Complex analytic geometry, an o-minimal viewpoint

Y. Peterzil *S. Starchenko*

Oberwolfach
July 2004

Setting (first try)

$\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, ...\rangle$ an o-minimal expansion of the real field

Goal: Investigate holomorphic functions, complex manifolds, analytic sets, analytic spaces.... which are definable in $\mathcal{R}$.

$$\begin{array}{rcl} \mathbb{C} & \sim & \mathbb{R}^2 \\ f : \mathbb{C}^n \to \mathbb{C} & \sim & f = (f_1, f_2) : \mathbb{R}^{2n} \to \mathbb{R}^2 \\ A \subseteq \mathbb{C}^n & \sim & A \subseteq \mathbb{R}^{2n} \end{array}$$

Note: $\mathcal{R}$ is a "parameter". Can substitute any o-minimal expansion of $\langle \mathbb{R}, +, \cdot \rangle$.

Examples of definable holomorphic functions

- $\mathcal{R} := \bar{\mathbb{R}} = \langle \mathbb{R}, +, \cdot \rangle$: Complex polynomials, complex algebraic functions.
- $\mathcal{R} := \mathbb{R}_{an} = \langle \mathbb{R}, +, \cdot, f \in \mathcal{F} \rangle$,
  $\mathcal{F} = \{f : [-1, 1]^n \to \mathbb{R} \text{ real-analytic}\}$:
  Every germ at 0 of a holomorphic function is definable on some neighborhood of 0 (using Re(f), Im(f)).
- $\mathcal{R} := \mathbb{R}_{exp} = \langle \bar{\mathbb{R}}, e^x \rangle$:
  (Bianconi): All definable holomorphic functions (on polydiscs) are already definable in $\bar{\mathbb{R}}$.
- In $\mathbb{R}_{an,exp} : \langle \mathbb{R}_{an}, e^x \rangle$: e^z is definable on:

(Why? $Re(e^{x+iy}) = e^x \sin y$, $Im(e^z) = e^x cosy$).

- e^z is not definable on all of $\mathbb{C}$.

Plan

- Functions of one and several variables.
- Properties of $\mathbf{K}$-manifolds, $\mathbf{K}$-analytic sets.
- A theorem on the closure of a $\mathbf{K}$-analytic set.
- Application: A theorem of Campana and Fujiki, an o-minimal viewpoint.

The general setting

$\mathcal{R} = \langle \mathbf{R}, <, +, \cdot, \cdots \rangle$ an arbitrary o-minimal expansion of a real closed field.

$$\mathbf{K} = \mathbf{R}(\sqrt{-1}) \sim \mathbf{R}^2 \; ; \; \mathbf{K}^n \sim \mathbf{R}^{2n}.$$

(Order topology on $\mathbf{R}$; product topology on $\mathbf{R}^{2n}$)

A K-holomorphic function (in $\mathcal{R}$)

Let $U \subseteq \mathbf{K}$ be definable and open. A definable $f : U \to \mathbf{K}$ is $\mathbf{K}$-*differentiable at* $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists in } \mathbf{K}.$$

The limit, if exists is called $f'(z)$.

If f is $\mathbf{K}$-differentiable at all points of U, it is called $\mathbf{K}$-*holomorphic.*

Note: All **K**-holomorphic functions are in particular definable in an o-minimal structure $\mathcal{R}$.

Standard examples, where $\mathbf{R} = \mathbb{R}; \mathbf{K} = \mathbb{C}$: If f is complex analytic and definable in $\mathcal{R}$ it is clearly **K**-holomorphic.

A nonstandard example, $\mathbb{R}_{an,exp} \preceq \mathcal{R}$: $\alpha \in \mathbf{R}$ an infinitesimal element, $f(z) = e^{\alpha z}$ is definable on $-1/\alpha < Im(z) < 1/\alpha$ and **K**-holomorphic.

Note: If f is definable then $\{z : f \text{ is } \mathbf{K}\text{-differentiable at } z\}$ is definable.

The main problems:

• No integration

• Power series don't converge in general.

Remedy: Topological analysis (in one variable).

See earlier papers for details. Here will only review the main properties of $\mathbf{K}$*-holomorphic functions.*

Main themes

• Good boundary behavior.

• No esssential singularities.

