Evelyn Nelson Lecture
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Definable groups in o-minimal structures

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References:

Tame topology and o-minimal structures, L. v.d. Dries, Cambridge University Press, London Mathematical Society Lecture Note Series 248 (1998).

Plan:

Part 1 First-order structures.

Part 2 O-minimality.

Part 3 Groups in o-minimal structures.

Part 4 Torsion-free abelian groups

• **A structure**

$$\mathcal{M} = \langle M; \overbrace{P_1, P_2, \ldots,}, \overbrace{f_1, f_2, \ldots} \rangle$$

• **First order formulas (over $\mathcal{M}$)**

Expressions built from

Relations, functions, Variables

Logical connectives: $\wedge, \vee, \neg, \rightarrow, \ldots$

Quantifiers: $\forall x$, $\exists x$

(and parameters from M)

Examples

(1) $\mathcal{M} = \langle R, +, \cdot \rangle$ a ring, $r \in R$.

A formula: $\forall y \exists x (x \cdot x = y + r)$ (no free-variables).

More examples

(2) $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, h(x) \rangle$.

$\psi(x) : \forall y \exists z \forall w (|x - w| < z \rightarrow |h(x) - h(w)| < y)$

$\psi(\mathcal{M}) = \{x \in \mathbb{R} : \psi(x)\} = \{x : h \text{ continuous at } x\}$

Definition The solution set of a formula is called a definable set.

• If $S \subseteq \mathbb{R}^n$ is a definable set in $\mathcal{M}$ then $Cl(S)$, ∂B, $Int(B)$ are definable as well.

• If h is differentiable then dh/dx is a definable function in $\mathcal{M}$ (i.e. its graph is a definable set).

(3) $\mathcal{G} = \langle G, * \rangle$ a group.

$Z(G) = \{g : \forall h \in G gh = hg\}$ is definable in $\mathcal{G}$.

BUT $[G, G] = \langle ghg^{-1}h^{-1} : g, h \in G \rangle$ is in general not definable.

Elementary equivalence

Definition $\mathcal{M} \equiv \mathcal{N}$ (for structures"of the same type") if every formula (with no free-variables) is true in $\mathcal{M}$ iff it is true in $\mathcal{N}$.

Examples

1. $F_1 \equiv F_2$ two rings. Then $CharF_1 = p$ / F_1 is algebraically closed IFF $CharF_2 = p$ / F_2 is algebraically closed.

2. $G_1 \equiv G_2$ two groups. Then G_1 is abelian/divisible IFF G_2 is abelian/divisible.

BUT

G_1 simple/perfect $\not\Rightarrow$ G_2 is simple/perfect.

Compactness in model theory

Assume $\psi(\bar{x})$ and $\{\phi_i(\bar{x}) : i \in I\}$ are formulas over $\mathcal{M}$. Assume that in every $\mathcal{N} \equiv \mathcal{M}$

$$\psi(N) = \bigcup_{i \in I} \phi_i(N).$$

Then there are $\phi_1, \ldots . \phi_k$ such that

$$\psi(M) = \bigcup_{i=1}^{k} \phi_i(M)$$

Example G a group. Assume that G **and every group elementarily equivalent to** G is perfect ($[G, G] = G$) then $[G, G]$ is obtained in finitely many steps.

Elimination Theory

In some structures all definable sets can be defined WITHOUT quantifiers. (or with a bounded number of them).

Examples (1) (Tarski-Seidenberg) Real closed fields: Every definable set is a boolean combination of polynomial inequalities $f(\bar{x}) < 0$.

(2) (Wilkie) Every definable set in $\mathbb{R}_{exp} = \langle \mathbb{R}, <, +, \cdot, e^x \rangle$ is of the form $\exists \bar{x} \phi(\bar{x}, \bar{y})$ (ϕ without quantifiers).

Corollary $\mathbb{R}_{exp}$ and every Real Closed Field are o-minimal:

Every definable set in one free-variable is a finite union of intervals.

O-minimal structures

Definition A linearly (densely) ordered $\mathcal{M} = \langle R, <, \ldots \rangle$ is *O(rder)-minimal* if every definable subset of R is a finite union of intervals, with end points in $R \cup \pm\infty$.

Examples

(1) Real closed fields.

(2) (Gabrielov, v.d. Dries)
$\mathbb{R}_{an} = \langle \mathbb{R}, <, +, \cdot, f : [-1,1]^n \to \mathbb{R} \rangle$, f real analytic on $[-1,1]^n$.

(2)(Wilkie, v.d.Dries-Miller)
$\mathbb{R}_{an,exp} = \langle \mathbb{R}, <, +, \cdot, \exp(t)$

(3) (v.d.Dries-Speissegger) $\langle \mathbb{R}_{an,exp}, \Gamma | (0,\infty)$.

(4) If $\mathcal{M}$ is o-minimal then there is an expansion of $\mathcal{M}$ in which every definable function has a definable primitive.

Properties of o-minimal structures:(Pillay, Steinhorn, Knight)

Assumption $\mathcal{M}$ an o-minimal expansion of a real closed field R.

