Tarski Lecture III

From closure to Hausdorff limits in tori (and nilmanifolds)

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Generalizing the closure problem

In the first two talks we discussed:

The closure problem

Given $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what is $\mathrm{cl}(\pi(X))$ in $\mathbb{T} = \mathbb{R}^n/\Lambda$?
The answer used linear spaces associated to complete types over $\mathbb{R}$, on X.

We want to extend the result in two directions:

From closure to Hausdorff limits

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The Hausdofff limits of a definable family

Question

Let $\{X_t : t \in T\}$ *be a family of subsets of* $\mathbb{R}^n$ *definable in an o-minimal structure on* $\mathbb{R}$. *For a lattice* $\Lambda \subseteq \mathbb{R}^n$, *we consider the possible* *Hausdorff limits* *of the family* $\{\pi_\Lambda(X_t) : t \in T\}$ *in* $\mathbb{T}_\Lambda$.

When are some (or all) Hausdorff limits equal to $\mathbb{T}_\Lambda$*?*

Hausdorff distance and limit

Definition

Given a metric space (M, d), and $X, Y \subseteq M$,

$$d_H(X, Y) = \inf\{\epsilon > 0 : X \subseteq Y^\epsilon \; Y \subseteq X^\epsilon\}, \text{where}$$
$$Y^\epsilon = \{x \in M : d(x, Y) \leqslant \epsilon\}.$$

We have $d_H(X, Y) = 0 \Leftrightarrow \mathrm{cl}(X) = \mathrm{cl}(Y)$.
Also, d_H is a metric on the collection of compact subsets of M.

Definition

Given a family $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ of relatively compact subsets of M, we say that a compact set $Y \subseteq M$ is a Hausdorff limit at ∞ of $\mathcal{F}$ if there is an unbounded sequence $t_n \in (0, \infty)$, such that $\lim_{n\to\infty} d_H(\mathrm{cl}(X_{t_n}), Y) = 0$.

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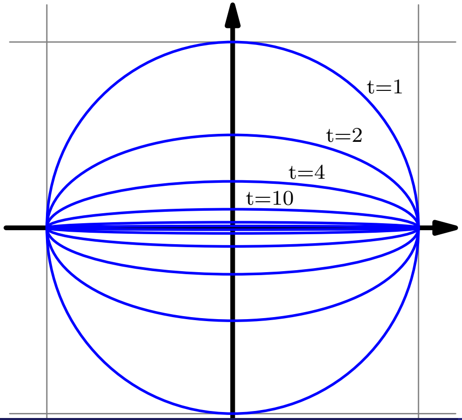
An Example

A family of ellipses

$$\mathcal{F}: \; X_t = \{(x,y) : x^2 + (ty)^2 = 1\}, \;\; t \in [1,,\infty).$$

The (unique) Hausdroff limit at ∞ is the interval $[-1,1] \times \{0\}$.



The new question

A question (A. Nevo)

Assume that $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ is a definable family of subsets of $\mathbb{R}^n$ in an o-minimal structure, and $\Lambda \subseteq \mathbb{R}^n$ a lattice,
Describe the family of Hausdorff limits of
$\pi_\Lambda(\mathcal{F}) := \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$ at ∞ inside $\mathbb{T}_\Lambda$.
In particular, when is $\mathbb{T}_\Lambda$ the unique Hausdorff limit at ∞, of $\pi_\Lambda(\mathcal{F})$?

Notice that the closure problem is a special case of the above (for $X \subseteq \mathbb{R}^N$, consider the constant family $X_t = X$, for all $t \in (0, \infty)$.

As in the closure problem, we may study the problem inside the fundamental domain $F_\Lambda \subseteq \mathbb{R}^n$.

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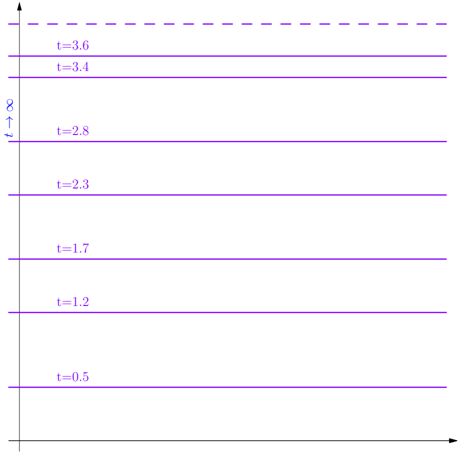
Horizontal lines in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Let $\mathcal{F} : X_t = \mathbb{R} \times \{t\}$, $t \in (0, \infty)$.



