Analytic-like functions in o-minimal structures

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Abstract

We prove a general partition theorem for definable functions of several variables in linearly ordered, o-minimal structures. Such function $F(\bar{u}, \bar{x})$ generates in a natural way a definable family of functions, $f_{\bar{u}}(\bar{x}) = F(\bar{u}, \bar{x})$. The theorem says that every function in the partition has the property that if two functions in the family it generates agree on some open set then they agree globally. We use this theorem to eliminate an assumption about analyticity of functions in an earlier result from [4].

1 Preliminaries

The objects we investigate are first order structures, linearly ordered by $<$, whose ordering is dense and without endpoints. We denote by M the universe of the structure $\mathcal{M}$. For the sake of simplicity we will assume that the language we consider is countable. Except in some particular cases, we always take 'definable' to mean 'definable with parameters' without necessarily mentioning the parameters we use. We remind the reader that such a structure $\mathcal{M}$ is called *o-minimal* if every definable (with parameters) subset of M is a finite union of points and open intervals (we always refer to the topology that is induced by the ordering on the M^n's). We refer the reader to [1] for some basic results on o-minimal structures.

*The writing of this paper has begun during my stay at Mcgill University, in 1992. (submitted but then retrieved by the author)

Let $\mathcal{M}$ be an o-minimal structure. Given $n \in \mathbb{N}$, every definable subset of M^n has an assigned dimension. Moreover, if $\{X_{\bar{u}}\}_{\bar{u}\in\mathcal{M}^k}$ is a uniformly definable family of subsets of M^n then for every $m \in \mathbb{N}$, the set $\{\bar{c} \in M^k : \dim(X_{\bar{c}}) = k\}$ is definable (for that and other facts on dimension of sets and tuples see [3]). It was also shown in [3] that $U \subseteq V$ are both of the same dimension if and only if U contains an open set in the relative toplogy of V. For $\bar{u} \in V$, we say that $U \subseteq V$ *is a neighborhood of* $\bar{u}$ if U is open in the relative topology of V.

The dimension of a tuple $\bar{a}$ over a set $B \subseteq M$, $\dim(\bar{a}/B)$, is defined as the least n such that $a_{i_1}, \ldots, a_{a_n} \subseteq \bar{a}$ and $\bar{a}$ is contained in the definable closure of $\{a_{i_1}, \ldots, a_{a_n}\} \cup B$ (which we denote by $dcl(\{a_{i_1}, \ldots, a_{a_n}\} \cup B)$). For $\bar{a}, \bar{b}$ and C, all from M, we have the following dimension formula:

$$\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/\bar{b}C) + \dim(\bar{b}/C).$$

If $\mathcal{M}$ is an $\omega-$saturated structure and X is A-definable in $\mathcal{M}$ then $\dim(X) = max\{\dim(\bar{a}/A) : \bar{a} \in X\}$. If $\dim(X) = \dim(\bar{a}/A)$, $\bar{a} \in X$, then we call $\bar{a}$ *a generic point of* X. For such a generic $\bar{a}$ in X, if $\bar{a} \in V \subseteq X$, V an A-definable set, then there is a neighborhood $U \in X$ of $\bar{a}$ that is contained in V.

One can extend the notion of dimension also to definable quotients over definable equivalence relations (see [5] for details): For $U \subseteq M^n$ a definable set and E a definable equivalence relation on U we let $U_k^E = \{\bar{u} \in U : \dim([\bar{u}]_E) = k\}$ (where $[\bar{u}]_E$ is the equivalence class of $\bar{u}$). Define

$$\dim\left(\frac{U}{E}\right) = max\{dimmU_k^E - k : k = 1, \ldots, n\}.$$

Again, if $\{U_{\bar{a}}\}_{\bar{\in}M^k}$ is a uniformly definable family of sets and if $\{E_{\bar{a}}\}_{\bar{a}\in M^k}$ is a uniformly definable family of equivalence relations on $U_{\bar{u}}$, respecively, then for every $m \in \mathbb{N}$, the set $\{bara \in M^k : \dim\left(\frac{U_{bara}}{E_{\bar{a}}}\right) = m\}$ is also definable.

2 Introduction

One of the difficulties in working with linearly ordered structures is the fact that local properties do not translate to global ones. The simplest example of this phenomenon is that of two definable functions of one variable which agree on some nonempty open interval but differ on the rest of their domain. In some particular cases this can be avoided. For example, if the structure in question has $\mathbb{R}$ as its universe and the two functions are real-analytic then, if they agree on any nonempty open interval, they agree on all of their common domain. So, if we work, say, in the structure $\mathcal{N} = \langle \mathbb{R}, <, +, \cdot, e^x \rangle$ in which every definable function is piecewise analytic, we can avoid some of the local-global difficulties that arise in that way. As we show below, in every o-minimal structure we have a partition theorem for definable functions that mimics some of the properties of the partition into real-analytic functions that we have in $\mathcal{N}$. For a more precise formulation we need some notation.

