K-manifolds and K-analytic sets

1. *A* $\mathbf{K}$*-manifold (in* $\mathcal{R}$*) of* $\mathbf{K}$*-dimension* n is a Hausdorff topological space M , a **finite** cover $M = \cup_{i \in I} U_i$ and $\forall i \in I$, $\phi_i : U_i \to V_i \subseteq \mathbf{K}^n$ a homeomorphism with an $\mathcal{R}$-definable open set V_i, such that

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are K-holomorphic.

Notation: $\dim_{\mathbf{K}}(M) = n$.

2. M a $\mathbf{K}$-manifold. A definable $A \subseteq M$ is *a* $\mathbf{K}$*-analytic subset* *of M* if

$\forall p \in M \,\exists$ definable $V \ni p, \exists f_1, \ldots, f_t : V \to \mathbf{K}$ $\mathbf{K}$-holomorphic such that $A \cap V = Z(f_1, \ldots, f_t)$ (so A is closed in M). A is called *finitely* $\mathbf{K}$*-analytic* if finitely many such V's cover A.

Remarks

1. Every $\mathbf{K}$-manifold is definably homeomorphic to a definable subset of $\mathbf{R}^k$ (v.d. Dries, Robeson)

2. A $\mathbf{K}$-manifold remains so in elementary extensions.

3. If A is a finitely $\mathbf{K}$-analytic subset of M it remains so in elementary extensions.

4. We will prove that $\mathbf{K}$-analytic sets are finitely $\mathbf{K}$-analytic.

Examples

1. Every definable open $U \subseteq \mathbf{K}^n$ is a $\mathbf{K}$-manifold.

2. $\mathcal{R} \succeq \bar{\mathbb{R}}$: $\mathbf{K}^n$, $\mathbb{P}^n(\mathbf{K})$, Nonsingular algebraic varieties over $\mathbf{K}$ are $\mathbf{K}$-manifolds. Algebraic affine (projective) varieties are K-analytic subsets of $\mathbf{K}^n$ ($\mathbb{P}^n(\mathbf{K})$).

3. $\mathcal{R} = \mathbb{R}_{an}$: Every compact complex manifold, (complex analytic subset) is a $\mathbf{K}$-manifold ($\mathbf{K}$-analytic subset).

4. Let $\{\mathcal{E}_\tau : \tau \in \mathcal{H}(\mathbb{C})\}$ be all 1-dimensional tori (parameterized by the upper half plane). This is a definable family of $\mathbf{K}$-manifolds.

In $\mathcal{R} \succeq \mathbb{R}_{an,exp}$, take $\tau_0 \sim$ transcend. real number. Then $\mathcal{E}_{\tau_0}$ is a $\mathbf{K}$-manifold, $\dim_{\mathbf{K}}(\mathcal{E}_{\tau_0}) = 1$, and $\mathcal{E}_{\tau_0}$ is not K-biholomorphic to an algebraic curve (see [3]).

Digression: Model theoretic link

Theorem (Zil'ber) (over $\mathbb{C}$) Compact complex manifolds, with their compact analytic subsets have quantifier elimination.
These are Zariski structures of finite Morley rank.

The main ingredient: Remmert proper mapping theorem.

Theorem The same is true for definably compact $\mathbf{K}$-manifolds and $\mathbf{K}$-analytic subsets.

The main ingredient: A definable strong version of Remmert's theorem:

Assume $f : M \to N$ is a $\mathbf{K}$-holomorphic map between K-manifolds. If A is a $\mathbf{K}$-analytic subset of M and $f(A)$ is closed in N then it is a $\mathbf{K}$-analytic subset of N.

Properties of K-analytic sets

Definition of regular points Assume $A \subseteq \mathbf{K}^n$ a definable set. $z_0 \in A$ is *a* $\mathbf{K}$*-regular point of* A*, of local dimension* d, if A is a d-dimensional $\mathbf{K}$-submanifold of $\mathbf{K}^n$ near z_0. Namely, $\exists$ a linear $\mathbf{K}$-subspace $L \subseteq \mathbf{K}^n$ of $\mathbf{K}$-dimension d, such that, in a neighborhood of z_0, A is the graph of a $\mathbf{K}$-holomorphic map from L into $\mathbf{K}^{n-d}$.

Notation $z_0 \in \mathrm{Reg}_{\mathbf{K}}(A))$.