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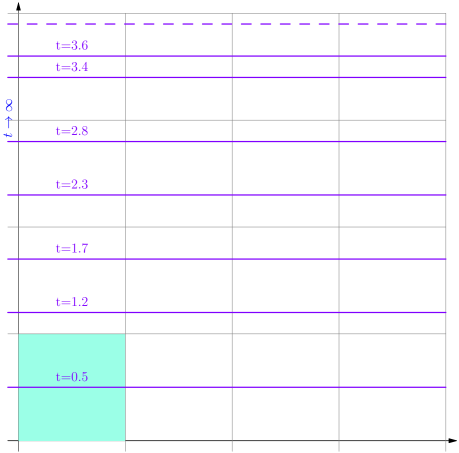
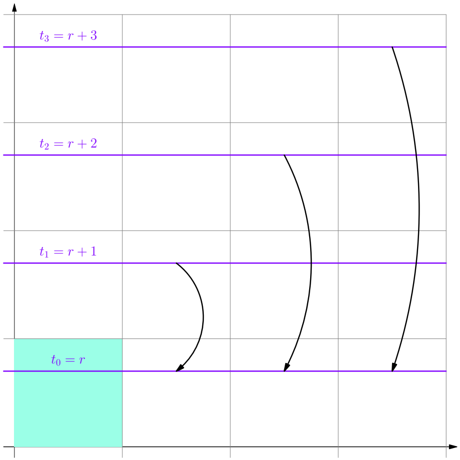


Consider $\pi_\Lambda(\mathcal{F}) = \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$.

Horizontal lines in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

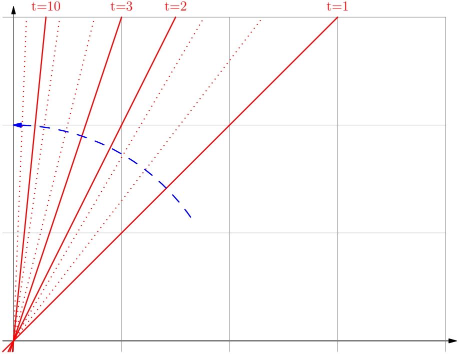
For $L = \mathbb{R} \times \{0\}$, the Hausdorff limits at ∞ are exactly the cosets of $\pi_\Lambda(L)$ in $\mathbb{T}_\Lambda$.



For each $r \in \mathbb{R}$, the sequence $(\pi_\Lambda(X_{r+n}))_{n=0}^{\infty}$ is constant.

Lines of increasing slope in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Consider the family $\mathcal{F} : X_t = \{(x, tx) : t \in (0, \infty)\}$.



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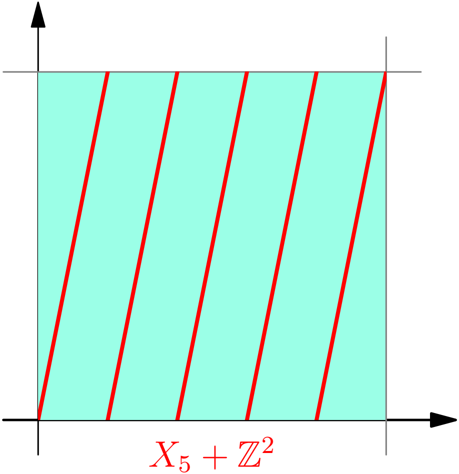
The (unique) Hausdorff limit of $\{\pi_\Lambda(X_t) : t \in (\infty)\}$ at ∞ is $\mathbb{T}_\Lambda$. This remains true for every lattice.



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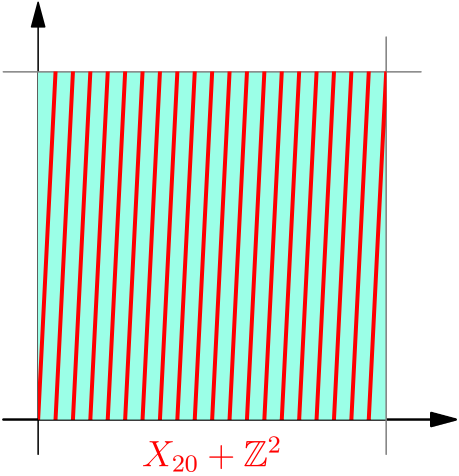