Let $W \subseteq M^n$ be a definable open set and $F : W \longrightarrow M$ a definable function. If U is the projection of W onto the first $n-1$ coordinates then we obtain a family of unary functions $\mathcal{F} = \{f_{\bar{u}} : \bar{u} \in U\}$ defined by $f_{\bar{u}}(x) = F(\bar{u}, x)$. Denote by $I_{\bar{u}}$ the domain of $f_{\bar{u}}$ (which clearly depends on W).

Definition 2.1 For F, W and U as above, we say that F is *analytic-like on W in x_n* if F is continuous on W and the following holds: For every $\bar{u}, \bar{v} \in U$, if there is a nonempty open interval $I \subseteq I_{\bar{u}} \cap I_{\bar{v}}$ such that $f_{\bar{u}}(x) = f_{\bar{v}}(x)$ for all $x \in I$ then $f_{\bar{u}}(x) = f_{\bar{v}}(x)$ for all x in $I_{\bar{u}} \cap I_{\bar{v}}$.

Clearly, if $M = \mathbb{R}$ and F is a real-analytic function on W then it is analytic-like in every variable. Notice also that by o-minimality, if F is analytic-like on W in x_n then there is $k \in \mathbb{N}$ such that for every $\bar{u}, \bar{v} \in U$, either $f_{\bar{u}}$ agrees with $f_{\bar{v}}$ on at most k points or they agree on every point of their common domain. Our goal is to prove:

Theorem 2.2 *Let $\mathcal{M}$ be an o-minimal structure and assume that $W \subseteq M^n$ is a definable open set and $F : W \longrightarrow \mathcal{M}$ is a definable function. Then, there are definable open sets $W_1, \ldots, W_k \subseteq W$ such that dim$(W \setminus \bigcup_{i=1}^k W_i) < n$, and for every $i = 1, \ldots, k$ the function $F|W_i$ is analytic-like on W_i in x_n. The definition of the W_i's uses no new parameters.*

In Corollary 5.1 we extend this result so as to obtain functions which behave well with respect to all variables simultaneously. We then give an analogue of the notion of analytic continuation for definable families of functions. In section 5 we show how to use Theorem 2.2 to strengthen a result from [4].

3 Preliminaries

The objects we investigate are first order structures, linearly ordered by $<$, whose ordering is dense and without endpoints. We denote by M the universe of the structure $\mathcal{M}$. For simplicity we will assume that the language considered is countable. Except in some particular cases, we always take 'definable' to mean 'definable with parameters' without necessarily mentioning the parameters we use. We remind the reader that such a structure $\mathcal{M}$ is called *o-minimal* if every definable subset of M is a finite union of points and open intervals (we always refer to the topology on the M^n's that is induced by the ordering). We refer the reader to [1] for some basic results on o-minimal structures.

Let $\mathcal{M}$ be an o-minimal structure. Given $n \in \mathbb{N}$, every definable subset of M^n has an assigned dimension. Moreover, if $\{X_{\bar{u}}\}_{\bar{u}\in\mathcal{M}^k}$ is a uniformly definable family of subsets of M^n then for every $m \in \mathbb{N}$, the set $\{\bar{c} \in M^k : \dim(X_{\bar{c}}) = m\}$ is definable (for that and other facts on dimension of sets and tuples see [3]). It was also shown in [3] that two definable sets $U \subseteq V$ are of the same dimension if and only if U contains an open set in the relative topology of V. For $\bar{u} \in V$, we say that $U \subseteq V$ *is a neighborhood of* $\bar{u}$ if U contains an open set around $\bar{u}$ in the relative topology of V.

The dimension of a tuple $\bar{a}$ over a set $B \subseteq M$, $\dim(\bar{a}/B)$, is defined as the least n such that $a_{i_1}, \ldots, a_{i_n} \subseteq \bar{a}$ and $\bar{a}$ is contained in the definable closure of $\{a_{i_1}, \ldots, a_{i_n}\} \cup B$ (which we denote by $dcl(\{a_{i_1}, \ldots, a_{i_n}\} \cup B)$). For $\bar{a}, \bar{b}$ and C, all from M, we have the following dimension formula:

$$\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/\bar{b} \cup C) + \dim(\bar{b}/C).$$

If $\mathcal{M}$ is an $\omega-$saturated structure and X is A-definable in $\mathcal{M}$ then $\dim(X) = max\{\dim(\bar{a}/A) : \bar{a} \in X\}$. If $\dim(X) = \dim(\bar{a}/A)$, $\bar{a} \in X$, then we call $\bar{a}$ *a generic point of* X *over* A. For such a generic $\bar{a}$ in X, if $\bar{a} \in V \subseteq X$ and V is an A-definable set, then V is a neighborhood of $\bar{a}$ and in particular $\dim(V) = \dim(X)$.

