The images of definable sets in the torus, and their associated Hausdorff limits.

Kobi Peterzil
(joint with Sergei Starchenko)

Department of Mathematics
University of Haifa

In celebration of Udi Hrushovski's 60's,
"Fields Institute" 2021

The setting:Discrete subgroups of $\mathbb{R}$-vector space

Basics

Let V be an $\mathbb{R}$-vector space of finite dim, $\Gamma \subseteq V$ a discrete (hence closed) subgroup.

- Γ is finitely generated, by $\mathbb{R}$-independent elements.
- The generators form a base for V **iff** V/Γ is compact in the quotient topology.

A lattice and a torus

- **A lattice** Γ in $\mathbb{R}^n$ is a discrete subgroup generated by a basis.
- The torus $\mathbb{T}_\Gamma := \mathbb{R}^n/\Gamma$ is a compact Lie group. The quotient map is $\pi_\Gamma : \mathbb{R}^n \to \mathbb{T}_\Gamma$.

The topology on $\mathbb{T}_\Gamma$

- A subset $Y \subseteq \mathbb{T}_\Gamma$ is closed **iff** $\pi_\Gamma^{-1}(Y)$ is closed in $\mathbb{R}^n$.
- Thus, for $X \subseteq \mathbb{R}^n$, $\pi_\Gamma(X)$ is closed in $\mathbb{T}_\Gamma$ **iff** $X + \Gamma$ is closed in $\mathbb{R}^n$.

The closure problem

We have $\pi_\Gamma : \mathbb{R}^n \to \mathbb{T}_\Gamma$

Problem I

Assume that $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure, describe $cl(\pi_\Gamma(X))$ in $\mathbb{T}_\Gamma$.

An important special case: a linear space

Assume $L \subseteq \mathbb{R}^n$ is a linear subspace, then

- $\pi_\Gamma(L)$ is closed in $\mathbb{T}_\Gamma$ **iff** L has a basis in $L \cap \Gamma$ (**iff** $L + \Gamma$ is closed in $\mathbb{R}^n$)
- So, the closure of $\pi_\Gamma(L)$ is obtained as follows: Let L^Γ be the smallest linear space $\supseteq L$ with a basis in Γ, then $cl(\pi_\Gamma(L)) = \pi_\Gamma(L^\Gamma)$.
- E.g. $\Gamma = \mathbb{Z}^3 \subseteq \mathbb{R}^3$. if $L = sp_\mathbb{R}(1, \sqrt{2}, -1)$ then $L^\Gamma = sp_\mathbb{R}\{(1,0,-1),(0,1,0)\}$ and $cl(\pi(L)) = \pi(L^\Gamma)$.

An answer to the closure problem

Previous work with S. Starchenko

Given $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure, we associate to every complete type over $\mathbb{R}$, $p \vdash X$, an $\mathbb{R}$-affine "nearest coset" $L_p + a_p \subseteq \mathbb{R}^n$, such that for every lattice $\Gamma \subseteq \mathbb{R}^n$,

$$cl(\pi_\Gamma(X)) = \bigcup_{p \vdash X} \pi_\Gamma(L_p^\Gamma + a_p) = \pi_\Gamma(Y)$$

In fact there is a definable set $Y \subseteq \mathbb{R}^n$ such that

Some model theory

Let $\Gamma \subseteq \mathbb{R}^n$ a lattice, $X \subseteq \mathbb{R}^n$ **arbitrary** and let

$$\langle \mathbb{R}; <, +, \cdot, X, \Gamma \rangle \prec \langle R^*; <, +, \cdot, X^*, \Gamma^* \rangle.$$

Then

- Then $cl(X) = st(X^* \cap O^n)$, where

$$O = \{x \in R^* : \exists r \in \mathbb{R}\ |x| \leqslant r\}.$$

- Thus

$$cl(X + \Gamma) = st((X^* + \Gamma^*) \cap O^n) = \bigcup_{p \vdash X} st((p(R^*) + \Gamma^*) \cap O^n).$$

