Tarski Lecture II

The interplay of o-minimality and discrete subgroups

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Recalling the problem

Fix $\mathbb{R}_{\mathrm{om}}$ an o-minimal expansion of the real field.

The closure problem

Given a definable set $X \subseteq \mathbb{R}^n$ in $\mathbb{R}_{\mathrm{om}}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what is the topological closure of $\pi(X)$ in $\mathbb{T} = \mathbb{R}^n/\Lambda$?

Equivalently, what is the closure of $X + \Lambda$ in $\mathbb{R}^n$?

We fixed $\mathcal{R} \succ \mathbb{R}_{full}$ which is $|\mathbb{R}|^+$-saturated. For $S \subseteq \mathbb{R}^n$ let $S^\sharp = S(\mathcal{R})$.

Let $\mathcal{O} = \{\alpha \in \mathcal{R} : \exists r \in \mathbb{R}\ |\alpha| < r\}$, the ring of finite elements,
and $\mu = \{\epsilon \in \mathcal{R} : \forall r \in \mathbb{R}\ |\epsilon| < r\}$, the ideal of infinitesimals.

We have $\mathcal{O} = \mathbb{R} \oplus \mu$ and denote by $\mathrm{st} \colon \mathcal{O} \to \mathbb{R}$ the “standard part map”.

Fact. For $S \subseteq \mathbb{R}^n$ we have $\mathrm{cl}(S) = \mathrm{st}(S^\sharp) := \mathrm{st}(S^\sharp \cap \mathcal{O}^n)$.

The non-standard formulation

What is $\mathrm{st}(X^\sharp + \Lambda^\sharp) \subseteq \mathbb{R}^n$?

Reducing the problem

We also have, $\mathrm{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$.

Localizing the problem

For a complete o-minimal type $p \in S_n(\mathbb{R})$ and a lattice $\Lambda \subseteq \mathbb{R}^n$ describe the set $\mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$.

Remark

For any set $Y \subseteq \mathcal{R}^n$ we have $\mathrm{st}(Y) = \mathrm{st}(\mu + Y)$.

Theorem (Λ-linearity of types)

For any complete o-minimal type $p \in S_n(\mathbb{R})$ there are $a_p \in \mathbb{R}^n$ and a linear subspace $L_p \subseteq \mathbb{R}^n$ such that for any lattice $\Lambda \subseteq \mathbb{R}^n$ we have

$$\mu + p(\mathcal{R}) + \Lambda^\sharp = \mu + a_p + L_p^\sharp + \Lambda^\sharp.$$

O-minimality vs. discrete subgroups

Recall

A structure on $\mathbb{R}$ is o-minimal if every definable subset of $\mathbb{R}$ is a finite union of intervals with end points in $\mathbb{R} \cup \{\pm\infty\}$.

In particular, every definable discrete subset of $\mathbb{R}^n$ is finite.

Thus, a lattice $\Lambda \subseteq \mathbb{R}^n$, and $\pi : \mathbb{R}^n \to \mathbb{T} = \mathbb{R}^n/\Lambda$ are **not** definable in any o-minimal structure.

So, in general, the set $X + \Lambda$ is not definable in any o-minimal structure.

How can we use o-minimality?

"Linearize" $\mu + p(\mathcal{R})$ independently of Λ.

μ-stabilizers of o-minimal types play major role.

For simplicity, we mostly consider one-dimensional types, i.e. types on o-minimal curves at ∞.

O-minimal detour I: one dimensional types

Let $\gamma : (0, \infty) \to \mathbb{R}^n$ be an $\mathbb{R}_{\text{om}}$-definable curve.

For any $\mathbb{R}_{\text{om}}$-definable set $X \subseteq \mathbb{R}^n$ exactly one of the sets $\{t \in (0, \infty) : \gamma(t) \in X\}$ for $\{t \in (0, \infty) : \gamma(t) \in \neg X\}$ is unbounded.

Thus there is a unique complete $\mathbb{R}_{\text{om}}$-type over $\mathbb{R}$, containing all sets $\{\gamma(t) : t > r\}$, for $r \in \mathbb{R}$.

Linearizing $\mu + p(\mathcal{R})$: first step

For $p \in S_n(\mathbb{R})$, we let

$$\mathrm{Stab}_\mu(p) = \{g \in \mathbb{R}^n : g + \mu + p(\mathcal{R}) = \mu + p(\mathcal{R})\},$$

and call it the μ-stabilizer of p.

Theorem (2015)

1. $\mathrm{Stab}_\mu(p)$ is a definable subgroup (linear subspace) of $\mathbb{R}^n$.
2. If p is unbounded then $\dim(\mathrm{Stab}_\mu(p)) > 0$.