One can extend the notion of dimension also to definable quotients over definable equivalence relations (see [5] for details): For $U \subseteq M^n$ a definable set and E a definable equivalence relation on U we let $U_k^E = \{\bar{u} \in U : \dim([\bar{u}]_E) = k\}$ (where $[\bar{u}]_E$ is the equivalence class of $\bar{u}$). Define

$$\dim\left(\tfrac{U}{E}\right) = max\{\dim(U_k^E) - k : k = 1, \ldots, n\}.$$

Again, if $\{U_{\bar a}\}_{\bar a\in M^k}$ is a uniformly definable family of sets and if $\{E_{\bar a}\}_{\bar a\in M^k}$ is a uniformly definable family of corresponding equivalence relations then for every $m\in\mathbb{N}$, the set $\{\bar a\in M^k : \dim\left(\frac{U_{\bar a}}{E_{\bar a}}\right)=m\}$ is also definable.

4 Germs of functions

The main tool in proving the theorem is the notion of *germs of functions*. We have started developing this notion in [2] and [4]. Let F, W, U and $\mathcal{F}$ be as in the Introduction, then they induce the following two equivalence relations on U:

Definition 4.1 (i) For $J\subseteq M$, $\bar u,\bar v\in U$, define $\bar u\sim_J \bar v$ iff $I_{\bar u}\cap J=I_{\bar v}\cap J$ and $f_{\bar u}(x)=f_{\bar v}(x)$ for all x in $I_{\bar u}\cap J$ (we say then that $f_{\bar u}|J=f_{\bar v}|J$). Let $[\bar u]_J$ denote the $\sim_J$-equivalence class of $\bar u$ in U.
(ii) For $a\in M$, $\bar u,\bar v\in U$ define $\bar u\approx_a \bar v$ iff there exists an open interval J containing a such that $f_{\bar u}|J=f_{\bar v}|J$.

Notice that if there is an open interval J containing a such that $f_{\bar u}, f_{\bar v}$ are not defined on any point of J then $\bar u_1\approx_a \bar u_2$. For $\mathcal{F}$ as above, the relation $\approx_a$ induces an obvious equivalence relation on the $f_{\bar u}$'s. We call the equivalence class of $\bar u$ (or of $f_{\bar u}$) with respect to this relation *the germ of $f_{\bar u}$ at a.* There are two other notions, of "positive" and "negative" germs, which are of use to us: For $\bar u\in U$ and $a\in M$, let

$$[\bar u]^+_a=\{\bar v\in U:\exists a_1>a(\bar v\sim_{(a,a_1)}\bar u)\},$$

$$[\bar u]^-_a=\{\bar v\in U:\exists a_1<a(\bar v\sim_{(a_1,a)}\bar u)\}.$$

All above notions are first-order definable over the same parameters which were used to define W, F, J and a. They clearly depend on F and W but when we use them it will be clear which function and set we refer to.

By taking an elementary extension of $\mathcal{M}$ we may assume from now on that $\mathcal{M}$ is ω-saturated.

Lemma 4.2 *Assume that F, W and U are as above, all A-definable.*
(i) If $a\in M$ and $\bar u_0$ is generic in U over aA then there is $a_1>a$ such that

$$dim([\bar u_0]_{(a,a_1)})=dim([\bar u_0]^+_a).$$

(ii) If $(\bar{u}_0, a)$ *is generic in* W *over* A *then*

$$\exists y_1 y_2 \left[\begin{array}{l} (y_1 < a < y_2) \,\&\, (y_1, y_2) \subseteq dom(f_{\bar{u}_0}) \,\&\, \\ \forall z \in (y_1, y_2)\ dim([\bar{u}_0]_{(y_1,y_2)}) = dim([\bar{u}_0]^+_z) \end{array} \right]. \tag{1}$$

(iii) The statement of (ii) holds with $[\bar{u}_0]^-_z$ *replacing* $[\bar{u}_0]^+_z$.

Proof. Without loss of generality $A = \emptyset$. (i) Let $\bar{v}_0$ be generic in $[\bar{u}_0]^+_a$ over $a\bar{u}_0$. By the dimension formula, it is easy to verify that $\bar{u}_0$ is also generic in $[\bar{u}_0]^+_a$ over $a\bar{v}_0$. Let $a_1 > a$ be such that $f_{\bar{v}_0}|(a, a_1) = f_{\bar{u}_0}|(a, a_1)$. By the saturation of $\mathcal{M}$ we can choose a_1 to be independent from $a\bar{v}_0\bar{u}_0$ (namely, we ensure that $a_1 \notin dcl(a\bar{u}_0\bar{v}_0)$ by taking a_1 close enough to a). Since $\bar{u}_0$ is still generic in $[\bar{u}_0]^+_a$ over $a_1 a\bar{v}_0$, there is a neighborhood $U_1 \subseteq [\bar{u}_0]^+_a$ of $\bar{u}_0$ such that for every $\bar{u} \in U_1$ we have $f_{\bar{u}}|(a, a_1) = f_{\bar{v}_0}|(a, a_1) = f_{\bar{u}_0}|(a, a_1)$. It follows that $U_1 \subseteq [\bar{u}_0]_{(a,a_1)} \subseteq [\bar{u}_0]^+_a$, and since $\dim(U_1) = \dim([\bar{u}_0]^+_a)$, we have

