examples with an example showing that our treatment of one-dimensional structures does not cover all geometric structures interpretable in o-minimal ones. Of course, strongly minimal sets interpretable in o-minimal theories are necessarily geometric structures, and need not be one-dimensional (take for example the complex field). Here is an unstable example.

**Example 2.7.** Let $\mathcal{N} := (\mathbb{Q}^2, \leq^*, P_i)_{i\in\omega}$ where $\leq^*$ is interpreted as the lexicographic order on $\mathbb{Q}^2$ and $P_i = \{(i, x) : x \in \mathbb{Q}\}$ for all $i \in \omega$. This description of $\mathcal{N}$ shows that it is definable in $(\mathbb{Q}, <)$ but as the $P_i$ are definable subsets with no supremum in $\mathcal{N}$ (with respect to $\leq^*$) and as there are infinitely many of them, it cannot be that $\mathcal{N}$ has a one-dimensional interpretation in any o-minimal structure. It is not hard to see that this is a geometric structure (indeed, it is þ-minimal, and has no algebraicity - i.e. $\mathrm{acl}_{\mathcal{N}}(A) = A$ for all $A \subseteq N$). By endowing $\mathbb{Q}^2$ with the usual group operation, we get a nontrivial variation.

## 3. Proof of the main theorem

We now return to the proof of Theorem 2.1. The proof uses, in addition to results of the previous sections, two main tools. The first is the Trichotomy Theorem for o-minimal structures of [11]. The second is a theorem from [2]:

**Theorem 3.1.** *Let $\mathcal{N}$ be definable in a saturated enough dense o-minimal structure $\mathcal{M}$. Let $N_0 \subseteq N$ be a small model and $p \in S_{\mathcal{N}}(N_0)$ a one-dimensional unstable type. Then $p$ has an almost o-minimal extension, namely there is an $\mathcal{N}$-definable set $X_0$ (possibly over new parameters) with $X_0 \cap p(N)$ infinite, and an $\mathcal{N}$-definable equivalence relation $E$ on $X_0$ with finite classes such that $X_0/E$ (with all its $\mathcal{N}$-induced structure) is o-minimal.*

**Remark 3.2.** The equivalence relation $E$ appearing in this theorem gives rise to the one appearing the statement of unstable part of Theorem 2.1.

We will need the following corollary of the above:

**Corollary 3.3.** *Let $\mathcal{N}$ be definable in a saturated enough dense o-minimal structure $\mathcal{M}$. Let $N_0 \subseteq N$ be a small model and $p \in S_{\mathcal{N}}(N_0)$ a one-dimensional unstable type. Then every complete non-algebraic extension of $p$ is unstable.*

*Proof.* This is a special case of the main result of [3], but we give a short self contained proof for the present setting. Let $B \supseteq N_0$ be a small set, $q \in S_{\mathcal{N}}(B)$ a non-algebraic extension of $p$, and assume that $b$ is an $\mathcal{M}$-generic realisation of $q$ over $B$.

By Theorem 3.1, there exists an almost o-minimal $X_0$ which is defined over $C_0 \supseteq N_0$ such that $X_0 \cap p(N)$ is infinite (we say that the almost o-minimal set $X_0$ *is defined over* $C_0$ if the set $X_0$, the equivalence relation $E$ and the linear ordering on $X_0/E$ are all defined in $\mathcal{N}$ over $C_0$). Let $b_0 \in X_0 \cap p(M)$ be $\mathcal{M}$-generic over $C_0$.

Next, we claim that there is $B_0 \subseteq N$ which is $\mathcal{M}$-independent from $C_0$ over $N_0$ and such that $b_0B_0 \equiv bB$ in $\mathcal{M}$. Indeed, we first take any $B_1 \supseteq N_0$ such that $b_0B_1 \equiv bB$ in $\mathcal{M}$ and then we let $B_0$ be an $\mathcal{M}$-generic realisation of $\mathrm{tp}_{\mathcal{N}}(B_1/b_0N_0)$ over $b_0C_0$. We now have $\dim_{\mathcal{M}}(B_0/b_0C_0) = \dim_{\mathcal{M}}(B_0/b_0N_0)$ as needed.