An analogue holds for **any** definable group in $\mathbb{R}_{\mathrm{om}}$.

The intuition

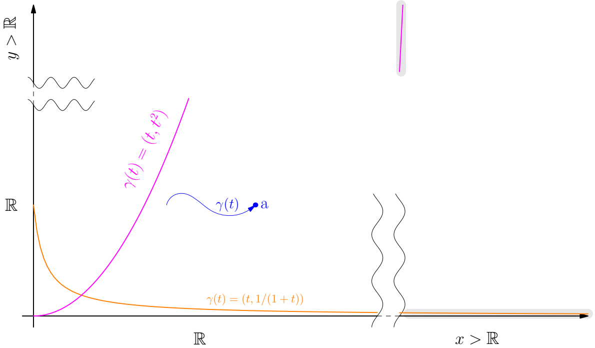
Unbounded o-minimal types are almost “flat”.

Example

Example

Below p is the type on a curve γ at ∞

- When $\lim_{t\to\infty}\gamma(t) = a$, for $a \in \mathbb{R}^2$, then $\mathrm{Stab}_\mu(p) = \{0\}$.
- When $\gamma(t) = (t, 1/(t+1))$, then $\mathrm{Stab}_\mu(p) = \mathbb{R} \times \{0\}$.
- When $\gamma(t) = (t, t^2)$, then $\mathrm{Stab}_\mu(p) = \{0\} \times \mathbb{R}$.



Linearizing $\mu + p(\mathcal{R})$: The nearest coset

For a type $p \in S_n(\mathbb{R})$, consider all affine subspaces, $A = a + L \subseteq \mathbb{R}^n$ (L a linear subspace) defined over $\mathbb{R}$, such that $p(\mathcal{R}) \subseteq \mu + A^\sharp$.

Definition+Fact

The intersection A_p of all the above affine spaces is itself an affine space defined over $\mathbb{R}$, and $p(\mathcal{R}) \subseteq \mu + A_p^\sharp$. We call it the nearest coset to p and denote by A_p.

Note

For a type $p \in S_n(\mathbb{R})$, A_p is invariant under $\mathrm{Stab}_\mu(p)$.

- Indeed, if $g \in \mathrm{Stab}_\mu(p)$, $g + \mu + p(\mathcal{R}) = \mu + p(\mathcal{R})$.
- Since $p(\mathcal{R}) \subseteq \mu + A_p^\sharp$, we have

$$\mu + p(\mathcal{R}) = g + \mu + p(\mathcal{R}) \subseteq g + \mu + A_p^\sharp = \mu + (g + A_p)^\sharp.$$

- From the minimality of A_p, we conclude $g + A_p = A_p$. □

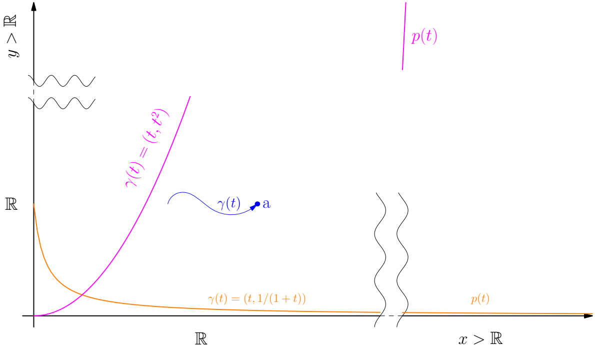
Example:

Below p is the type on a curve γ at ∞

When $\lim_{t\to\infty}\gamma(t) = a$, for $a \in \mathbb{R}^2$, then $A_p = \{0\}$.

When $\gamma(t) = (t, 1/(t+1))$, then $A_p = \mathbb{R} \times \{a\}$.

When $\gamma(t) = (t, t^2)$, then $A_p = \mathbb{R}^2$.



Notice: for $\alpha \models p$ we have $\alpha + \mathrm{Stab}_\mu(p) \subseteq \mu + p(\mathcal{R}) \subseteq \mu + A_p^\sharp$.

O-minimal types are linear mod lattices

Theorem (Λ-linearity of types)

Let $p \in S_n(\mathbb{R})$, with $A_p = a_p + L_p$. Then for every lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mu + p(\mathcal{R}) + \Lambda^\sharp = \mu + a_p + L_p^\sharp + \Lambda^\sharp.$$

Proof.