$$\dim([\bar{u}_0]_{(a,a_1)}) = \dim([\bar{u}_0]^+_a). \tag{2}$$

Notice that if a_2 is between a and a_1 then (2) still holds with a_2 replacing a_1.

To prove (ii), let $(\bar{u}_0, a)$ be generic in W and pick $a_1 > a$ as in (i), i.e. $\dim([\bar{u}_0]_{(a,a_1)}) = \dim([\bar{u}_0]^+_a)$. By the comment above we can choose such $a_1 \notin dcl(\bar{u}_0, a)$ and hence $(\bar{u}_0, a)$ is still generic in W over a_1. The statement '$(\bar{u}_0, x) \in W \,\&\, \dim([\bar{u}_0]_{(x,a_1)}) = \dim([\bar{u}_0]^+_x)$' can be written as a first order formula $\psi(x, \bar{u}_0, a_1)$, which is satisfied by a. Since a is independent from $\bar{u}_0 a_1$ there is an interval J containing a such that for every $x \in J$ the formula $\psi(x, \bar{u}_0, a_1)$ holds.

Choose $b_1, b_2 \in J$ such that $b_1 < a < b_2 < a_1$. Then the interval (b_1, b_2) is contained in the domain of $f_{\bar{u}_0}$ (by the first clause of ψ) and for every $x \in (b_1, b_2)$ we have

$$[\bar{u}_0]_{(b_1,a_1)} \subseteq [\bar{u}_0]_{(b_1,b_2)} \subseteq [\bar{u}_0]^+_x.$$

Since $\dim([\bar{u}_0]_{(b_1,a_1)}) = \dim([\bar{u}_0]^+_x)$ we must have $\dim([\bar{u}_0]_{(b_1,b_2)}) = \dim([\bar{u}_0]^+_x)$.

To prove (iii), we first prove the "negative" analogue of (i) and then repeat he same proof as in (ii). □

5 Proof of 1.2 and some corrolaries

We go back now to the proof of Theorem 2.2. The idea of the proof is first to replace W with a set W' such that if $(\bar{u}, a) \in W'$ then $[\bar{u}]^+_a$ and $[\bar{u}]^-_a$ split into only finitely many $\approx_a$-classes. Next we choose in a uniform way one $\approx_a$-class from each $[\bar{u}]_a^+$ and $[\bar{u}]_a^-$, and obtain a set W'' such that if $(\bar{u}, a) \in W''$ then $[\bar{u}]_a^+ = [\bar{u}]_a^-$. F will be analytic-like on W'' in x_n.

By adding constants to the language we may assume that W and F are 0-definable. By the Cell decomposition Theorem (see [1]) we may assume that F is continuous on W. As before, we let U be the projection of W on the first $n-1$ coordinates.

We denote by $\phi^+(\bar{u}_0, a)$ statement (1) in Lemma 4.2 and let $\phi^-(\bar{u}_0, a)$ be the corresponding statement obtained by replacing $[\bar{u}_0]_z^+$ with $[\bar{u}_0]_z^-$. By (ii) and (iii) of the above lemma, if $(\bar{u}_0, a)$ is generic in W then $\phi^+(\bar{u}_0, a)$ and $\phi^-(\bar{u}_0, a)$ hold and hence there is k such that $\dim([\bar{u}_0]_a^-) = \dim([\bar{u}_0]_a^+) = k$. For every $k \in \{0, \ldots, n-1\}$, define

$$W_k = \{(\bar{u}, x) \in W : \mathcal{M} \models \phi^+(\bar{u}, x) \& \phi^-(\bar{u}, x) \& \dim([\bar{u}]_x^+) = \dim([\bar{u}]_x^-) = k\}.$$

Each W_k is 0-definable and every generic point in W is in one of the W_k's, hence $\dim(W \setminus \bigcup W_i) < n$. We denote by F_k the restriction of F to W_k and let U_k be the projection of W_k onto the first $n-1$ coordinates. Clearly, it will be sufficient to prove the theorem for each F_k and W_k.

