Complex analytic geometry, an o-minimal viewpoint

Part III

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Let $\widetilde{\mathcal{M}}$ be a saturated elementary extension of the many-sorted structure $\mathcal{M}$ of compact complex spaces.

The following theorem is a model-theoretic interpretation of a result proved independently by Campana and Fujiki.

Theorem (Pillay) *Let* $a, b \in \widetilde{\mathcal{M}}$ *and* $c = \mathrm{Cb}(\mathrm{tp}(a/b))$*. Then the type* $\mathrm{tp}(c/a)$ *is "algebraic", i.e. internal to* $\widetilde{\mathbb{P}(\mathbb{C})}$*.*

In this talk is we will discuss some analytic ingredients of the above theorem.

We will not consider analytic spaces.

"Nonstandard algebraicity" in CCM

Let X be a compact complex manifold and $\pi : V \to X$ a holomorphic vector bundle. Let $\mathbb{P}(V)$ be the corresponding projective vector bundle. (It is a compact complex manifold.)

Fact *If* $c \in \widetilde{\mathbb{P}(V)}$ *and* $a = \tilde{\pi}(c) \in \widetilde{X}$ *then the type* $\mathrm{tp}(c/a)$ *is internal to* $\widetilde{\mathbb{P}(\mathbb{C})}$*.*

Let B be also a compact complex manifolds, $Z \subseteq B \times X$ an irreducible analytic subset. We denote by $\pi_X : Z \to X$ the standard projection.

Corollary *Assume that there is a bimeromorphism* μ *between* Z *and an analytic subset of* $\mathbb{P}(V)$ *such that the following diagram is commutative.*

$$\begin{tikzcd} Z \arrow[rr, "\mu"] \arrow[dr, "\pi_X"'] & & \mathbb{P}(V) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Then there is a Zariski open subset $S \subseteq Z$ *such that for all* $(c, a) \in \widetilde{S}$ *the type* $\mathrm{tp}(c/a)$ *is internal to* $\widetilde{\mathbb{P}(\mathbb{C})}$*.*

From now on we will work in our usual setting, i.e. we fix an o-minimal expansion of a real closed field $\mathbf{R}$, $\mathbf{K} = \mathbf{R}(\sqrt{-1})$, etc..

Definable Families

Let M, C be definable sets.

We say that a family $Z_c \subseteq M, c \in C$, is *definable* if the set $Z = \{(c, a)\,;\, a \in Z_c\}$ is a definable subset of $C \times M$.

Very often we will identify the family $Z_c, c \in C$, with this set Z.
Thus if L, M are definable sets and $Z \subseteq L \times M$ definable, then, for $(c, a) \in Z$, we will often denote by Z_c the set $\{x \in M\,;\, (c, x) \in Z\} \subseteq M$.

Theorem 1 *Let L, M be $\mathbf{K}$-manifolds, and $Z \subseteq L \times M$ an irreducible $\mathbf{K}$-analytic subset.*
Then there is a $\mathbf{K}$-holomorphic vector bundle $\pi : V \to M$ and $\mathbf{K}$-meromorphic map $\mu : Z \to \mathbb{P}(V)$ and Zariski open $S \subseteq Z$ such that the following diagram is commutative

$$\begin{tikzcd} & S \subseteq Z \arrow[dl, dashed, "\pi_L"'] \arrow[dr, "\pi_M"] \arrow[rr, "\mu"] & & \mathbb{P}(V) \arrow[dl, "\pi"] \\ L & & M & \end{tikzcd}$$

μ is $\mathbf{K}$-holomorphic on S, and, for $(c, a), (c', a) \in S$, $\mu((c, a)) = \mu((c', a))$ implies $S_c = S_{c'}$ near a.

Corollary 2 *Assume in addition that Z and M are definably compact, and for some Zariski open $C \subseteq \pi_L(Z)$ all $Z_c, c \in C$, are irreducible and pairwise distinct as subsets of M. Then μ is a $\mathbf{K}$-bimeromorphism between Z and a $\mathbf{K}$-analytic subset of $\mathbb{P}(V)$.*

An Easy Case

Assume that for every $(c,a) \in Z$ the set Z_c is a $\mathbf{K}$-holomorphic curve on a $\mathbf{K}$-manifold M, and for $(c,a) \neq (c',a) \in Z$ the curves Z_c and $Z_{c'}$ have different tangent lines at a.