Note: $\mathrm{Reg}_{\mathbf{K}}(A)$ is a definable, open subset of A.

Define $\mathrm{Sing}_{\mathbf{K}}(A) := X \setminus \mathrm{Reg}_{\mathbf{K}}(A)$.

For M a $\mathbf{K}$-manifold, $A \subseteq M$, $\mathrm{Reg}_{\mathbf{K}}(A)$ is defined using charts.

We omit $\mathbf{K}$ *in* $\mathrm{Reg}(A)$ *and* $\mathrm{Sing}(A)$*.*

Fact If A is a $\mathbf{K}$-analytic subset of M then

1. $\mathrm{Reg}(A)$ is dense in A.
2. $\dim_{\mathbf{R}}(\mathrm{Sing}(A)) \leqslant \dim_{\mathbf{R}} A - 2$. ($\dim_{\mathbf{R}}()$ is the o-minimal dimension).
3. $\mathrm{Sing}(A)$ is a $\mathbf{K}$-analytic subset of M.

Let $\mathrm{Reg}(A) = \bigcup_{i=1}^{r} M_i$, where the M_i's are definably connected $\mathbf{K}$-submanifolds of M. Define:

$\dim_{\mathbf{K}} A := \max\{\dim_{\mathbf{K}}(M_i) : i = 1, \ldots, r\}$ (the "complex" dimension of A).

Note: $\dim_{\mathbf{K}} A = 2\dim_{\mathbf{R}} A$.

Main Theorem Let M be a $\mathbf{K}$-manifold, $A \subseteq M$ a definable closed subset. Then the following are equivalent:

1. A is a $\mathbf{K}$-analytic subset of M.

2. $\dim_{\mathbf{R}}(\mathrm{Sing}(A)) \leqslant \dim_{\mathbf{R}} A - 2$, and the same is true in every open subset of M.

3. A is finitely $\mathbf{K}$-analytic subset of M.

Remarks: (a) Notice that (2) and (3) are first order statements, preserved in elementary extensions.

(b) (2) $\Rightarrow$ (3) implies that the functions which witness the analyticity of A are definable in $\langle \mathbf{R}, <, +, \cdot, M, A \rangle$.

Proof of Main Theorem

Topological background

Definition X, Y definable sets, $f : X \to Y$ is a definable continuous map, $y \in Y$.

f is *definably proper over* y if for every definable curve $\gamma : (0, \infty) \to X$ such that $f(\gamma(t)) \to_{t\to\infty} y$, $\gamma(t)$ has a limit in X.

Examples

1. The orthogonal projection of $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ onto the x-axis is definably proper over $x \neq 0$. Any nonorthogonal projection is definably proper over the x-axis.

2. $X = S^1 \setminus \{(0, 1)\} \subseteq \mathbb{R}^2$. Any projection is not proper over the image of $(0, 1)$.

Fact If $X \subseteq \mathbf{R}^m$ is locally closed then set of $y \in Y$ such that f is definably proper over y is definable and open.

The main o-minimal tool

Partition Theorem Let $A \subseteq \mathbf{K}^n$ be a definable, locally closed set, $\dim_{\mathbf{R}} A = 2d$. Then there are definable open sets $U_1, \ldots, U_t \subseteq \mathbf{K}^n$ and $\mathbf{K}$-linear subspaces $L_1, \ldots, L_t \subseteq \mathbf{K}^n$ of $\mathbf{K}$-dimension d, such that:

1. $A \subseteq \bigcup_{i=1}^t U_i$.
2. The orthogonal projection $\pi_i : U_i \cap A \to L_i$ is definably proper over $\pi(A)$.

Moreover, we can choose $t \leqslant 2n + 1$.

Proof Let $L_1, \ldots, L_{2n+1} \subseteq \mathbf{K}^n$ be a generic, independent (model theoretically) sequence of $\mathbf{K}$-linear subspaces of dimension d. $\pi_i : \mathbf{K}^n \to L_i$ orthogonal projection.

Let $U_i = \pi_i^{-1}\{y \in L_i : \pi_i|A \text{ is proper over } y\}$.

For every $z \in A$, there is some L_i which is generic over z. $\pi_i|A$ is definably proper over $\pi(z)$, hence $z \in U_i$. $\square$

Proof of (2) $\Rightarrow$ (3) in Main Theorem:
We assume $\dim_{\mathbf{R}}(\mathrm{Sing}A) \leqslant \dim_{\mathbf{R}} A - 2$.