Claim. Given $k \in \{0, \ldots, n-1\}$ and $(\bar{u}, a)$ in W_k, then

$$\dim\left(\frac{[\bar{u}]_a^+}{\approx_a}\right) = \dim\left(\frac{[\bar{u}]_a^-}{\approx_a}\right) = 0,$$

where the germs are taken now with respect to F_k and W_k. Namely, the collection of functions which have the same positive (or negative) germ at a splits to only finitely many $\approx_a$-classes, all with respect to F_k and W_k.

Proof. One concern before we prove the lemma is the fact that all notions of germs which appear in the definition of W_k are taken with respect to F and W. We want to replace these with conditions which are taken with respect to F_k and W_k (For example, if the domain of $f_{\bar{u}}$ collapsed from an interval to a point a as we moved from W to W_k then the germ $[\bar{u}]_a^+$ would have a different meaning with respect to W or W_k).

We first argue with respect to F and W. Fix $(\bar{u}, a)$ in W_k and let (a_1, a_2) be an interval we obtain from $\phi^+(\bar{u}, a)$ and $\phi^-(\bar{u}, a)$. Namely, $a_1 < a < a_2$ and, all with respect to F and W, (a_1, a_2) is contained in the domain of $f_{\bar{u}}$ and for all $z \in (a_1, a_2)$ we have $\dim([\bar{u}]_{(a_1,a_2)}) = \dim([\bar{u}]_z^+) = \dim([\bar{u}]_z^-) = k$.

For $a_1' \in (a_1, a_2)$ and $\bar{v} \in [\bar{u}]^+_{a_1'}$, there is $a_1' < a_2' < a_2$ such that $\bar{v} \sim_{(a_1', a_2')} \bar{u}$. So, $(a_1', a_2') \subseteq dom(f_{\bar{v}})$ and if $x \in (a_1', a_2')$ then $[\bar{v}]^+_x = [\bar{u}]^+_x$, $[\bar{v}]^-_x = [\bar{u}]^-_x$ and

$$\dim([\bar{v}]_{(a_1', a_2')}) = \dim([\bar{v}]^+_x) = \dim([\bar{v}]^-_x) = k,$$

hence $(\bar{v}, x) \in W_k$. It folllows that for any $a_1' \in (a_1, a_2)$ we have $(\bar{u}, a_1') \in W_k$. Moreover, we showed that $[\bar{u}]^+_{a_1'}$ (and similarly $[\bar{u}]^-_{a_1'}$) does not change when we move from W to W_k. In particular, $\dim([\bar{u}]_{(a_1,a_2)}) = \dim([\bar{u}]^+_a) = \dim([\bar{v}]^-_a) = k$, where now we take the germs with respect to F_k and W_k.

From now on all the germs we consider are taken with respect to F_k and W_k. We still work with a, a_1 and a_2 as above.

Consider the germ of $\bar{u}$ at a, denoted by $[\bar{u}]_{\approx_a}$. Since $[\bar{u}]_{(a_1,a_2)} \subseteq [\bar{u}]_{\approx_a} \subseteq [\bar{u}]^+_a$, we have $\dim([\bar{u}]_{\approx_a}) = k$. $(\bar{u}, a)$ was an arbitrary element in W_k, hence for every $(\bar{v}, a) \in W_k$ we have $\dim([\bar{v}]_{\approx_a}) = k$. But this means that every $\approx_a$-class which is contained in $[\bar{u}]^+_a$ has the same dimension as that of $[\bar{u}]^+_a$, which implies that there can be only finite many such classes, namely $\dim\left(\frac{[\bar{u}]^+_a}{\approx_a}\right) = 0$. Similarly, $\dim\left(\frac{[\bar{u}]^-_a}{\approx_a}\right) = 0$, which ends the proof of the Claim.

By replacing W with W_k we may assume from now on that W itself satisfies the above claim. We continue to partition W.

As the claim implies,

$$\forall (\bar{u}, a) \in W (\dim\left(\frac{[\bar{u}]^+_a}{\approx_a}\right) = \dim\left(\frac{[\bar{u}]^-_a}{\approx_a}\right) = 0). \tag{3}$$

This can be written as a first order statement, hence should be true in every elementary extension of $\mathcal{M}$. So, by compactness, there is $k \in \mathbb{N}$ such that for every $(\bar{u}, a) \in W$, there are at most k $\approx_a$-classes which are contained in $[\bar{u}]^+_a$. It is easily seen, by o-minimality, that the $\approx_a$-classes in $[\bar{u}]^+_a$ are linearly ordered in a uniform way by the formula

$$[\bar{u}]_{\approx_a} \preceq [\bar{v}]_{\approx_a} \Leftrightarrow \begin{array}{l} \exists a_1 < a (dom(f_{\bar{v}}) \cap (a_1, a) = \emptyset) \text{ or} \\ \exists a_1 < a \forall x \in (a_1, a)(f_{\bar{v}}(x) \leq f_{\bar{u}}(x)) \end{array}.$$

By the same argument, we may assume that for every $(\bar{u}, a) \in W$ there are at most k $\approx_a$-classes in $[\bar{u}]^-_a$ and that we can order them in a uniform way.

