# Topological closure of algebraic and o-minimal flows in complex tori

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# The setting and problem

- Let $\Lambda = \bigoplus_{i=1}^{2n} \mathbb{Z}\omega_i$ be a lattice in $\mathbb{C}^n$. I.e. $\{\omega_1, \ldots, \omega_{2n}\}$ is a basis for the real vector space $\mathbb{C}^n$.
- Let $T$ be the quotient group $\mathbb{C}^n/\Lambda$.
- The group $T$ admits the structure of a compact complex Lie group, an $n$-dimensional complex torus.
  (If $\Lambda$ satisfies the **Riemmann period relations** then $T$ admits also the structure of a projective algebraic group, i.e. *an abelian variety*. E.g. when $n = 1$, $T$ is an elliptic curve)
- Let $\pi : \mathbb{C}^n \to T$ be the quotient map. It is a holomorphic group homomorphism and $ker(\pi) = \Lambda$. $\pi$ **is a transcendental map**.

## The problem

Given a "tame" set $X \subseteq \mathbb{C}^n$, what can we say about $\pi(X) \subseteq T$? More precisely, what is the closure of $\pi(X)$ in $T$?

# A related result-the Zariski closure

**Assume that $T$ is an abelian variety, so algebraic in some $\mathbb{P}^k(\mathbb{C})$.**

“Ax-Lindemann-Weierstrass”

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety then the **Zariski closure** of $\pi(X)$ is a coset of an abelian subvariety of $T$. I.e., $\pi(X) = V + a$, where $V \leqslant A$ is an alegrbaic subgroup and $a \in A$.

# Ullmo-Yafaev's question-the topological closure

We have $\pi : \mathbb{C}^n \to T$ (a torus), with $\Lambda = ker(\pi)$ a lattice in $\mathbb{C}^n$.

For $X \subseteq \mathbb{C}^n$ algebraic, or definable in an o-minimal structure.
What is the **topological closure** of $\pi(X)$ in $T$?

## Observation

The closure of $\pi(X)$ in $T$ equals $\pi\left(\mathrm{cl}_{\mathbb{C}^n}(\Lambda + X)\right)$.

Indeed, this follows from:

- $\pi$ is continuous.
- $\pi$ is a quotient map.

## The new question

What is the (topological) closure of $\Lambda + X$ (in $\mathbb{C}^n$) ?

# Examples

Some pictures

## An important example

- Assume that $V \subseteq \mathbb{C}^n$ is an $\mathbb{C}$-subspace (or even an $\mathbb{R}$-subspace).
- Then $\mathrm{cl}(\Lambda + V)$ is a real Lie subgroup of $\mathbb{C}^n$!
- Its connected component is an $\mathbb{R}$-subspace, with a basis in $\Lambda$. Denote it by $V^\Lambda$. It is the smallest $\mathbb{R}$-subspace of $\mathbb{C}^n$, containing $V$ with a basis in $\Lambda$.
- $\pi(V^\Lambda)$ is a (closed) real subtorus of $T$.

# Theorem-algebraic case

## Theorem

Let $X \subseteq \mathbb{C}^n$ be an algebraic subvariety. Then there are $\mathbb{C}$-subspaces $V_1, \ldots, V_r \subseteq \mathbb{C}^n$ of positive dimension, and algebraic varieties $C_1, \ldots, C_r \subseteq \mathbb{C}^n$ such that for every lattice $\Lambda \subseteq \mathbb{C}^n$,

1. $Cl(\Lambda + X) = (\Lambda + X) \cup \bigcup_{i=1}^{r} (\Lambda + V_i^{\Lambda} + C_i)$. I.e. $Cl(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^{r} (T_i + \pi(C_i))$, for $T_i$ real subtori of $T$.
2. For each $i = 1, \ldots, r$, $\dim C_i < \dim X$.
3. If $V_i$ is maximal among $V_1, \ldots, V_r$ then $C_i$ is finite.

