Tarski lectures 2024
UC Berkeley

Kobi Peterzil
U. of Haifa

Sergei Starchenko
U. of Notre Dame

April 2024

Plan of talks

Lecture I: Closures and flows in real tori: a model theoretic approach

Lecture II: The interplay of o-minimality and discrete groups

Lecture III: From closures to Hausdorff limits, in tori and nilmanifolds

Lecture I

Closures and flows in real tori: a model theoretic approach

Kobi Peterzil
U. of Haifa

Sergei Starchenko
U. of Notre Dame

April 22, 2024

The closure problem

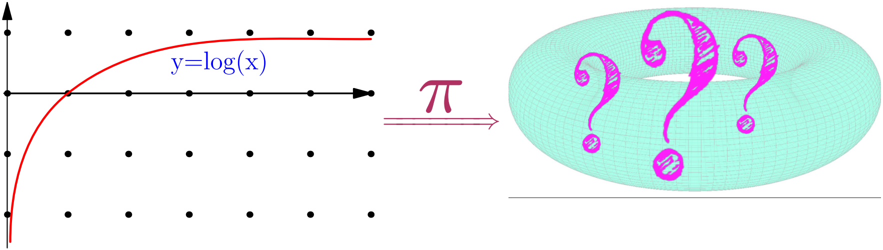
The general question

Let $\mathbb{T}$ be an abelian variety or a compact torus.

For $K = \mathbb{R}$ or $K = \mathbb{C}$, let $\pi : K^n \to \mathbb{T}$ be the covering map.

Given a "tame" set $X \subseteq K^n$ (e.g. semialgebraic).

What is the closure of $\pi(X)$ in $\mathbb{T}$?



Some history: Ax, Lindemann, Weierstrass

Theorem (Lindemann (1882)-Weierstrass (1885))

Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ be algebraic numbers. If $\alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly independent then $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are algebraically independent over $\mathbb{Q}$.

Theorem (Ax (1972))

Let $X \subseteq \mathbb{C}^n$ be an irreducible complex algebraic variety and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}[X]$ regular functions on X. If $\alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ then $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are algebraically independent over $\mathbb{C}$.

Theorem (Ax-Lindemann for complex tori, geometric version)

Let $\mathbb{T}$ be a compact complex torus and $\pi : \mathbb{C}^n \to \mathbb{T}$ be a covering map. If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety then the complex analytic Zariski closure of $\pi(X)$ in $\mathbb{T}$ is a translate of a complex subtorus of $\mathbb{T}$

From Zariski closure to topological closure

Theorem (Ax-Lindemann for complex tori, geometric version)

Let $\mathbb{T}$ be a compact complex torus and $\pi : \mathbb{C}^n \to \mathbb{T}$ be a covering map. If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety then the complex analytic Zariski closure of $\pi(X)$ in $\mathbb{T}$ is a translate of a complex subtorus of $\mathbb{T}$

Ullmo-Yafaev, 2015

What can be said about the topological closure of $\pi(X)$ in $\mathbb{T}$?

Does a version of Ax-Lindemann Theorem hold for it?

When $X \subseteq \mathbb{C}^n$ is an algebraic **curve**, Ullmo and Yafaev described the closure $\pi(X)$ in terms of cosets of real subtori of A.

Placing the problem in $\mathbb{R}^n$

Even for $X \subseteq \mathbb{C}^n$ algebraic, the closure of $\pi(X)$ brings-in real tori, hence the problem fits better into the real (not complex) setting:

- Let $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$ be a lattice in $\mathbb{R}^n$, i.e. Λ is a subgroup generated by a basis $(\omega_1, \ldots, \omega_n)$ of $\mathbb{R}^n$. Let $\mathbb{T}_\Lambda$ be the quotient group $\mathbb{R}^n/\Lambda$.
- The group $\mathbb{T}_\Lambda$ is called an n-dimensional (real) torus, and admits the structure of a compact Lie group.
- Let $\pi_\Lambda : \mathbb{R}^n \to \mathbb{T}$ be the quotient map. It is a smooth group homomorphism and $\ker(\pi_\Lambda) = \Lambda$.

Reformulating the problem

Given a “tame” set $X \subseteq \mathbb{R}^n$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what can be said about the topological closure of $\pi_\Lambda(X)$ in $\mathbb{T}_\Lambda$?

Tameness and o-minimality

For the rest of the talks, we take "tame" to mean o-minimal.

Recall that the following structures are o-minimal:

$\langle \mathbb{R}; <, +, \cdot \rangle$ (Tarski)

The definable sets are semi-algebraic. E.g. solutions to $p(\bar{x}) > 0$, for $p(\bar{x}) \in \mathbb{R}[\bar{x}]$.

$\mathbb{R}_{\exp} = \langle \mathbb{R}; <, +, \cdot, e^x \rangle$ (Wilkie)

E.g. solutions to $\exists x\, p(e^x, e^{e^y}, x, y, z) > 0$, for $p(\bar{x}) \in \mathbb{R}[\bar{x}]$.