We define now the following partition of W: For $i, j \in \{0, \dots, k\}$, let

$$W_{i,j} = \left\{ (\bar{u}, a) \in W : \begin{array}{l} [\bar{u}]_{\approx_a} \text{ is the i-}th \text{ class in } [\bar{u}]^+_a \\ \text{and the j-}th \text{ class in } [\bar{u}]^-_a \end{array} \right\}.$$

Consider the $W_{i,j}$'s of dimension n. It is easy to verify now that if $(\bar{u},a),(\bar{v},a)$ are interior points of $W_{i,j}$ and if $[\bar{u}]^+_a = [\bar{v}]^+_a$ then $\bar{u}{\approx_a}\bar{v}$ (in the sense of $F|W_{i,j}$ and $W_{i,j}$). Also, if $[\bar{u}]^-_a = [\bar{v}]^-_a$ then $\bar{u}{\approx_a}\bar{v}$ (in the sense of $F|W_{i,j}$ and $W_{i,j}$). By restricting ourselves to those $W_{i,j}$'s which have dimension n in M^n, we may replace W with one of the $W_{i,j}$'s. At the final step we partition W into finitely many cells (see [1] for the right definition and theorem). Notice that when we carry out this partition, we do not lose the last property of W that we have established, i.e., if $(\bar{u},a)$ is in one of the open cells then $[\bar{u}]^+_a = [\bar{u}]^-_a = [\bar{u}]_{\approx_a}$ where now the germs are taken with respect to this cell and to the restriction of F to it. Without loss of generality then, W is a cell and for every $(\bar{u},a) \in W$ we have $[\bar{u}]^+_a = [\bar{u}]^-_a = [\bar{u}]_{\approx_a}$. We claim that $F|W$ is analytic-like on W in the n-th variable: Recall that $f_{\bar{u}}(x), f_{\bar{v}}(x)$ are given by the functions $F(\bar{u},x)$ and $F(\bar{v},x)$, respectively. Notice that since W is a cell, the domains of $f_{\bar{u}}$ and of $f_{\bar{v}}$, $I_{\bar{u}}$ and $I_{\bar{v}}$, respectively, are nonempty open intervals and hence also $I_{\bar{u}} \cap I_{\bar{v}}$. Assume that $f_{\bar{u}}(x) = f_{\bar{v}}(x)$ for all x in some nonempty open interval in M. Let $I_{\bar{u}\bar{v}} = \{x \in I_{\bar{u}} \cap I_{\bar{v}} : f_{\bar{u}}(x) = f_{\bar{v}}(x)\}$. By the continuity of F, $I_{\bar{u}\bar{v}}$ is a closed set in $I_{\bar{u}} \cap I_{\bar{v}}$. Since $I_{\bar{u}\bar{v}}$ is infinite it is a finite union of points and nonemepty closed intervals. Assume, towards contradiction, that $I_{\bar{u}} \cap I_{\bar{v}} \neq I_{\bar{u}\bar{v}}$ and let $x_0 \in I_{\bar{u}\bar{v}}$ be an endpoint of a nonempty interval of $I_{\bar{u}\bar{v}}$. If x_0 is a left endpoint of the interval, then $[\bar{v}]^+_{x_0} = [\bar{u}]^+_{x_0}$. But then, by our assumption on W, $\bar{v} \approx_a \bar{u}$, which implies that x_0 is not an interior point of $I_{\bar{u}\bar{v}}$. $I_{\bar{u}\bar{v}}$ cannot have then such a left endpoint of an interval and similarly it cannot have a right endpoint of an interval. But then $I_{\bar{u}\bar{v}} = I_{\bar{u}} \cap I_{\bar{v}}$. □

We can conclude now a similar result to Theorem 2.2, for functions of more then one variable. We need first to introduce some notation. Let $W \subseteq M^n$ be a definable open set and let $G : W \longrightarrow M$ be definable function. For $S = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ and $\bar{u} \in M^{n-r}$, the function G gives rise to a definable function $G_{\bar{u}} : M^r \longrightarrow M$, by: $G_{\bar{u}}(\bar{x}) = G(\bar{y})$, where $y_{i_k} = u_k$ if $i_k \notin S$ and $y_{i_k} = x_k$ if $i_k \in S$. A definable set U is called *definably connected* if it has no definable subset which is open and closed in its relative topology.