• Entire holomorphic functions/analytic sets are algebraic.

• Finiteness theorems and uniformity in parameters.

Functions of one variable

Analogues to classical results (no proofs):

$U \subseteq \mathbf{K}$ a definable, definably connected open set.

1. **Maximum Principle** If $f : U \to K$ is $\mathbf{K}$-holomorphic and nonconstant then $|f|$ does not attain its maximum on U.

2. **Infinite differentiability**
If f is $\mathbf{K}$-holomorphic on U then $f'(z)$ is also $\mathbf{K}$-holomorphic on U.

3. **Strong removing of singularities**
Assume $L \subseteq U$ is definable closed,
$f : U \setminus L \to \mathbf{K}$ $\mathbf{K}$-holomorphic.

A. If $\dim L = 1$ and f is continuous on all of U then f is $\mathbf{K}$-holomorphic on U.

B. If $\dim L = 0$ and f is locally bounded at L then f is $\mathbf{K}$-holomorphic on U.

Good boundary behavior

Theorem If $f : U \to \mathbf{K}$ is a nonconstant $\mathbf{K}$-holomorphic function then the set

$$Z(f) = \{z \in bd(U) : \lim_{z' \to z} f(z') = 0\}$$

has dimension smaller than 1.

Proof Assume $\dim Z(f) = 1$.

Define $H(z) = f(z)f(iz)f(-z)f(-iz)$.

By Maximum Principle, $H = 0$ and so $f = 0$. $\square$

Theorem (no essential singularities)

If f is $\mathbf{K}$-holomorphic in a punctured neighborhood of z_0, f nonconstant, then $\exists n \in \mathbb{Z}$, such that the limit of $f(z)/(z-z_0)^n$ exists in $\mathbf{K}$ and is nonzero.

The number n is denoted by $ord_{z_0}(f)$.

(**Note:** This is false for definable real differentiable functions: Consider $x \mapsto x^\alpha$ for infinite $|\alpha|$, in elementary extensions of $\mathbb{R}_{exp}$).

The number n can be determined topologically and definably, using the notion of a winding number.

A proof over $\mathbb{C}$

If no such n exists then 0 is an essential singularity. Hence, $\forall$ open U containing 0, $f(U)$ is dense in $\mathbb{C}$.

Consider $G = \text{Graph}(f) \subseteq (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$.

$\Rightarrow \text{Cl}(G) \cap (\{0\} \times \mathbb{C}) = \mathbb{C}$,

$\Rightarrow \dim \text{Cl}(G) \setminus G = 2 = \dim G$, contradicting o-minimality. $\square$

Corollaries

1. **Ring of germs.**

Let $\mathcal{O}_0^{\mathcal{R}}$=All germs of $\mathbf{K}$-holomorphic functions near 0. The ring $\mathcal{O}_0^{\mathcal{R}}$ injectively embeds into $\mathbf{K}[[z]]$, via the map

$$f \mapsto \Sigma \frac{f^{(n)}(0)}{n!} z^n.$$

(but the power series might not converge!)

2. **Uniform definability of order.**

Assume $\{f_s : s \in S\}$ is a definable family of $\mathbf{K}$-holomorphic functions on $U_s \setminus \{0\}$. Then there is $n \in \mathbb{Z}$,
$\forall s \in S$, if $|ord_0 f_s| > n$ then $f \equiv 0$.

3. **No new entire K-holomorphic functions.**

If $f : \mathbf{K} \to \mathbf{K}$ if $\mathbf{K}$-holomorphic (on all of $\mathbf{K}$) then f is a polynomial over $\mathbf{K}$.

Functions of several variables

Definition $U \subseteq \mathbf{K}^n$ open. A definable $f : U \subseteq \mathbf{K}^n \to \mathbf{K}$ is $\mathbf{K}$-*holomorphic on* U if it is continuous on U, and $\mathbf{K}$-holomorphic in each variable separately.

Remarks

1. Over $\mathbb{C}$, this is equivalent to other classical definitions which use convergent power series.

2. We develop the theory in several variables through reduction to the 1-variable case ("taking fibers").

Basic results for several variables

If $f : U \subseteq \mathbf{K}^n \to \mathbf{K}$ is $\mathbf{K}$-holomorphic, U definably connected, then:

- Every partial derivative of f is $\mathbf{K}$-holomorphic.
- The zero set of f has dimension $\leqslant 2n-2$, or else $f \equiv 0$.
- If at some point of U, the function f and all partial derivatives of f vanish then $f \equiv 0$.