In the o-minimal setting, using v.d.Dries-Lewenberg, we find a definable $Y \subseteq \mathbb{R}^n$ such that $cl(X + \Gamma) = \pi_\Gamma(Y)$

Hausdorff limits

Definition

Assume that $(X_k)_{k\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^n$. A set $Y \subseteq \mathbb{R}^n$ is **a Hausdorff limit** of (X_k) if for every $R > 0$ and $\epsilon > 0$, there exists N such that for all $k \geqslant N$, inside the ball $||x|| < R$ we have

$$Y \subseteq B(X_k;\epsilon) \text{ and } X_k \subseteq B(Y;\epsilon).$$

If Y_1, Y_2 are closed Hausdorff limits of (X_k) then $Y_1 = Y_2$.

From now on, all Hausdorff limits are assumed to be closed.

Hausdorff limits and model theory

Recommended

“Limit sets in o-minimal structures”, v.d. Dries, Proceedings of the RAAG Summer school i Lisbon, 2003

Non-standard view of Hausdorff limits

Assume that $\{X_t : t \in T\}$ is a definable family of subsets of $\mathbb{R}^n$ in **some** structure $\mathcal{M}$ on $\mathbb{R}$. Let $\mathcal{M} \prec \mathcal{M}^*$ be an $|\mathbb{R}|^+$-saturated extension. Then, a closed set $Y \subseteq \mathbb{R}^n$ is a Hausdorff limit of some sequence X_{t_n}, $t_n \in T$, **iff** there exists $\alpha \in T^*$ such that $Y = st(X_\alpha \cap O^n)$.

Related theorem by v.d. Dries, using definability of types

If $\mathcal{F} = \{X_t : t \in T\}$ be a family of subsets of $\mathbb{R}^n$ definable in an o-minimal structure $\mathcal{M}$. Then the family of all Hausdorff limits of sequences from $\mathcal{F}$ is itself definable in $\mathcal{M}$.

An example:

Back to lattices: $\pi_\Gamma : \mathbb{R}^n \to \mathbb{T}_\Gamma = \mathbb{R}^n/\Gamma$

Several article in dynamical systems study families given by **dilations** e.g. $\{tX : t \in (0, \infty)\}$ of a set in $X \subseteq \mathbb{R}^n$ and more generally on nilmanifolds.
(Randol(1984), Bjorklund and Fish (2009), Kra, Shah and Sun (2017))

Their goal: Give conditions under which a sequence of measures μ_{t_n} on $\mathbb{T}_\Gamma$, associated to $\pi_\Gamma(t_n X)$, converges to the Haar measure on $\mathbb{T}_\Gamma$.

Remark: If μ_{t_n} converges to the Haar measure on $\mathbb{T}_\Gamma$ then the Hausdorff limit of $\pi_\Gamma(X_{t_n})$ equals $\mathbb{T}_\Gamma$.

A question (A. Nevo)

Assume that $\{X_t : t \in (0, \infty)\}$ is **any** definable family of subsets of $\mathbb{R}^n$ in an o-minimal structure.

Describe the family of Hausdorff limits of $\pi_\Gamma(X_{t_n}) \subseteq \mathbb{T}_\Gamma$, as $t_n \to \infty$?

Example:

Let $\mathcal{R}$ be an o-minimal structure over $\mathbb{R}$.

Theorem 1 (P-Starchenko)

Let $\{X_t : t \in (0,\infty)\}$ be an $\mathcal{R}$-definable family of subsets of $\mathbb{R}^n$.
Then there are $\mathbb{R}$-linear spaces $L_1,\ldots,L_s \subseteq \mathbb{R}^n$, definable compact sets $K_1,\ldots,K_s \subseteq \mathbb{R}^n$ and functions $a_1,\ldots,a_s : (0,\infty) \to \mathbb{R}^n$, such that for all sufficiently large t,