Corollary 5.1 *Let $\mathcal{M}$ be an o-minimal structure and assume that $W \subseteq M^n$ is a definable open set. If $F : W \longrightarrow M$ is a definable function then there are definable open sets $W_1, \ldots, W_k \subseteq W$ such that dim$(W \setminus \bigcup_{i=1}^k W_i) < n$, and such that for every i the following holds: (i) $F|W_i$ is a continuous function. (ii) Denote $F|W_i$ by G, and assume that $S = \{j_1, \ldots, j_r\} \subseteq \{x_1, \ldots, x_n\}$, $\bar{u}$*

and $\bar{v}$ are in M^{n-r}. If $G_{\bar{u}}(\bar{x}) = G_{\bar{v}}(\bar{x})$ for all $\bar{x}$ in some nonempty open set $D \subseteq M^r$ then $G_{\bar{u}}$ and $G_{\bar{v}}$ agree on every definably connected open set which contains D and is contained in the common domain of $G_{\bar{u}}$ and $G_{\bar{v}}$. The definition of the W_i's uses no new parameters.

Again, that if $M = \mathbb{R}$ and the $F|W_i$'s are real-analytic functions then the conclusion of the theorem is clearly true (moreover, in that case we can omit the 'definably connected' restriction).

Notice that if F satisfies (ii) for a particular $S \subseteq \{1, \ldots, n\}$ and if $W_1 \subseteq W$ then $F|W_1$ still satisfies (ii) with respect to the same S. So, by applying Theorem 2.2, each time for a different x_i, we may assume F is analytic-like on W in every x_i, $i = 1, \ldots, n$. As we will show, W satisfies (ii).

Let $S \subseteq \{1, \ldots, n\}$, $|S| = r$, and let $\bar{u}, \bar{v}$ be in M^{n-r}. Assume that the functions $F_{\bar{u}}$ and $F_{\bar{v}}$ agree on a nonempty open set $D \subseteq M^r$ and let $D \subseteq C$, where C is an open, definably connected subset of $dom(F_{\bar{u}}) \cap dom(F_{\bar{v}})$. We will show that $F_{\bar{u}}|C = F_{\bar{v}}|C$.

We let $E = \{\bar{x} \in C : F_{\bar{u}}(\bar{x}) = F_{\bar{v}}(\bar{x})\}$. D is contained in the interior of E in C, call it $Int(E)$. It is sufficient to show that $Int(E)$ is a closed subset of C. Let $\bar{x} \in C$ be in the closure of $Int(E)$ and let $R \subseteq C$ be an open rectangular box containing $\bar{x}$. $R \cap Int(E)$ is a nonempty open set, hence it contains an open rectangular box R_1. The functions $F_{\bar{u}}$ and $F_{\bar{v}}$ agree on R_1 and it easily follows that $F_{\bar{u}}$ and $F_{\bar{v}}$ must agree then on all of R, hence $R \subseteq E$ and hence $\bar{x}$ is in $Int(E)$. $\square$

* * *

Theorem 2.2 and Corollary 5.1 eliminate some of the local-global discrepancies that were discussed in the Introduction. However, even as far as definable families of functions go, difficulties still remain. For example, two functions $f_{\bar{u}}(x), f_{\bar{v}}(x)$ which arise from an analytic-like function $F(\bar{u}, x)$ may still agree locally, i.e. on an open interval, yet be different functions globally. However, now this can happen only for the reason that $f_{\bar{u}}$ and $f_{\bar{v}}$ have different domains. A small improvement on that is the following construction which mimics the notion of analytic continuation over the reals (however, this is not a first-order notion!).

We assume that $F : W \longrightarrow M$ is a definable function on $W \subseteq M^n$, an open set, and that U is the projection of W on the first $n-1$ coordinates.

F is analytic-like in x_n on W. We first define on U the following equivalence relation: $\bar{u} \sim \bar{v}$ if there are $\bar{u} = \bar{u}_0, \ldots, \bar{u}_m = \bar{v}$ in U such that for $i = 0, \ldots, m-1$, $I_{\bar{u}_i} \cap I_{\bar{u}_{i+1}} \neq \emptyset$ and $f_{\bar{u}_i} | I_{\bar{u}_i} \cap I_{\bar{u}_{i+1}} = f_{\bar{u}_{i+1}} | I_{\bar{u}_i} \cap I_{\bar{u}_{i+1}}$. Since F is analytic-like in x_n, if $\bar{u} \sim \bar{v}$ and $I_{\bar{u}} \cap I_{\bar{v}} \neq \emptyset$ then $f_{\bar{u}}$ and $f_{\bar{v}}$ agree on every x in $I_{\bar{u}} \cap I_{\bar{v}}$. We associate with the $\sim$-class of $\bar{u} \in U$ a convex subset of M, $D_{\bar{u}}$, and a function $h_{\bar{u}} : D_{\bar{u}} \longrightarrow M$ as follows:

Let $D_{\bar{u}} = \cup\{I_{\bar{v}} : \bar{v} \sim \bar{u}\}$ and define $h_{\bar{u}}(x) = f_{\bar{v}}(x)$ whenever $\bar{u} \sim \bar{v}$ and x is in $I_{\bar{v}}$. $D_{\bar{u}}$ is convex by the definition of $\sim$ and by the previous comment $h_{\bar{u}}$ is well-defined on it. Clearly, $D_{\bar{u}}$ and $h_{\bar{u}}$ depend only on the $\sim$-class of $\bar{u}$ and It is easy to see that for $\bar{u}, \bar{v} \in U$, either $h_{\bar{u}} = h_{\bar{v}}$ (so in particular $D_{\bar{u}} = D_{\bar{v}}$) $h_{\bar{u}}(x) = h_{\bar{v}}(x)$ for only finitely many points x.