In particular, if $\dim X = 1$ (i.e. $X$ is a complex curve) then each $C_i$ is finite, so

$$Cl(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^{r} T_i + a_i, \text{ for } a_i \in T$$

(a theorem of Ullmo-Yafaev).

# Theorem-the O-minimal case

**The real setting**
For $\Lambda$ a lattice in $\mathbb{R}^n$, $\pi : \mathbb{R}^n \to T$ is the quotient map on a compact **real** torus.

## Theorem

Let $X \subseteq \mathbb{R}^n$ be a closed definable set in an o-minimal structure. Then there are $\mathbb{R}$-subspaces $V_1, \ldots, V_r \subseteq \mathbb{R}^n$ of positive dimension, and definable closed sets $C_1, \ldots, C_r \subseteq \mathbb{R}^n$ such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

1. $Cl(\Lambda + X) = (\Lambda + X) \cup \bigcup_{i=1}^r (\Lambda + V_i^\Lambda + C_i)$. I.e. $Cl(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^r (T_i + \pi(C_i))$, for $T_i$ real subtori of $T = \mathbb{R}^n/\Lambda$.
2. For each $i = 1, \ldots, r$, $\dim_{\mathbb{R}} C_i < \dim_{\mathbb{R}} X$.
3. If $V_i$ is maximal among $V_1, \ldots, V_r$ then $C_i$ is bounded, (so $\pi C_i$ closed).

# A non-standard view point

- Let $\mathcal{R}$ be an $(2^{\aleph_0})^+$-saturated extension of **all structure** on $\mathbb{R}$. In particular, $\langle \mathbb{R}, <, +, \cdot \rangle \prec \langle R, <, +, \cdot \rangle$.
- For $B \subseteq \mathbb{R}^n$, we let $B^\sharp$ be its realization in $\mathcal{R}$.
- Let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in $\mathcal{R}$, a valuation ring.
- Let $\mu = \{\epsilon \in \mathcal{O} : \forall r \in \mathbb{R}\ |\epsilon| < r\}$ be the infinitesimals.
- Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ be the residue map, we also write $\mathrm{st} : \mathcal{O}^n \to \mathbb{R}^n$. For $Y \subseteq \mathcal{R}^n$, let

$$\mathrm{st}(Y) := \mathrm{st}(\mathcal{O}^n \cap Y).$$

- **Note: If $B \subseteq \mathbb{R}^n$ then $\mathrm{st}(B^\sharp) =$ the closure of $B$.**

**The ACF point of view**

- $\hat{\mathcal{O}} := \mathcal{O} + i\mathcal{O}$ is a valuation ring in the ACF $\hat{\mathcal{R}} = \mathcal{R}(i)$. $\hat{\mu} := \mu + i\mu$ a maximal ideal. $\mathrm{st} : \hat{\mathcal{O}} \to \mathbb{C}$ the residue map.

## The non-standard formulation of the problem

1. For $X \subseteq \mathbb{C}^n$ an algebraic variety, what is $\text{st}(\Lambda^\sharp + X^\sharp) \subseteq \mathbb{C}^n$?
2. For $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure, what is $\text{st}(\Lambda^\sharp + X^\sharp) \subseteq \mathbb{R}^n$?

Fix $\bar{\mathbb{R}}$ an o-minimal expansion of $\mathbb{R}$. For $\alpha \in \mathcal{R}^n$, let $tp(\alpha/\mathbb{R})$ denote the o-minimal type of $\alpha$ over $\mathbb{R}$.

$$\text{st}(\Lambda^\sharp + X^\sharp) = \bigcup_{\alpha \in X^\sharp} \text{st}(\Lambda^\sharp + \alpha).$$

We may divide this union into complete types.

$$\text{st}(\Lambda^\sharp + X) = \bigcup_{p \in S_X(\mathbb{R})} \text{st}(\Lambda^\sharp + p(\mathcal{R})).$$

Here $S_X(\mathbb{R})$ is the space of complete o-minimal types in $X$ over $\mathbb{R}$.