$$X_t \subseteq \bigcup_{j=1}^{s} L_j + K_j + a_j(t),$$

and in addition, **for every lattice** $\Gamma \subseteq \mathbb{R}^n$,
(i) if $L_j^\Gamma = \mathbb{R}^n$ for some $j = 1,\ldots,s$ then for every sequence $t_n \to \infty$,

$$\lim_{n\to\infty} \pi_\Gamma(X_{t_n}) = \mathbb{T}_\Gamma.$$

(ii) if for all j, $L_j^\Gamma \neq \mathbb{R}^n$, then for all large enough $K \in \mathbb{N}$, every Hausdorff limit of the family $\{\pi_{K\Gamma}(X_t) : t \in (0,\infty)\}$ is a proper subsets of $\mathbb{T}_{K\Gamma}$.

The collection of **all** Hausdorff limits

Theorem 2 (P-Starchenko)

Let $\{X_t : t \in (0,\infty)\}$ be an $\mathcal{R}$-definable family of subsets of $\mathbb{R}^n$. Then there are $\mathbb{R}$-linear spaces $L_1,\dots,L_k \subseteq \mathbb{R}^n$, definable sets $Y_1,\dots,Y_k \subseteq \mathbb{R}^n$ and functions $a_1,\dots,a_k : (0,\infty) \to \mathbb{R}^n$, such that **for every lattice** $\Gamma \subseteq \mathbb{R}^n$, and every closed $Z \subseteq \mathbb{T}_\Gamma$ the following are equivalent:

1. $Z \subseteq \mathbb{T}_\Gamma$ is a Hausdorff limit of a sequence $(\pi_\Gamma(X_{t_n}))_n$, for some sequence $t_n \to \infty$.
2. $$Z = \bigcup_{j=1}^{k} \pi_\Gamma(Y_j) + \pi_\Gamma(L_j^\Gamma) + \lim_{n\to\infty} \pi_\Gamma(a_j(s_n)),$$

   for some sequence $s_n \to \infty$.

The connection to model theory

Let $\langle \mathcal{R}, \Gamma \rangle \prec \langle \mathcal{R}^*, \Gamma^* \rangle$.

- As we noted, every Hausdorff limit of $\{\pi_\Gamma(X_t) : t \in (0, \infty)\}$ can be obtained as follows: For $\alpha >> 0$ in R^*,

$$Z = st(\pi_{\Gamma^*}(X_\alpha) = \pi_\Gamma(st((X_\alpha + \Gamma^*) \cap O^n)).$$

- We now consider complete types $p \vdash X_\alpha$ over $\mathbb{R}\langle \alpha \rangle$ and associate to each such type a coset of the form $L_p + b_p$, where $L_p \subseteq \mathbb{R}^n$ is $\mathbb{R}$-linear and $b_p \in \mathbb{R}\langle \alpha \rangle$.
- The main observation: Each type $p \vdash X_\alpha$, contributes to Z a coset $\pi_\Gamma(L_p^\Gamma) + \pi_\Gamma(c_p)$, with $c_p \in st((b_p + \Gamma^*) \cap O^n)$.

Some comments on the theorem

- If $Z_1, Z_2 \subseteq \mathbb{T}_\Gamma$ are two Hausdorff limits as above then, up to a finite partition, Z_1 and Z_2 are translates of each other.
- Every Hausdorff limit is of the form $\pi(W)$ for an $\mathcal{R}$-definable $W \subseteq \mathbb{R}^n$. In fact, we can find an $\mathcal{R}$-definable $D \subseteq \mathbb{R}^k$ such that for every $\Gamma \subseteq \mathbb{R}^n$, the following are equivalent:
  1. $Z \subseteq \mathbb{T}_\Gamma$ is a (closed) Hausdorff family of the family $(\pi_\Gamma(X_t))_t$
  2. There is $(b_1, \ldots, b_k) \in D$ such that

$$Z = \bigcup_{j=1}^{k} \pi_\Gamma(Y_j) + \pi_\Gamma(L_j^\Gamma) + b_j.$$

- One may recover the topological content some of the dynamical systems results on dilations.