$\mathbb{R}_{\mathrm{an,exp}} = \langle \mathbb{R}_{\exp}, (f \restriction [0,1]^n)_{f \in \mathcal{F}} \rangle$ (van den Dries - Miller)

The expansion of $\mathbb{R}_{\exp}$ by all restricted real analytic functions. E.g. solutions to $\forall z\ \arctan(e^{\sin x} - y^2 + 3z) > 0$, for $x \in [-1, 1], y \in \mathbb{R}$.

O-minimality and closure

From now on, we fix an o-minimal structure $\mathbb{R}_{\text{om}} = \langle \mathbb{R}; <, +, \cdot, \cdots \rangle$.

The o-minimal formulation

Given a definable set $X \subseteq \mathbb{R}^n$ in $\mathbb{R}_{\text{om}}$, and a lattice $\Lambda \subseteq \mathbb{R}^n$, what can we say about the topological closure of $\pi_\Lambda(X)$ in $\mathbb{T}_\Lambda$?

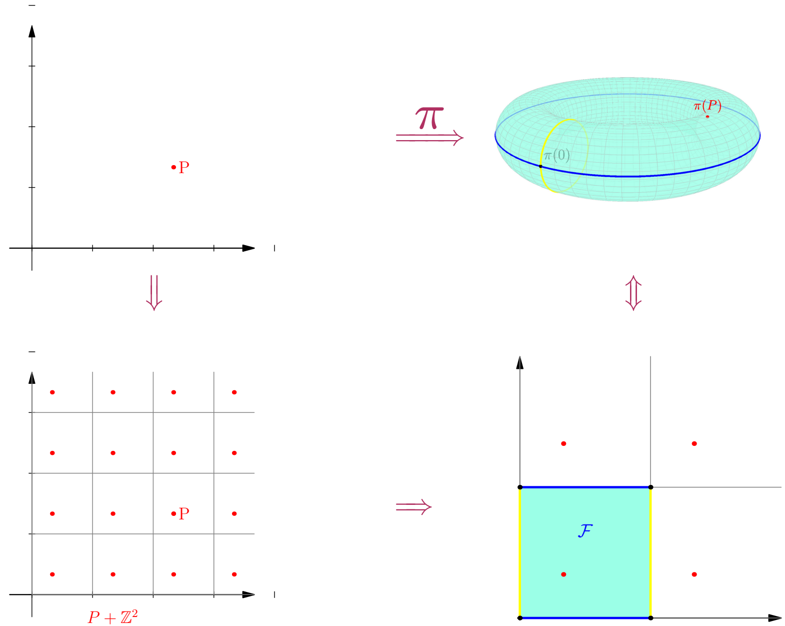
When the setting is clear we use π and $\mathbb{T}$ instead of π_Λ and $\mathbb{T}_\Lambda$.

Observation

For any $X \subseteq \mathbb{R}^n$ we have $\text{cl}(\pi(X)) = \pi(\text{cl}(X + \Lambda))$.

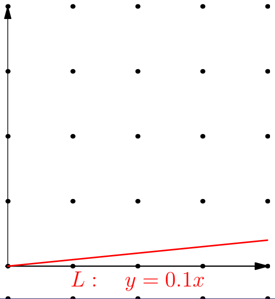
So, from now one we work with $\text{cl}(X + \Lambda)$ in $\mathbb{R}^n$ (instead of $\text{cl}(\pi(X))$).

X vs $X + \Lambda$ vs $\pi({}_{\Lambda}(X))$

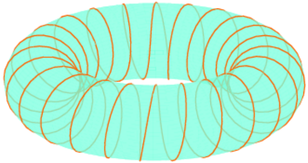


An important example: linear spaces

- Assume that $L \subseteq \mathbb{R}^n$ is an $\mathbb{R}$-subspace.
- Then $\mathrm{cl}(L+\Lambda)$ is a real Lie subgroup of $\mathbb{R}^n$
- Its connected component is an $\mathbb{R}$-subspace, with a basis in Λ. Denote it by L^Λ. It is the smallest $\mathbb{R}$-subspace of $\mathbb{R}^n$, containing L with a basis in Λ and $\mathrm{cl}(L+\Lambda) = L^\Lambda + \Lambda$.
- $\pi(L^\Lambda)$ is a (closed) real subtorus of $\mathbb{T}$.



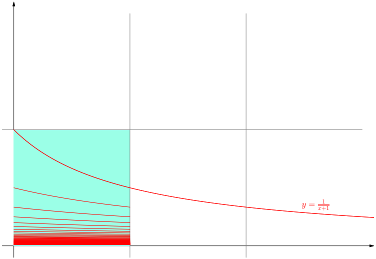
$\Longrightarrow$



Some Examples: Curves on Tori

Let X be the semialgebraic curve $y = \dfrac{1}{x+1}, \quad x \geqslant 0$.

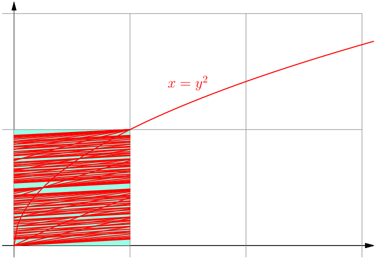
We translate it to the fundamental domain by elements of $\mathbb{Z}^2$



We have $\mathrm{cl}(X + \mathbb{Z}^2) = \Big(X \cup (x\text{-axis})\Big) + \mathbb{Z}^2$.