Because $b$ and $B$ were $\mathcal{M}$-independent over $N_0$, the same is true for $b_0$ and $B_0$ and hence $B_0$ is $\mathcal{M}$-independent from $b_0C_0$ over $N_0$. It follows that

$$\dim_{\mathcal{M}}(b_0B_0C_0/N_0) = \dim_{\mathcal{M}}(b_0/N_0) + \dim_{\mathcal{M}}(B_0/N_0) + \dim_{\mathcal{M}}(C_0/N_0),$$

and therefore $\dim_{\mathcal{M}}(b_0/B_0C_0) = 1$. This implies that $\mathrm{tp}_{\mathcal{N}}(b_0/B_0C_0)$ is a non-algebraic type containing the formula $x \in X_0$. Because $X_0/E$ is an o-minimal structure defined over $C_0$, it easily follows that $\mathrm{tp}_{\mathcal{N}}(b_0/B_0C_0)$ is unstable.

We now conjugate $b_0, B_0$ and $C_0$, over $N_0$, with $b, B$ and some $C$ (using an $\mathcal{N}$-automorphism over $N_0$). It follows that $\mathrm{tp}_{\mathcal{N}}(b/BC)$ is unstable and therefore $q = \mathrm{tp}_{\mathcal{N}}(b/B)$ is also unstable. □

Note that Corollary 3.3 is false if the type $p$ is not assumed to be one-dimensional. E.g. consider the structure $\mathcal{N} = \langle \mathbb{R}^2, E, < \rangle$, where $E$ is the equivalence relation $\langle x, y\rangle E \langle x', y'\rangle$ iff $x = x'$, and $<$ is the natural linear ordering on $\mathbb{R}^2/E$. The unique two-dimensional type in this structure is unstable but the type of every element over another, $E$-equivalent, element is stable.

We are now ready to prove Theorem 2.1.

*Proof.* We assume that $p \in S_{\mathcal{N}}(N_0)$ is a one-dimensional nontrivial type.

**Case 1**: There is a stable $\mathcal{N}$-formula in $p$.

Since every formula implying a stable formula is itself stable we can find a stable $\varphi \in p$ with $\dim_{\mathcal{M}}(\varphi(N)) = 1$. Let $X = \varphi(N)$. It follows from 1.9 (2), that $X$ is stably embedded in $\mathcal{N}$. In this case, we may replace $\mathcal{N}$ by $X$ and assume that $\mathcal{N}$ is one-dimensional and $\mathrm{U}(\mathcal{N}) = 1$. By Theorem 1, $\mathcal{N}$ is necessarily 1-based. Obtaining a type-definable group in this situation is by now fairly well known, so we will be brief.

Because $p$ is non-trivial and $U(p) = 1$, there is a type-definable abelian group $G$ in $\mathcal{N}$ whose generic is nonorthogonal to $p$ (see, Theorem 3 of [5]). Moreover, $G$ is defined over a realization of $p$. Since $\mathcal{N}$ is now superstable, there is an $\mathcal{N}$-definable (rather than type-definable) group $G_0 \supseteq G$, with $U(G_0) = 1$. By a theorem of Gagelman, [1], $G_0$ has finite Morley rank, and moreover the proof shows that this Morley rank is at most $U(G_0) = 1$. Hence $\mathrm{MR}(G_0) = 1$ which implies that $G$ itself was definable and can be chosen to be strongly minimal. Regarding the parameters defining $G$, the type $p$ does not fork over a finite subset of $N_0$, so we can choose this realization already in $N_0$. It follows that $p$ is strongly minimal and that $G$ is a locally modular group. The structure of $G$ is given by [6].