- If $p(\mathcal{R})$ is bounded, i.e. $p(\mathcal{R}) \subseteq a + \mu$ for some $a \in \mathbb{R}^n$, then $A_p = \{a\}$ and the theorem follows.
- Asume $p(\mathcal{R})$ is unbounded. Then $\mathrm{Stab}_\mu(p)$ is infinite, and both sides are invariant under $\mathrm{Stab}_\mu(p)$.
- Use factorization by $\mathrm{Stab}_\mu(p)^\Lambda$ to reduce dimension.

□

Back to the closure theorem

Corollary (First Main Step)

If $X \subseteq \mathbb{R}^n$ *is definable then for any lattice* $\Lambda \subseteq \mathbb{R}^n$*,*

$$\mathrm{cl}(X+\Lambda) = \mathrm{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp) =$$

$$\bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(a_p + L_p^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{cl}(a_p + L_p + \Lambda) =$$

$$\bigcup_{p \in S_X(\mathbb{R})} (a_p + L_p^\Lambda + \Lambda).$$

Next step:

Simplify

$$\bigcup_{p \in S_X(\mathbb{R})} (a_p + L_p^\Lambda + \Lambda).$$

The definability of the collection of nearest cosets

Let $X \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}_{\mathrm{om}}$. Applying the theory of Tame Pairs (v.d. Dries), we obtain

Theorem

The family of nearest cosets $\{A_p : p \in S_X(\mathbb{R})\}$ is definable in $\mathbb{R}_{\mathrm{om}}$.

Corollary (Definability)

There is a definable family of affine subspaces of $\mathbb{R}^n$, $\{a_t + L_t : t \in T\}$ *(depending only on* X*) such that for any lattice* $\Lambda \subseteq \mathbb{R}^n$,

$$\mathrm{cl}(X + \Lambda) = \bigcup_{t \in T} (a_t + L_t^{\Lambda} + \Lambda).$$

The closure theorem

Using Baire Category Theorem and o-minimal cell decomposition, we conclude:

Theorem

Let $X \subseteq \mathbb{R}^n$ *be closed, definable in* $\mathbb{R}_{\text{om}}$.
Then there are infinite $\mathbb{R}$*-subspaces* $L_1, \ldots, L_r \subseteq \mathbb{R}^n$*, and definable closed sets* $C_1, \ldots, C_r \subseteq \mathbb{R}^n$ *such that for every lattice* $\Lambda \subseteq \mathbb{R}^n$

$$\mathrm{cl}_{\mathbb{R}^n}(X + \Lambda) = \left[X \cup \bigcup_{i=1}^{r} (L_i^{\Lambda} + C_i)\right] + \Lambda.$$

Hence for $\pi : \mathbb{R}^n \to \mathbb{T} = \mathbb{R}^n/\Lambda$ *we have*

$$\mathrm{cl}_{\mathbb{T}}(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^{r} (T_i + \pi(C_i)),$$

where $T_i = \pi(L_i^{\Lambda})$ *are real subtori of* $\mathbb{T}$.

A curious corollary

Corollary

Given $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{\mathrm{om}}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, there is an $\mathbb{R}_{\mathrm{om}}$-definable set $Y_\Lambda \subseteq \mathbb{R}^n$, such that

$$\mathrm{cl}(\pi(X)) = \pi(Y_\Lambda).$$

Addendum: on equidistribution

Let $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ be a torus and $\pi\colon \mathbb{R}^n \to \mathbb{T}$ the projection map.

Let $\gamma(t)\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^n$ be a definable curve in $\mathbb{R}_{\mathrm{om}}$, and for $R \geqslant 0$ let $\gamma_R = \gamma \cap B(0, R)$.

For $X \subseteq \mathbb{T}$ let $\mu_{\gamma,R}(X) = \dfrac{\text{length of } (\gamma_R \cap \pi^{-1}(X))}{\text{length of } \gamma_R}$.

Each $\mu_{\gamma,R}$ is a probability measure on $\mathbb{T}$.

Theorem (P-S, Ulmo-Yafaev for semialgebraic curves)

Assume $\mathbb{R}_{\mathrm{om}}$ is polynomially bounded.
Then $\mathrm{cl}_{\mathbb{T}}(\pi(\gamma)) = \mathbb{T}$ ***if and only if***

$$\lim_{R\to\infty} \mu_{\gamma,R} = \mu_{\mathbb{T}},$$

where $\mu_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$.

Namely, $\pi(\gamma)$ is dense in $\mathbb{T}$ iff it is "equidistributed" in $\mathbb{T}$.

On Equidistribution

Equidistribution fails in general o-minimal structures.

Example

Let $\gamma(t)\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}^2$ be given by $x = t$, $y = e^t$.
Then $\pi(\gamma)$ is dense in $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$.

But, the family of measures $\mu_{\gamma,R}$ does not converge (as R goes to infinity).