Some Examples: Curves on Tori

Let X be the semialgebraic curve $x = y^2, \quad y \geqslant 0$.
We translate it to the fundamental domain by elements of $\mathbb{Z}^2$.



after 100 translates

We have $\mathrm{cl}(X + \mathbb{Z}^2) = \mathbb{R}^2$.

A uniform closure theorem

Recall thal if $L \subseteq \mathbb{R}^n$ is a linear subspace and $\Lambda \subseteq \mathbb{R}^n$ is a lattice then L^Λ is the smallest linear subspace containing L with a basis in Λ.

Theorem

Let $X \subseteq \mathbb{R}^n$ be a closed definable set in $\mathbb{R}_{\mathrm{om}}$. Then there are $\mathbb{R}$-subspaces $L_1, \ldots, L_k \subseteq \mathbb{R}^n$, and definable closed sets $C_1, \ldots, C_k \subseteq \mathbb{R}^n$ such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\mathrm{cl}_{\mathbb{R}^n}(X + \Lambda) = \left[X \cup \bigcup_{i=1}^{k} (L_i^\Lambda + C_i)\right] + \Lambda.$$

I.e.

$$\mathrm{cl}_{\mathbb{T}_\Lambda}(\pi_\Lambda(X)) = \pi_\Lambda(X) \cup \bigcup_{i=1}^{k} (T_i + \pi_\Lambda(C_i)),$$

where $T_i = \pi(L_i^\Lambda)$ are real subtori of $\mathbb{T}_\Lambda$.

A model theoretic approach

- Let $\mathbb{R}_{full}$ be the expansion of $\mathbb{R}$ by all subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$.

  And let $\mathcal{R} \succ \mathbb{R}_{full}$ be an $|\mathbb{R}|^+$-saturated elementary extension.
- For $X \subseteq \mathbb{R}^n$, let $X^\sharp$ denote its realization in $\mathcal{R}$.
- Let $\mathcal{O} = \{\alpha \in \mathcal{R} : \exists r \in \mathbb{R}\ |\alpha| < r\}$, the ring of finite elements.
- Let $\mu = \{\epsilon \in \mathcal{R} : \forall r \in \mathbb{R}\ |\epsilon| < r\}$, the infinitesimals, a maximal ideal in $\mathcal{O}$. We have $\mathcal{O} = \mathbb{R} \oplus \mu$.
- Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ denote the standard part map (also, the residue map). Namely, $\mathrm{st}(\alpha) =$ the unique $r \in \mathbb{R}$ such that $\alpha \in r + \mu$.

  We extend it coordinate-wise to $\mathrm{st} : \mathcal{O}^n \to \mathbb{R}^n$.
- For $Y \subseteq \mathcal{R}^n$, let $\mathrm{st}(Y) := \mathrm{st}(\mathcal{O}^n \cap Y)$.

Closure through the standard part map

A key observation

If $Y \subseteq \mathbb{R}^n$ then $\mathrm{cl}(Y) = \mathrm{st}(Y^\sharp)$.

Proof

- Assume $a \in \mathrm{cl}(Y)$.
  Then $B(a,r) \cap Y \neq \varnothing$ for every $r \in \mathbb{R}^{>0}$.
  By saturation, there is $\alpha \in \bigcap_{r \in \mathbb{R}^{>0}} B^\sharp(a,r) \cap Y^\sharp$.
  Obviously $\mathrm{st}(\alpha) = a$. Hence $\mathrm{cl}(Y) \subseteq \mathrm{st}(Y^\sharp)$.
- Assume $a = \mathrm{st}(\alpha)$, for $\alpha \in Y^\sharp$. Then $|a - \alpha| \in \mu$, hence $Y^\sharp \cap B(a,r) \neq \varnothing$ for every $r \in \mathbb{R}^{>0}$. Thus, the same is true in $\mathbb{R}_{full}$, so $a \in \mathrm{cl}(Y)$.

The non-standard formulation of the question

For $X \subseteq \mathbb{R}^n$ definable in the o-minimal structure $\mathbb{R}_{\mathrm{om}}$ and for a lattice $\Lambda \subseteq \mathbb{R}^n$, what is $\mathrm{st}(X^\sharp + \Lambda^\sharp) \subseteq \mathbb{R}^n$?

Complete o-minimal types appear

We have $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{\mathrm{om}}$.

Partition into types

Let $S_X(\mathbb{R})$ be the collection of all complete $\mathbb{R}_{\mathrm{om}}$-types over $\mathbb{R}$, on X (i.e. containing the formula $x \in X$).

For $p(x) \in S_X(\mathbb{R})$, we let $p(\mathcal{R})$ be its set of realizations in $\mathcal{R}$.

We have:

$$\mathrm{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp).$$

The new question

For a complete type $p \in S_X(\mathbb{R})$, what is $\mathrm{st}(p(\mathcal{R}) + \Lambda^\sharp)$?