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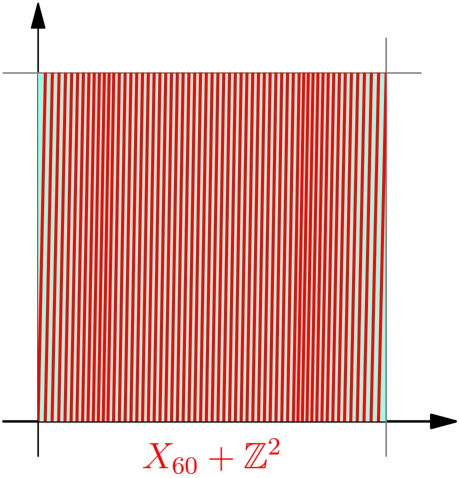
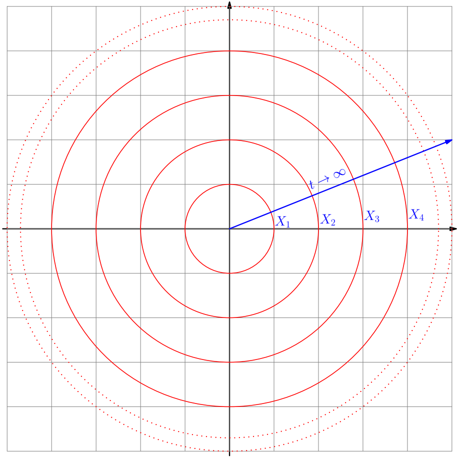


Circles of increasing radius in $\mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

Consider the family $X_t = \{(x, y) : x^2 + y^2 = t^2\}$, for $t \in (0, \infty)$.



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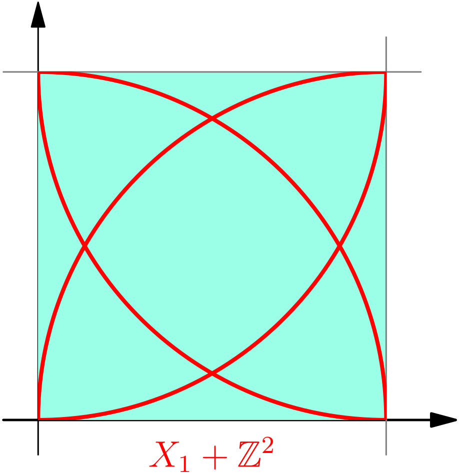

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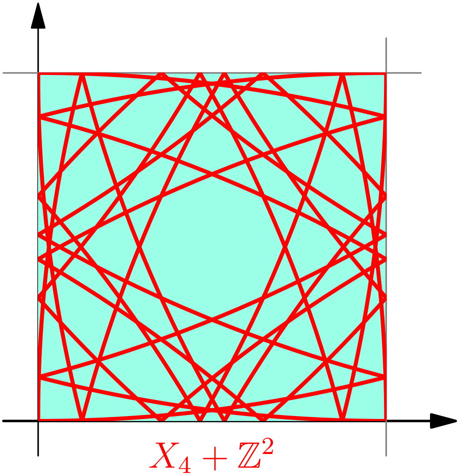

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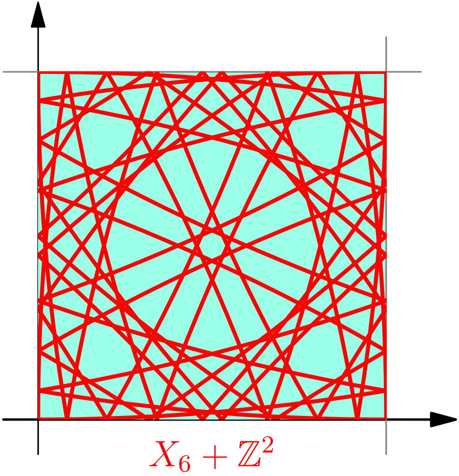

A model theoretic approach

We let $\mathcal{R} \succ \mathbb{R}_{full}$, $\mathcal{O} \subseteq \mathcal{R}$ the ring of finite elements, μ the ideal of infinitesimals in $\mathcal{O}$ and $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ the standard part map.
For $S \subseteq \mathbb{R}^n$, we let $S^\sharp = S(\mathcal{R})$ and $\mathrm{st}(S^\sharp) = \mathrm{st}(S^\sharp \cap \mathcal{O}^n)$

Fact (based on L. Narens, 1972)

Let $\{X_t : t \in (0, \infty)\}$ be a family of subsets of $\mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^n$ a lattice. For a compact $Y \subseteq \mathbb{T}_\Lambda$, the following are equivalent

1. *Y is a Hausdorff limit at ∞ of $\{\pi_\Lambda(X_t) : t \in (0, \infty)\}$.*
2. *There is $\xi > \mathbb{R}$ such that*

$$Y = \pi_\Lambda(\mathrm{st}(X_\xi^\sharp + \Lambda^\sharp)).$$

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Summary-Closure vs. Hausdorff limit

The closure of X

We started with $X \subseteq \mathbb{R}^n$ defined over $\mathbb{R}$ and then $\mathrm{cl}(X + \Lambda) = \mathrm{st}(X^\sharp + \Lambda^\sharp)$.