In this case we can take $V = T(M)$, the tangent bundle of M, with the standard projection $\pi : T(M) \to M$, and μ to be the map that assigns to each (c,a) the element of $\mathbb{P}(T_a(M))$ that represents the line $T_a(Z_c)$.

Idea for the general case.

Let M be a $\mathbf{K}$-manifold, $S_1, S_2 \subseteq M$ $\mathbf{K}$-submanifolds, and $a \in S_1 \cap S_2$. Assume $S_1 \neq S_2$ near a, and let F_1, F_2 be K-holomorphic functions defining S_1 and S_2 at a. Since $F_1 \neq F_2$ near a, for sufficiently large $r \in \mathbb{N}^+$, the components of F_1 and F_2 have different Taylor polynomials of degree r at a.

We need to consider higher order tangent bundles $T^{(r)}(M), r \in \mathbb{N}^+$.

Minimal Requirements:

I. Standard.

(Ia) For a $\mathbf{K}$-manifold M and $r \in \mathbb{N}^+$, the r-th tangent bundle $\pi : T^{(r)}(M) \to M$ should be a $\mathbf{K}$-holomorphic vector bundle over M.

(Ib) If $f : M \to N$ is a $\mathbf{K}$-holomorphic map then the r-the differential $D^{(r)}f : T^{(r)}(M) \to T^{(r)}(N)$ of f should be a $\mathbf{K}$-holomorphic vector bundle morphism over f.

(Ic) $D^{(r)}(f \circ g) = (D^{(r)}f) \circ (D^{(r)}g)$, and the r-differential of the identity map on M should be the identity map on $T^{(r)}(M)$.
In particular, if $f : M \to N$ is a $\mathbf{K}$-biholomorphism then $D^{(r)}f : T^{(r)}(M) \to T^{(r)}(N)$ should be also $\mathbf{K}$-biholomorphism.

II. Submanifolds.

If S is a $\mathbf{K}$-submanifold of a $\mathbf{K}$-manifold M then $D^{(r)}(\mathrm{id}_S) : T^{(r)}(S) \to T^{(r)}(M)$ should be an embedding and the image of $T^{(r)}(S)$ should be a $\mathbf{K}$-submanifold of $T^{(r)}(M)$.

II. Uniformity.

If $F_b : S_b \to M, b \in B$, is a definable family of $\mathbf{K}$-holomorphic maps between $\mathbf{K}$-manifolds then the family $D^{(r)}(F_b)$ should be also definable.

III. Separation.

If S_1, S_2 are $\mathbf{K}$-submanifolds of a $\mathbf{K}$-manifold M, $a \in S_1 \cap S_2$, and $T_a^{(r)}(S_1) = T_a^{(r)}(S_2)$ for all $r \in \mathbb{N}^+$, then $S_1 = S_2$ near a.

For $a \in M$, as usual, $T_a^{(r)}(M) = \pi^{-1}(a)$ is the r-th tangent space to M at a.

We take $T_a^{(r)}(M)$ to be the bundle of differential operators of order r. They are the same as r-jet spaces of Pillay and Ziegler (inspired by the sheaves of differential operators used by Campana).

Higher order tangent bundles of affine sets.

We fix $r \in \mathbb{N}^+$. Let $x = (x_1, \ldots, x_n)$ be the standard coordinate functions on $\mathbf{K}^n$.
If $U \subseteq \mathbf{K}^n$ is definable open then $T^{(r)}(U) = U \times \mathscr{D}_x^{(r)}$ with $\pi = \pi_1$, where $\mathscr{D}_x^{(r)}$ is the vector space of all differential operators of the form

$$\sum_{0<|\alpha|\leqslant r} c_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad c_\alpha \in \mathbf{K}.$$

Since $\mathscr{D}_x^{(r)}$ is a finitely-dimensional vector space over $\mathbf{K}$, it has a canonical structure of a $\mathbf{K}$-manifold, and so does $T^{(r)}(U)$.