Reduction step 1 (By working in charts and partitioning $\mathrm{Reg}A$) A is a relatively closed subset of $U \subseteq \mathbf{K}^{\ell}$, $\mathrm{Reg}A$ is definably connected.

For simplicity, we assume $\ell = n + 1$, $\dim_{\mathbf{R}} A = 2n$.

Reduction step 2 (Using the Partition Theorem) The projection $\pi : A \to \mathbf{K}^n$ is definably proper on $\pi(A)$.

Remove the set $B = \mathrm{Sing}A \cup (\mathrm{Sing}(\pi))$.

$\dim_{\mathbf{R}} B \leqslant 2n - 2$ and π is a local $\mathbf{K}$-biholomorphism on $A^* = A \setminus B$.

Notice that $\pi(A^*)$ is a definably connected and open in $\mathbf{K}^n$.

By properness, $\pi|A^*$ is everywhere m-to-1, for some m.

By definable choice, there are definable $\phi_1, \ldots, \phi_m : \pi(A^*) :\to \mathbf{K}$, such that $(\bar{z}, y) \in A^*$ iff $\phi_i(\bar{z}) = y$ for some i. Define

$$\Psi(\bar{z}, y) = \Pi_{i=1}^{m}(Y - \phi_i(\bar{z})).$$

$\Psi : \pi(A^*) \times \mathbf{K} \to \mathbf{K}$ is a polynomial in Y which is zero on A^*.

By the symmetry of Ψ in the ϕ_i's, it is $\mathbf{K}$-holomorphic on $\pi(A^*) \times \mathbf{K}$.

We want to extend Ψ holomorphically to $\pi(A) \times \mathbf{K}$: ($\pi(A)$ is actually open!).

By properness, each ϕ_i, and hence also Ψ is bounded near all points of $\pi(A) \times \mathbf{K}$.

By Removal of Singularities, Ψ can be extended holomorphically to $\pi(A) \times \mathbf{K}$, and we get $A = \Psi^{-1}(0)$. Hence A is finitely $\mathbf{K}$-analytic in M. $\square$

Some corollaries

1. A **K**-analytic set is irreducible iff Reg$A$ is definably connected. It thus remains irreducible in elementary extensions and o-minimal expansions.

2. If $A \subseteq M$ is a **K**-analytic subset of M and $A_1, \ldots, A_r$ are the definably connected components of Reg$A$ then $\mathrm{Cl}(A_1), \ldots, \mathrm{Cl}(A_r)$ are the irreducible components of A. In particular, the number of irreducible components is bounded in definable families of **K**-analytic sets.

Closure Theorems

A Corollary of Main Theorem: Assume that M is a $\mathbf{K}$-manifold, $U \subseteq M$ open and definable and $A \subseteq U$ an irreducible $\mathbf{K}$-analytic subset of U. If $\dim_{\mathbf{R}}(\mathrm{Cl}_M(A) \setminus A) \leqslant \dim_{\mathbf{R}} A - 2$ then $\mathrm{Cl}_M(A)$ is a $\mathbf{K}$-analytic subset of M.

(Over $\mathbb{C}$, this follows from the Shiffman's Theorem, generalizing Remmert-Stein).

Examples

1. If $A = U \subseteq M = \mathbb{C}$ is the open unit disc then A is $\mathbb{C}$-analytic in U but its closure in $\mathbb{C}$ is not $\mathbb{C}$-analytic in $\mathbb{C}$
($\dim_{\mathbf{R}}(\mathrm{Cl}(A) \setminus A) = 1 = dim_{\mathbf{R}} A - 1.$)

2. $M = \mathbb{C}^2$, $U = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$;
A=the graph of $e^{1/z}$ in U.

Then $\mathrm{Cl}(A)$ in $\mathbb{C}^2$ is not an analytic set in $\mathbb{C}^2$ ($\dim_{\mathbf{R}}(\mathrm{Cl}(A) \setminus A) = 2 = \dim_{\mathbf{R}} A$).

In the o-minimal setting we have:

Definition If $F \subseteq M$ is a $\mathbf{K}$-analytic subset then $M \setminus F$ is called *a Zariski open subset of* M. If X is a $\mathbf{K}$-analytic subset of M then $X \setminus F$ is a Zariski open subset of X.