The Hausdorff limits of $\{X_t : t \in (0, \infty)\}$

For each non-standard $\xi > \mathbb{R}$, $\mathrm{st}(X^\sharp_\xi + \Lambda^\sharp)$ is a Hausdorff limit at ∞.

Again, we may partition into types but now not over $\mathbb{R}$, but over $\mathbb{R}\langle\xi\rangle$, the o-minimal structure generated by $\mathbb{R}$ and ξ.
For simplicity, below let $\mathcal{X} = X^\sharp_\xi$.

$$\mathrm{st}(\mathcal{X} + \Lambda^\sharp) = \bigcup_{p \in S_{\mathcal{X}}(\mathbb{R}\langle\xi\rangle)} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$$

Here, $S_{\mathcal{X}}(\mathbb{R}\langle\xi\rangle) =$ the o-minimal types on $\mathcal{X} = X^\sharp_\xi$, over $\mathbb{R}\langle\xi\rangle$.
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Nearest coset to a type

Proposition

For a type $p \in S_n(\mathbb{R}\langle\xi\rangle)$, there is a smallest linear subspace $L_p \subseteq \mathbb{R}^n$, and $\alpha \in \mathbb{R}\langle\xi\rangle$, such that $p(\mathcal{R}) \subseteq \mu + \alpha + L_p^\sharp$.

We call such translate $\alpha + L_p$ a nearest coset of p.

Theorem (Λ-linearity of types)

Assume that $p(x) \in S_n(\mathbb{R}\langle\xi\rangle)$, and $a_p + L_p$ is a nearest coset of p. Then, for every lattice $\Lambda \subseteq \mathbb{R}^n$ we have

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The uniform Hausdorff limits theorem

Theorem

Let $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ be an $\mathbb{R}_{om}$-definable family of subsets of $\mathbb{R}^n$.

Then there are $\mathbb{R}$-linear spaces $L_1, \ldots, L_s \subseteq \mathbb{R}^n$, such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

1. If $L_j^\Lambda = \mathbb{R}^n$ for some $j = 1, \ldots, s$ then $\mathbb{T}_\Lambda$ is the only Hausdroff limit at ∞ of $\pi_\Lambda(\mathcal{F}) := \{\pi_\Lambda(X_t) : t \in (0, \infty)\}$. This remains true if Λ is replaced by a finite index subgroup.
   Call it $\mathcal{F}$ is strongly Λ-dense in $\mathbb{T}_\Lambda$.
2. If $L_j^\Lambda \neq \mathbb{R}^n$ for all $j = 1, \ldots, s$, then exists $K \in \mathbb{N}$ such that for every subgroup $\Lambda_0 \subseteq K \cdot \Lambda$, no Hausdorff limit at ∞ of $\pi_{\Lambda_0}(\mathcal{F})$ is equal to $\mathbb{T}_{\Lambda_0}$.

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The uniform Hausdorff limits theorem

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Let $\mathcal{F} = \{X_t : t \in (0, \infty)\}$ be an $\mathbb{R}_{om}$-definable family of subsets of $\mathbb{R}^n$.

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Example

$\mathcal{F} : X_t = t + [0, 2] \subseteq \mathbb{R}\, , t \in (0, \infty)$.

Since all X_t are bounded, the only associated nearest coset is $L = \{0\}$, so $L^{\mathbb{Z}} = \{0\} \neq \mathbb{R}$.

Still, for $\Lambda = \mathbb{Z}$¡ or $\Lambda = 2\mathbb{Z}$, for all t, $\pi_\Lambda(X_t) = \mathbb{T}_\Lambda$ (so all Hausdroff limits equal $\mathbb{T}_\Lambda$).

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Nilmanifolds

We denote by $U_n(\mathbb{R})$ the group of real upper-triangular $n \times n$ matrices with 1's on the main diagonal.

By a unipotent group we mean a real algebraic subgroup $G \subseteq U_n(\mathbb{R})$. It is a nilpotent group and when abelian $G \cong (\mathbb{R}^k, +)$, for some k.

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The nilmanifold case-reduction to the abelian case

- Let $G_{ab} := G/[G, G]$, an abelian group.
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Further problems

- Uniformity of the closure problem, in parameters.
- Equidistribution for $X \subseteq \mathbb{R}^n$ with $\dim X > 1$.
- Describe explicitly all the family of Hausdorff limits of a definable family $\mathcal{F}$ (we only knew then the family is strongly Λ-dense in $\mathbb{T}_\Lambda$).
- Replace unipotent groups by reductive groups: E.g. the closure problem for definable subsets of $SL(n, \mathbb{R})$ and quotients by lattices.