$D_{\bar{u}}$ and $h_{\bar{u}}$ are not in general first-order definable. However, in the case that finitely many $I_{\bar{v}}$'s, $\bar{v} \sim \bar{u}$, cover $D_{\bar{u}}$ this set and function will be definable.

Example 5.2 *Let $\mathcal{M}$ be the structure $\langle \mathbb{R}, <, +, \lambda_\pi(x) \rangle$, where λ_π is the restriction of scalar multiplication by π to the the interval $[-1, 1]$. More precisely, $\lambda_\pi(x) = \pi x$ if $-1 \leq x \leq 1$ and is 0 for all other x's. Let*

$$F(u_1, u_2, x) = \lambda_\pi(x - u_1) + u_2.$$

Using the usual notation we have $W = \mathbb{R}^3$ and $U = \mathbb{R}^2$. Let $W_1 = \{(u_1, u_2, x) : |x - u_1| \leq 1\}$ and $W_2 = \mathbb{R}^3 \setminus W_1$. Then $F|W_1, F|W_2$ are both analytic-like in x.

Consider $F|W_1$ and let $\sim$ be as we have just described. We get that $(u_1, u_2) \sim (v_1, v_2)$ iff $\pi u_1 - u_2 = \pi v_1 - v_2$. In particular, $h_{(u_1, \pi u_1)}(x) = \pi x$ and the domain of $h_{(u_1, \pi u_1)}$, i.e. $D_{(u_1, \pi u_1)}$, is all of $\mathbb{R}$. However, one can show that scalar multiplication by π is not definable in $\mathcal{M}$, i.e. $h_{(u_1, \pi u_1)}$ is not definable. If we go to a nonstandard extension of $\mathcal{M}$ and take u_1 finite then $D_{(u_1, \pi u_1)}$ is the collection of all finite elements of $\mathcal{M}$, clearly an undefinable set.

6 Application

In [2], [4] and [5] we used a property of structures which we called "Collapse of Families of functions", or the CF property.

Definition 6.1 A structure $\mathcal{M}$ is said to have *the CF property* if for every definable F, W and U as before and generic $a \in M$ we have $\dim\left(\frac{U}{\approx_a}\right) \leq 1$.

The CF property is a slight weakening of the notion of local modularity which was used in the investigation of stable structures. As is shown in [2], every nontrivial example of such structures will be made up of reducts of ordered vector spaces. In [4] we showed that if $\mathcal{M}$ is a reduct of some o-minimal expansion of $\langle \mathbb{R}, <, +, \cdot \rangle$, $\mathcal{M}$ without the CF property, and if every definable function in $\mathcal{M}$ is piecewise analytic then a real closed field is definable in $\mathcal{M}$. At the end of that paper we announced that one could do away with the assumption that definable functions are piecewise analytic. With Theorem 2.2 at hand we can now omit one of the two usages of the analyticity in [4].

For F, W and U as above and $x, y \in M$, let $U_{xy} = \{\bar{u} \in U : f_{\bar{u}}(x) = y\}$.

Lemma 6.2 *Let $\mathcal{M}$ be an o-minimal structure. Then the following are equivalent.*
(1) $\mathcal{M}$ does not have the CF property.
(2) There is an $\mathcal{M}$-definable family of functions $\mathcal{F} = \{f_{\bar{u}} : \bar{u} \in U\}$ and open intervals $I_1, I_2 \subseteq M$ such that for all $\bar{u} \in U$ $dom(f_{\bar{u}}) = I_1$ and: (i) For every $\bar{u}_1, \bar{u}_2 \in U$, either $f_{\bar{u}_1}|I_1 = f_{\bar{u}_2}|I_1$ or there are at most finitely many points $x \in I_1$ for which $f_{\bar{u}_1}(x) = f_{\bar{u}_2}(x)$. (ii) For every point $(x, y) \in I_1 \times I_2$, there are infinitely many different germs at x among the $f_{\bar{u}}$'s for which $f_{\bar{u}}(x) = y$ (said differently, $dim\left(\frac{U_{xy}}{\approx_x}\right) \geq 1$).

Namely, the negation of the CF property is equivalent to the existence of a definable object similar to a pseudoplane.

Proof. Exactly as in Lemma 3.8 in [4], since we only use there the fact that analytic functions are analytic-like in all variables.

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