Example. *For* $r = 1$, $\mathscr{D}_x^{(1)} = \{\sum_{i=1}^n c_i \frac{\partial}{\partial x^i}\}$, *and* $T^{(1)}(U)$ *is the usual tangent bundle.*

If $a \in U$ then the r-th tangent space to U at a is $T_a^{(r)}(U) = \pi^{-1}(a) = \{a\} \times \mathscr{D}_x^{(r)}$. It is identified with

$$\mathscr{D}_a^{(r)} = \left\{ \sum_{0<|\alpha|\leqslant r} c_\alpha \left. \frac{\partial^{|\alpha|}}{\partial x^\alpha} \right|_{x=a} ; c_\alpha \in \mathbf{K}. \right\}$$

If $a \in \mathbf{K}^n$ and $\mathcal{D} \in \mathscr{D}_a^{(r)}$ then $\mathcal{D}$ acts on $\mathbf{K}$-holomorphic functions from open neighborhoods of a into $\mathbf{K}$ via: $\mathcal{D}(h) = \left. \frac{\partial^{|\alpha|} h}{\partial x^\alpha} \right|_{x=a} \in \mathbf{K}$.

Differentials.

Let $r \in \mathbb{N}^+$. For definable open $U \subseteq \mathbf{K}^n$ and $\mathbf{K}$-holomorphic $F : U \to \mathbf{K}^m$, the r-th differential $D^{(r)}F$ is defined to satisfy the following:
Let $a \in U$, $b = F(a)$, and $\mathcal{D} \in T_a^{(r)}(U) = \mathscr{D}_a^{(r)}$.
Then $D^{(r)}F\,(\mathcal{D})$ is the element $\mathcal{D}_1 \in \mathscr{D}_b^{(r)} = T_b^{(r)}(\mathbf{K}^m)$ such that $\mathcal{D}_1(h) = \mathcal{D}(h \circ F)$ for every $\mathbf{K}$-holomorphic function $h : V \to \mathbf{K}$, $b \in V$, $V \subseteq \mathbf{K}^m$ definable open.

Example *Let* $U \subseteq \mathbf{K}$ *be definable open* $F : U \to \mathbf{K}$ $\mathbf{K}$*-holomorphic,* $a \in U$ *and* $b = F(a)$*. Let* h *be a definable* $\mathbf{K}$ *holomorphic function into* $\mathbf{K}$ *defined near* b*. Applying the Chain Rule we obtain:*

$$\left.\frac{\partial(h \circ F)}{\partial x}\right|_{x=a} = F'(a)h'(b), \text{ and}$$

$$\left.\frac{\partial^2 h \circ F}{\partial x^2}\right|_{x=a} = F''(a)h'(b) + [F'(a)]^2 h''(b)$$

Thus, under $D^{(2)}F$*,*

$$\left.\frac{\partial}{\partial x}\right|_{x=a} \mapsto F'(a) \left.\frac{\partial}{\partial x}\right|_{x=b} \text{ and}$$

$$\left.\frac{\partial^2}{\partial x^2}\right|_{x=a} \mapsto F''(a) \left.\frac{\partial}{\partial x}\right|_{x=b} + [F'(a)]^2 \left.\frac{\partial^2}{\partial x^2}\right|_{x=b}.$$