**Case 2**: Every $\mathcal{N}$-formula in $p$ is unstable.

We already saw that it is possible to have stable types all of whose formulas are unstable. However, we will first show that nontriviality forces the instability of $p$. By Theorem 3.1 this will provide us, for every $a \models p$, with an almost o-minimal set $X_0$ containing $a$, with $X_0 \cap p(\mathcal{N})$ infinite. We will then show that the o-minimal structure associated with $X_0$ is nontrivial, and complete the proof using the trichotomy theorem for o-minimal structures.

We fix an $\mathcal{N}$-formula $\varphi \in p$ with $X = \varphi(N)$ and $\dim_{\mathcal{M}}(X) = 1$.

**Claim** $p$ is unstable.

*Proof.* By a recent result in [3], in a dependent theory every non-trivial stable type of U-rank 1 must contain a stable formula, therefore $p$ must be unstable. However, for the sake of completeness we give here a direct proof.

We write $p = p(x)$ and assume towards a contradiction that $p$ is stable.

We first observe that if $q \in S_{\mathcal{N}}(N_1)$ is a non-algebraic extension of $p$ over a model $N_1 \supseteq N_0$ then all of its formulas are unstable (although $q$ is still a stable type). Indeed, since $p$ is stable with U(p) = 1 and since $q$ is non-algebraic, it is a non-forking extension of $p$. Moreover, if $\phi(x, d) \in q$ defined a stable set then, because $q$ is a non-forking extension of $p$, by the Open Mapping Theorem (see 1.9(1)) some positive boolean combination $\psi(x)$ of $N_0$-conjugates of $\phi(x, d)$ is in $p(x)$. But then $\psi(x)$ itself defines a stable set, contradicting our Case 2 assumption.

Because $p$ is nontrivial there is $A \supseteq N_0$ and $a, b, c$ realising $p$ such that $\{a, b, c\}$ realise a strong nontrivial configuration over $A$. Let $R \subseteq X^3$ be a ternary relation which is $\mathcal{N}$-definable over $A$, witnessing this nontriviality. Namely,
(i) $R(a, b, c)$ holds.
(ii) The projection map of $R$ on any of its two coordinates is a finite-to-one map and its image is two dimensional.
(iii) For every $a_1$, the set $\{\langle b_1, c_1\rangle : R(a_1, b_1, c_1)\}$ is either empty or $\mathcal{M}$-one-dimensional.
(All of this can be obtained for $R$ because the notion of $\mathcal{M}$-dimension for $\mathcal{N}$-definable subsets of $X^n$ is $\mathcal{N}$-definable using the fact that $\exists^\infty$ is first order in $\mathcal{N}$).

The formula $\phi(x, y) := \exists z R(x, y, z)$ is an $\mathcal{N}$-formula over $A$ with $\mathcal{N} \models \phi(a, b)$. Let $\mathcal{N}_1 \subseteq \mathcal{N}$ be a small model containing $aA$ such that $\dim_{\mathcal{M}}(b/N_1) = 1$. Because $a$ is $\mathcal{M}$-independent from $b$ over $N_0$, we can also find $a_0 \models p$ with $\phi(a_0, b)$, such that $\dim_{\mathcal{M}}(a_0, b/N_1) = 2$ and in particular, the partial type $\{\phi(a_0, y) \wedge \phi(a, y)\} \cup p(y)$ is non-algebraic. It follows from our above observation that $\phi(a_0, y) \wedge \phi(a, y)$ is unstable and therefore there is a complete (non-algebraic) unstable type $r(y) \in S_{\mathcal{N}}(a_0 N_1)$, with $\phi(a_0, y) \wedge \phi(a, y) \in r$. Fix $b_0$ an $\mathcal{M}$-generic realisation of $r$ over $a_0 N_1$.

By assumption, $\mathcal{N} \models \phi(a_0, b_0)$ and hence there exists $c_0$ such that $R(a_0, b_0, c_0)$ and therefore $b_0 \in \mathrm{acl}_{\mathcal{N}}(a_0 c_0 A)$. Since $\dim_{\mathcal{M}}(b_0 a_0/N_1) = 2$, it is easy to see that $\{a_0, b_0, c_0\}$ realize a strong nontrivial configuration over $N_1$.