(1) If $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{N}$ also o-minimal.

(2) Every definable subset of M^n has finitely many (definably) connected components.
Admits cell decomposition.

(3) Definable functions are piecewise C^n w.r. to R.

(4) Definable subsets of M^n have dimension in $\mathbb{N}$ (a-la manifold dimension). Tuples in M^n have dimension (a-la transcendence degree).

Definable groups

$\mathcal{M}$ an o-minimal expansion of a R.C. Field R.

A group $\langle G, * \rangle$ is definable in $\mathcal{M}$ if $G \subseteq M^n$ is a definable set and $*$ a definable function.

Setting: G definable in $\mathcal{M}$.

1. G admits (unique) a definable C^p-manifold structure w.r. to R, with finitely many definable charts, making it a C^p-group.

2. G is (definably) connected-by-finite.

3. G DCC on definable subgroups.

4. G infinite $\Rightarrow$ G has unbounded exponent.

5. If $\dim G = 1$ and (definably) connected then G is abelian and divisible.

Examples of definable groups

• (1-dimensional) $R_a = \langle R, + \rangle$, $R_m = \langle R^{>0}, \cdot \rangle$,

$SO(2, R)$ (real algebraic group). $\langle [0, 1), +(mod 1) \rangle$ (semi-algebraic group),

• $G(R)$, G an algebraic group/R (real algebraic). Or, $G(R)^0$ (semi-algebraic group).

• $G(K)$, G an algebraic group/K a.c.f. of characteristic zero.

• Compact real analytic Lie groups (definable in $\mathbb{R}_{an}$).

•

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^{\sqrt{2}} & v \\ 0 & 0 & 1 \end{pmatrix} t > 0, u, v \in \mathbb{R} \right\}$$

(solvable group) (definable in $\langle \mathbb{R}, <, +, \cdot, t^{\alpha} \rangle_{\alpha \in \mathbb{R}}$.)

• $\langle \mathbb{Z}, + \rangle$ is NOT definable.

Definably compact groups

Setting G definable in an o-minimal expansion of a real closed field.

G is called *definably compact* if every definable $f : (0,1)^R \to G$ has a limit point in G as t tends to 0. (Extension of classical compactness).

Examples: Compact real Lie groups, (Real points of) Abelian varieties. Nonstandard tori.

Theorem(Edmundo) Every definably compact group has torsion points.

Theorem(P-Steinhorn) Every G which is not definably compact has a definable 1-dimensional subgroup.

Definably simple groups in o-minimal structures (P-Pillay-Starchenko)

Definition A nonabelian G is *definably simple* if it has no definable normal subgroups.

Theorem 1 If G is definably simple then there is a real close field R such that:

(1) $G \simeq H(\widetilde{R})$, H an algebraic simple group/$\widetilde{R}$. Moreover, $\widetilde{R}$, H and the isomorphism can be definably recovered from $\langle G, * \rangle$.

OR

(2) $G \simeq H(R)^0$, H an R-simple algebraic group/R. Moreover, R, H and the isomorphism can be definably recovered from $\langle G, * \rangle$.

Theorem 2 A definably simple G has NO normal subgroups over all fields iff it is either of type (1) or it is "noncompact" of type (2) ($\Rightarrow$ Knesser-Tits Conjecture for real closed fields).

Compact := *Closed and bounded in* $GL(n, R)$

Examples

(1) $PSL(2, \widetilde{R})$.

(2a) Compact type: $SO(3, R)$ (if R is nonarchimedean then the infinitesimals of e form a normal subgroup).

(2b) Noncompact type: $PSL(2, R)$.

Definable linear groups in o-minimal structures (P-P-S)

Assumption $G < GL(n, R)$ definable in an o-minimal expansion of a real closed field R.

Theorem (1) Either G is semi-algebraic or G is definable in $\langle \mathbb{R}, <, +, \cdot, t^{\alpha_1}, \ldots, t^{\alpha_k} \rangle$, $\alpha_i \in \mathbb{R}$, or G is definable in $\langle \mathbb{R}, <, +, \cdot, \exp(x) \rangle$.

(2) If G is semi-simple, or "compact" then G is semi-algebraic.

(3) If G is nilpotent then G is definably isomorphic to a semi-algebraic group.

(4) There are definable solvable groups which are not even group- isomorphic to a semi-algebraic group:

$$G = \left\{ \begin{pmatrix} t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} t > 0, u, v \in \mathbb{R} \right\}$$

Abelian torsion-free groups

Theorem(Miller-Starchenko) If $\mathcal{M}$ is "polynomially bounded" then every 1-dimensional, torsion-free definable group is definably isomorphic to R_a or R_m.

Theorem(P-starchenko) Let G be definable in $\mathbb{R}_{an}^{\mathbb{R}}$ or an elementarily equivalent structure. If G is abelian and torsion-free then $G \simeq R_a^k \oplus R_m^n$, with the isomorphism definable in $\langle R, <, +, \cdot, G \rangle$.