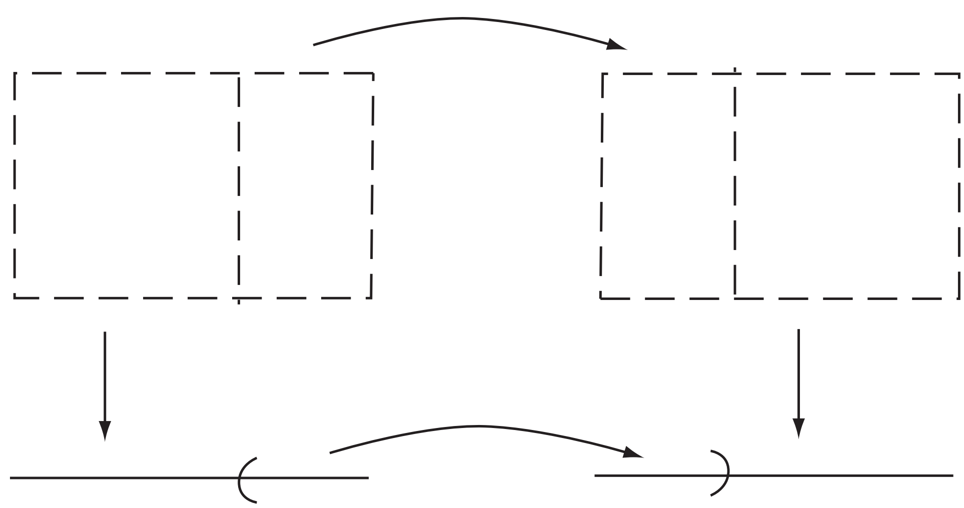
If we take $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ *as a basis for* $\mathscr{D}_x^{(2)}$ *then* $D^{(2)}F : (x, v) \mapsto (F(x), Av)$ *where* A *is the matrix*

$$\begin{pmatrix} F'(x) & 0 \\ F''(x) & F'(x)^2 \end{pmatrix}$$

In general, if $U \subseteq$ is definable open and $F : U \to \mathbf{K}^m$ a $\mathbf{K}$-holomorphic function then, for any $r \in \mathbb{N}^+$, the component functions of $D^{(r)}F$ are polynomials in the components of F and their partial derivatives. In particular, $D^{(r)}F$ is $\mathbf{K}$-holomorphic.

Directly from the definition of the r-th differential it follows that $D^{(r)}(\mathrm{id}_U) = \mathrm{id}_{T^{(r)}(U)}$ and $D^{(r)}(g \circ f) = (D^{(r)}g) \circ (D^{(r)}f)$.

If M is a $\mathbf{K}$-manifold then the r-th tangent bundle $T^{(r)}(M)$ is obtained by working in charts U_i, and using the r-th differentials of the transition maps to glue $T^{(r)}(U_i)$.



Uniformity

Claim 3 *Let $F_b : M_b \to N_b, b \in B$, be a definable family of* $\mathbf{K}$*-holomorphic maps between* $\mathbf{K}$*-manifolds. Then the family $D^{(r)}F_b : T^{(r)}(M_b) \to T^{(r)}(N_b)$ is also definable.*

Claim 4 *Let B, M, N be* $\mathbf{K}$*-manifolds and $F(x, y) : B \times M \to P$ a* $\mathbf{K}$*-holomorphic map. For $b \in B$ let $G(b, -)$ be the r-th differential of $F(b, -) : T^{(r)}(M) \to T^{(r)}(N)$. Then the map $G(x, u) : B \times T^{(r)}(M) \to T^{(r)}(N)$ is also* $\mathbf{K}$*-holomorphic.*

Submanifolds

Claim 5 *Let S, M be $\mathbf{K}$-manifolds and $F : S \to M$ $\mathbf{K}$-holomorphic. If F is an immersion on S then, for every $r \in \mathbb{N}^+$, the r-th differential $D^{(r)}F$ is injective and the image of $T^{(r)}(S)$ under $D^{(r)}F$ is a $\mathbf{K}$-submanifold of $T^{(r)}(M)$.*

Proof Let $F : \mathbf{K}^n \to \mathbf{K}^m$ be the embedding $x \mapsto (x, 0)$, $a \in \mathbf{K}^n$, $r \in \mathbb{N}^+$, and α a multi-index with $|\alpha| \leqslant r$. Then

$$D^{(r)}F : \left.\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\right|_a \mapsto \left.\frac{\partial^{|\beta|}}{\partial y^{\beta}}\right|_{(a,0)}, \text{ where } \beta_i = \begin{cases} \alpha_i & i \leqslant n \\ 0 & i > 0 \end{cases}$$

$\square$

If S is a $\mathbf{K}$-submanifold of a $\mathbf{K}$-manifold M then the r-th differential of the identity map id_S is an embedding. Thus we can consider $T^{(r)}(S)$ as a $\mathbf{K}$-submanifold of $T^{(r)}(M)$.