Because $r = \mathrm{tp}_{\mathcal{N}}(b_0/a_0 N_1)$ is unstable so is $\mathrm{tp}_{\mathcal{N}}(b_0/N_1)$ and therefore, by Corollary 3.3, the type $\mathrm{tp}_{\mathcal{N}}(b_0/c_0 N_1)$ (which is nonalgebraic) is also unstable.

Finally, $a_0$ and $b_0$ are inter-algebraic over $c_0 N_1$ hence $\mathrm{tp}_{\mathcal{N}}(a_0/c_0 N_1)$ is also unstable. Because this last type extends $p$, the type $p$ is unstable as well. **End of Claim**.

We now proceed with the proof of Theorem 2.1. Since $p$ is unstable, for every $a \models p$ we can find, (using the same arguments as in the proof of Corollary 3.3), an almost o-minimal $X_0$ which is $\mathcal{N}$-definable over a parameter set $C \supseteq N_0$, with $\dim_{\mathcal{N}}(a/C) = 1$. Let $E$ be the associated $\mathcal{N}$-definable equivalence relation with finite classes, such that $X_0/E$ (with all the induced $\mathcal{N}$-structure) is o-minimal.

Let $\mathcal{X}$ denote the $\mathcal{N}$-interpretable o-minimal structure induced on $X_0/E$. Note that we do not know whether $X_0/E$ is stably embedded in $\mathcal{N}$. Therefore, it could be that the structure $\mathcal{X}$ depends on the model $\mathcal{N}' \prec \mathcal{N}$) in which we consider its induced structure. Thus, we fix such a (saturated enough) $\mathcal{N}'$ once and for all and consider the structure it induces on $X_0/E$. The model $\mathcal{N}'$ will not play any role from now on.

Let $d := a/E$ denote the $E$-class of $a$ in $X_0$, and let $q_0 := \mathrm{tp}_{\mathcal{X}}(d/C)$. Our aim is to show that (regardless of our choice of $N'$) $q_0$ is rich (resp. linear) iff $p$ is rich (resp. linear). Because $p$ is nontrivial, it follows from Lemma 1.7 that $\mathrm{tp}_{\mathcal{N}}(a/C)$ is also nontrivial and therefore (since the classes of $E$ are finite) that $q_0$ is nontrivial as well in $\mathcal{X}$.

We now consider $d$ as a generic element in the o-minimal structure $\mathcal{X}$. The Trichotomy Theorem for o-minimal structures [12], implies that the structure which $\mathcal{X}$ induces on some interval $I \subseteq X_0$ around $d$ is either that of an interval in an ordered vector space or that of an o-minimal expansion of a real closed field.

Assume that $p$ is a rich type in $\mathcal{N}$. We are still working with a fixed one-dimensional $\mathcal{N}$-definable $X \supseteq p(N)$ defined over $N_0$. Then there exists a set $A \supseteq N_0$, with $\dim_{\mathcal{N}}(a/A) = 1$ and an $A$-definable infinite almost normal family of curves in $X \times X$, all going through $\langle a, a\rangle$. We may assume (using automorphism arguments) that $A$ is $\mathcal{M}$-independent from $C$ over $N_0$ and hence that all the curves are actually contained in $X_0 \times X_0$. Because the family is normal, there exists a number $m$ such that every curve in the family is definable by any $m$-distinct points on it. It follows that if we endow $X_0$ with all $\mathcal{N}$-definable sets over $A$ then the type of $a$ is rich in this structure. It easily follows that $q_0$ is rich, and hence the $\mathcal{N}$-induced structure on a neighbourhood of $d$ is that of an o-minimal expansion of a real closed field.

If $p$ is linear then it easily follows that $q_0$ must be linear as well and therefore the structure of $\mathcal{X}$ around $d$ is that of an interval in an ordered vector space over an ordered division ring. This completes the proof of Case 2, and with it the proof of Theorem 2.1. □

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