The Closure Theorem Assume that M is a definably connected $\mathbf{K}$-manifold, and A a $\mathbf{K}$-analytic subset of a Zariski open subset of M. Then $\mathrm{Cl}_M(A)$ is a $\mathbf{K}$-analytic subset of M.

Note By example (2) above, this fails outside o-minimality.

Proof of the Closure Theorem

Without loss of generality, A is irreducible subset of $M \setminus F$, $\dim_{\mathbf{K}}(A) = d$. Let $\overline{A} = \mathrm{Cl}_M(A)$.

Enough to show: $\dim_{\mathbf{R}}(\overline{A} \setminus A) \leqslant 2d - 2$.

Enough to show: $\dim_{\mathbf{R}}(\overline{A} \cap F) \leqslant 2d - 2$.

Reduction step 1 $A = \mathrm{Reg}(A)$ is a $\mathbf{K}$-submanifold of $U \cap \mathbf{K}^n$, F a $\mathbf{K}$-analytic subset of U.

Reduction step 2 The projection $\pi : A \to \mathbf{K}^d$ is definably proper and m-to-1 on $\mathrm{Reg}(A)$, with $\phi_1, \ldots, \phi_m : \pi(A) \to \mathbf{K}^{n-d}$ giving its branches.

Assume that $\dim_{\mathbf{R}}(\overline{A} \cap F) = 2d - 1$. Take $(z_0, w_0) \in \mathbf{K}^d \times \mathbf{K}^{n-d}$ generic in $\overline{A} \cap F$ and assume that $F = \Psi^{-1}(0)$ near (z_0, w_0).

For some $i = 1, \ldots, m$, $w_0 = lim_{z \to z_0} \phi_i(z)$. Hence $z_0 \in bd(\pi A)$.

Define on $\pi(A)$ the function $\Phi(z) = \Psi(z, \phi_i(z))$. Since Ψ is continuous. $\Phi(z)$ tends to zero as z tends to z_0 in $\pi(A)$.

But then the set
$\{z' \in bd(\pi A) : \lim_{z \to z'} \Phi(z) = 0\}$
has $\mathbf{R}$-dimension $\geqslant 2d - 1$. By the "boundary behavior" of $\mathbf{K}$-holomorphic maps, $\Phi(z) \equiv 0$.

$\Rightarrow A \subseteq F$. Contradiction. $\square$

A corollary to the Closure Theorem

M, N are $\mathbf{K}$-manifolds, $F \subseteq X \subseteq M$ are $\mathbf{K}$-analytic subsets of M, and assume that $X \setminus F$ is a dense susbet of $\mathrm{Reg}(X)$.

If $f : X \setminus F \to N$ is $\mathbf{K}$-holomorphic then the closure of the graph of f is a $\mathbf{K}$-analytic subset of $M \times N$. Namely, f is a $\mathbf{K}$-meromorphic map.

Proof $Graph(f)$ is $\mathbf{K}$-analytic in $(M \times N) \setminus (F \times N)$.

By the Closure Theorem, the closure is $\mathbf{K}$-analytic in $M \times N$. $\square$.

Remark Because of the flexibility to work in any definable manifold, the Closure Theorem and its Corollary easily generalize to "$\mathbf{K}$-analytic spaces" and their subsets.

A generalization of Chow's theorem

Theorem If A is a $\mathbf{K}$-analytic subset of $\mathbf{K}^n$ then A is algebraic.

Proof For simplicity, we assume $\dim_{\mathbf{K}} A = n-1$ and A is irreducible.

Choose a generic $\mathbf{K}$-linear plane $L \subseteq \mathbf{K}^n$ of dimension $n-1$.

$\Rightarrow$ The projection $\pi : A \to L$ is definably proper (over all of L!).

$\Rightarrow$ $\pi(A)$ is closed in L AND $\pi(A)$ is open in L.

$\Rightarrow$ $\pi(A) = L$.

As before, there is $\Psi : \pi(A) \times \mathbf{K} \to \mathbf{K}$ $\mathbf{K}$-holomorphic, such that $A = \Psi^{-1}(0)$.

Ψ is $\mathbf{K}$-holomorphic on $\mathbf{K}^n$ therefore a polynomial. $\square$

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Related work:

A glimpse at isoalgebraic spaces, R. Huber and M. Knebusch, Not di Matematica, vol X, Suppl. n. 2 (1990) 315-336.