**Question** For $p \in S_X(\mathbb{R})$, what is $\text{st}(\Lambda^\sharp + p)$?

# The $\mu$-stabilizer of a type

For $p$ an o-minimal $n$-type over $\mathbb{R}$, we let

$$Stab_\mu(p) = \{g \in \mathbb{R}^n : g + \mu + p(\mathcal{R}) = \mu + p(\mathcal{R})\}.$$

## Theorem-PS 2015

1. The group $Stab_\mu(p)$ is a definable subgroup of $\mathbb{R}^n$.
2. The dimension of $Stab_\mu(p)$ is at most $\dim p$.
3. If $p$ is unbounded then $\dim(Stab_\mu(p)) > 0$.

## Examples

**A.** If $p$ is a bounded type (hence, infinitesimally close to an algebraic type) then $Stab_\mu(p) = \{0\}$.
**B.** If $p$ a type at infinity on a curve that has $y = mx + b$ as an asymptote then $Stab_\mu(p) = \{(x, mx) : x \in \mathbb{R}\}$
**C.** If $p$ is the type at $+\infty$ of $(x, x^2)$ then $Stab_\mu(p) = \{0\} \times \mathbb{R}$.

# $\mu$-stabilizers and the closure of $\Lambda + X$

We saw that $\text{cl}(\Lambda + X) = \bigcup_{p \in S_X(\mathbb{R})} \text{st}(\Lambda + p(\mathcal{R}))$.

## Fact

For every type $p$, the set $\text{st}(\Lambda^\sharp + p)$ contains a translate of the linear space $Stab_\mu(p) \subseteq \mathbb{R}^n$.

**Proof**

- We show that $\text{st}(\Lambda^\sharp + X^\sharp)$ is invariant under $Stab_\mu(p)$.
- Take $g \in Stab_\mu(p)$, and assume $a \in \text{st}(\Lambda^\sharp + p)$.
- I.e. $a = \text{st}(\lambda^* + \alpha)$, for $\lambda^* \in \Lambda^\sharp$ and $\alpha \models p$.
- Since $g \in \mathbb{R}^n$, $g + a = \text{st}(g + \lambda^* + \alpha) =$
- $\text{st}(\lambda^* + g + \alpha) = \text{st}(\lambda^* + \epsilon + \alpha')$, for $\epsilon \in \mu$ and $\alpha' \models p$.
- Hence, $g + a = \text{st}(\lambda^* + \alpha') \in \text{st}(\Lambda^\sharp + p)$. □

## back to example

The set $X = \{(x, x^2) : x \in \mathbb{R}\}$. The type $tp((\alpha, \alpha^2)/\mathbb{R})$.
It follows that for any lattice $\Lambda$, $\text{cl}(\Lambda + X)$ contains a coset of $\{0\} \times \mathbb{R}$.

# Asymptotic affine spaces

Given $\alpha \in \mathcal{R}^n$, consider all affine subspaces, $A = V + a \subseteq \mathbb{R}^n$ defined over $\mathbb{R}$, such that $(\alpha + \mu) \cap A \neq \varnothing$ I.e., $\alpha$ is infinitesimally close to $A$.

## Definition+Fact

Let $A_\alpha$ be the intersection of all the above affine spaces.Then $A_\alpha$ is itself an affine space defined over $\mathbb{R}$, and $(\alpha + \mu) \cap A_\alpha \neq \varnothing$.

**Example**

- When $\alpha \sim a \in \mathbb{R}^2$, then $A_\alpha = \{a\}$.
- When $\alpha = (t, 1/t)$, for $t >> 0$. $A_\alpha = \mathbb{R} \times \{0\}$.
- When $\alpha = (s, s^2)$, $s >> 0$, $A_\alpha = \mathbb{R}^2$.

## Note

For $p = tp(\alpha/\mathbb{R})$, $A_\alpha$ contains a coset of $Stab_\mu(p)$.