Separation

Claim 6 *Let S be a $\mathbf{K}$-submanifold of a $\mathbf{K}$-manifold M, $a \in S, U \subseteq M$ definable open containing a, and $h : U \to \mathbf{K}$ a $\mathbf{K}$-holomorphic function with $f(a) = 0$. The following are equivalent.*
1. h vanishes on S near a
2. For all $r \in \mathbb{N}^+$ and $\mathcal{D} \in T_a^{(r)}(S)$, $\mathcal{D}(f) = 0$

Proof Assume $M = \mathbf{K}^2$, $a = (0, 0)$ and $S = \{(x, y) \in \mathbf{K}^2 \,;\, y = 0\}$. For each $r \in \mathbb{N}^+$, $T_a^{(r)}(S)$ is spanned by $\{\frac{\partial}{\partial x}|_{(0,0)}, \ldots, \frac{\partial^r}{\partial x^r}|_{(0,0)}\}$. In this case the statement of the claim is:
Let h be $\mathbf{K}$-holomorphic near $(0, 0)$ with $h(0, 0) = 0$. Then $h(x, 0) \equiv 0$ if and only if all derivatives of $h(x, 0)$ vanish at $x = 0$. $\square$

Corollary 7 *Let S_1, S_2 be $\mathbf{K}$-submanifolds of a $\mathbf{K}$-manifold N and $a \in S_1 \cap S_2$. The following are equivalent.*

1. *For all $r \in \mathbb{N}^+$ the vector spaces $T_a^{(r)}(S_1)$ and $T_a^{(r)}(S_2)$ are equal, as subspaces of $T_a^{(r)}(M)$.*
2. *S_1 is equal to S_2 near a.*

Uniform Bound

Claim 8 *Let* $S_b, b \in B$, *be a definable family of* $\mathbf{K}$-*submanifolds of a* $\mathbf{K}$-*manifold* M. *Then the family* $T^{(r)}(S_b), b \in B$, *is also definable.*

Theorem 9 *Let* $S_b, b \in B$, *be a definable family of* $\mathbf{K}$-*submanifolds of a* $\mathbf{K}$-*manifold* M. *Then there is* $r \in \mathbb{N}$ *such that for all* $b, b' \in B$ *and all* $a \in S_b \cap S_{b'}$, *if* $T_a^{(r)}(S_b) = T_a^{(r)}(S_{b'})$ *then* $S_b = S_{b'}$ *near* a.

Embedding into Grassmannian.

We fix a $\mathbf{K}$-manifold M and a definable family $S_b, b \in B$, of $\mathbf{K}$-submanifolds of M. We will assume that all S_b have the same dimension n.

Let r be as in Theorem 9, d the dimension of the vector space $T_a^{(r)}(S_b)$, and $S = \{(b, a) \in B \times M\ ;\ a \in S_b\}$.

Consider the function $\mu : S \to \mathrm{Grass}(d, T^{(r)}(M))$ that assigns to (b, a) the element of $\mathrm{Grass}(d, T_a^{(r)}(M))$ corresponding to $T_a^{(r)}(S_b)$. Clearly, μ is definable.

By the choice of r we have that $\mu((b, a)) = \mu((b', a))$ if and only if $S_b = S_{b'}$ near a.

Theorem 10 *Let L, M be $\mathbf{K}$-manifolds, $S \subseteq L \times M$ a definably connected $\mathbf{K}$-submanifold such that the projection $\pi_L : S \to L$ has constant rank on S.*
Then there is a $\mathbf{K}$-holomorphic vector bundle $\pi : V \to M$ and a $\mathbf{K}$-holomorphic map $\mu : S \to \mathbb{P}(V)$ such that the following diagram is commutative

$$\begin{tikzcd} & S \arrow[dl, dashed, "\pi_L"'] \arrow[dr, "\pi_M"] \arrow[rr, "\mu"] & & \mathbb{P}(V) \arrow[dl, "\pi"] \\ L & & M & \end{tikzcd}$$

and $\mu((b, a)) = \mu((b', a))$ if and only if $S_b = S_{b'}$ near a.