Good boundary behavior

If $U \subseteq \mathbf{K}^n$ open and definably connected, $f : U \to \mathbf{K}$ is $\mathbf{K}$-holomorphic and nonconstant. Then the set

$$Z(f) = \{z \in bd(U) : \lim_{z' \to z} f(z') = 0\}$$

has dimension at most $2n - 2$.

Proof Assume that z is a generic element of $Z(f)$. If $\dim Z(f) = 2n - 1$ then there is a generic (over z) one-dimensional $\mathbf{K}$-subspace $\ell \subseteq \mathbf{K}^n$ such that $(\ell + z) \cap bd(U)$ is infinite. By the 1-variable case $f|U \cap (\ell + z) \equiv 0$. Since ℓ is generic get an open subset of U where f is 0. Finish using the 1-variable case. $\square$

Removing singularities

Theorem $U \subseteq \mathbf{K}^n$ is definable and open, $L \subseteq U$ definable and closed, $f : U \setminus L \to \mathbf{K}$ is a $\mathbf{K}$-holomorphic function.

- **Case A.** $\dim L \leqslant \dim U - 1$**.** If f is continuous on all of U then it is $\mathbf{K}$-holomorphic on all of U.
- **Case B.** $\dim L \leqslant \dim U - 2$. If f is locally bounded on L then f is $\mathbf{K}$-holomorphic on all of U.
- **Case C.** $\dim L \leqslant \dim U - 3$. In this case f is necessarily $\mathbf{K}$-holomorphic on all of U.

The ring of germs of K-holomorphic functions

Definition $\mathcal{O}_n^{\mathcal{R}}$ = The ring of germs of (definable) $\mathbf{K}$-holomorphic functions at $0 \in \mathbf{K}^n$.

Properties

• $\mathcal{O}_n^{\mathcal{R}}$ is injectively embedded into $\mathbf{K}[[\bar{z}]]$.

• $\mathcal{O}_n^{\mathcal{R}}$ is a local ring (the maximal ideal contains all germs which vanish at 0).

• Weierstrass Preparation and Division Theorems hold in $\mathcal{O}_n^{\mathcal{R}}$.

• $\mathcal{O}_n^{\mathcal{R}}$ is Noetherian.

No new entire functions

Theorem If $f : \mathbf{K}^n \to \mathbf{K}$ is $\mathbf{K}$-holomorphic then f is a polynomial function.

Proof Induction on n. Write $f = f(\bar{w}, z)$.

- $\forall w \in \mathbf{K}^{n-1}$, $f(\bar{w}, -)$ is a polynomial of degree $n(\bar{w})$ (by the 1-variable case).
- This remains true in elementary extensions, so $n(\bar{w})$ is bounded: $f(\bar{w}, z) = \Sigma_{k=0}^{n} a_k(\bar{w}) z^k$.
- $a_k(\bar{w}) = \dfrac{1}{k!} \dfrac{\partial^k}{\partial z^k} f(\bar{w}, 0)$. Thus, each a_k is $\mathbf{K}$-holomorphic on $\mathbf{K}^{n-1}$.
- By induction, each a_k is a polynomial. $\square$

References:

1. Expansions of algebraically closed fields in o-min. struct. ,P-S, *Selecta Mathematica*, NS 7 (2001) 409-445.
2. Expansions of ACF's II: Functions of several variables, P-S, *JML* v. 3. #1 (2003) 1-35.
3. Uniform definability of the Weierstrass $\wp$-functions and generalized tori of dimension one. To appear in *Selecta Mathematica.*
4. **Topological Analysis**, G.T. Whyburn, Princeton University Press, Princeton, 1964.
5. **Complex Analytic sets** E. M. Chirka, Kluwer Academic Publishers, 1989.
6. **Complex analytic varieties** H. Whitney, Addison-Wesley 1972.

Related work:

A glimpse at isoalgebraic spaces, R. Huber and M. Knebusch, Not di Matematica, vol X, Suppl. n. 2 (1990) 315-336.