# An important ingredient

## Theorem

For any type $p = tp(\alpha/\mathbb{R})$, and for any lattice $\Lambda$,

$$\mathrm{st}(\Lambda^\sharp + p(\mathcal{R})) = \mathrm{cl}(\Lambda + A_\alpha).$$

So, if $A_\alpha = V_\alpha + a_\alpha$ then

$$\mathrm{st}(\Lambda^\sharp + p(\mathcal{R})) = \Lambda + V_\alpha^\Lambda + a_\alpha.$$

## Corollary

If $X \subseteq \mathbb{R}^n$ is definable then for any lattice $\Lambda$,

$$\mathrm{cl}(\Lambda + X) = \bigcup_{p \in S_X(\mathbb{R})} \mathrm{st}(\Lambda^\sharp + p(\mathcal{R})) = \bigcup_{\alpha \in X(\mathcal{R})} (\Lambda + V_\alpha^\Lambda + a_\alpha).$$

# The definability of $\mathcal{A}(X)$

We consider the family of affine spaces $\mathcal{A}(X) = \{A_\alpha : \alpha \in X(\mathcal{R})\}$. We also consider the pair of structures $(\bar{\mathcal{R}}, \bar{\mathbb{R}})$. It is a *Tame pair* (v.d. Dries).

## Theorem

1. Every subset of $\mathbb{R}^n$ that is definable in $(\bar{\mathcal{R}}, \bar{\mathbb{R}})$ is definable in $\bar{\mathbb{R}}$ (v.d. Dries).
2. The family $\mathcal{A}(X)$ is definable in $(\mathcal{R}, \bar{\mathbb{R}})$ (exercise).So, definable in $\bar{\mathbb{R}}$

## Corollary

There is a definable family of affine spaces $\{V_t + a_t : t \in T\}$ (depending only on $X$), such that for any lattice $\Lambda \subseteq \mathbb{R}^n$,

$$\text{cl}(\Lambda + X) = \bigcup_{t \in T} (\Lambda + V_t^\Lambda + a_t).$$

# The final step-obtaining finitely many subspaces

Using Bair Category Theorem, we obtain:

**Theorem**

Given a definable family of affine subspaces $\{V_t + a_t : t \in T\}$, there are finitely many linear spaces $V_1, \ldots, V_r$, and a partition $T = T_1 \cup \cdots \cup T_r$, such that for every lattice $\Lambda$ and $i = 1, \ldots, r$,the set $\bigcup_{t \in T_i}(\Lambda + V_t)$ is dense in $\Lambda + V_i^{\Lambda}$.

The main theorem follows:

$$\mathrm{cl}(\Lambda + X) = \bigcup_{i=1}^{r}(\Lambda + V_i^{\Lambda} + C_i),$$

where $C_i = \{a_t : t \in T_i\}$.

# The algebraic case

When $X \subseteq \mathbb{C}^n$ is an algebraic variety we use

- The algebraically closed valued field $\langle \hat{\mathcal{R}}, +, \cdot, \hat{\mathcal{O}} \rangle$.
- Affine $\mathbb{C}$-sets that are $\mu$-close to $\alpha \in X(\hat{\mathcal{R}})$.
- A theorem on ACVF with a section (Delon), instead of Tame pairs.

# What about other groups and discrete subgroups?

**Work in progress**
**When we weaken the lattice assumption**

## Theorem

Let $\Lambda$ be a discrete subgroup of $\mathbb{R}^n$ and let $X \subseteq \mathbb{R}^n$ be a definable set in an o-minimal structure. Let $V_0 \subseteq \mathbb{R}^n$ be the real subspace generated by $\Lambda$ (when $\Lambda$ a lattice, $V_0 = \mathbb{R}^n$).
Then

$$\mathrm{cl}(\Lambda + X) = \bigcup_{\alpha \in X(\mathcal{R}) \cap (V_0 + \mathcal{O})} \mathrm{cl}(\Lambda + A_\alpha)).$$

**When we replace $\mathbb{R}^n$ by any real algebraic commutative group**

## Theorem

An analogous theorem for lattices and definable sets in arbitrary **commutative** real algebraic groups.