

Complex geometry in tame model theoretic settings: definability, analyticity and algebraicity

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Failure in affine spaces

Clearly, the theorem fails in \mathbb{C}^n : $V = \{(z, e^z) : z \in \mathbb{C}\} \subseteq \mathbb{C}^2$.

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Assume that V is a complex analytic subvariety of \mathbb{C}^n . TFAE:

- (1) V is a \mathbb{C} -algebraic set.
- (2) V is **definable** in the real field $\langle \mathbb{R}; <, +, \cdot \rangle$ (to be discussed soon).

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- (1) V is semi-algebraic with respect to two distinct maximal real closed subfields of \mathbb{C} (namely, two identifications of \mathbb{C} with \mathbb{R}^2).
- (2) V is a \mathbb{C} -constructible set (i.e. a finite boolean combination of \mathbb{C} -algebraic varieties).

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4. **If $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_r$ then $A \times B \in \mathcal{D}_{k+r}$.**

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Write $\mathcal{D} = \langle U; \tilde{\mathcal{S}} \rangle$ for the structure spanned by $\tilde{\mathcal{S}}$.

The sets in \mathcal{D} are called \mathcal{D} -definable, or (first-order) definable in \mathcal{D} .

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A group as a structure

Every group $\langle G; \cdot \rangle$ can be viewed as a structure. The complexity of definable sets will depend on the specific group.

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1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is \mathfrak{D} -definable. The collection $\{C_b : b \in U^r\}$ is called a **definable family of sets**.

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Fact (bringing back Logic)

A set $X \subseteq U^k$ is \mathfrak{D} -definable **iff** it can be obtained from the sets in \check{S} via a finite sequence of Boolean operations and quantifiers “*there exist* $x \in U \dots$ ”, “*for all* $x \in U \dots$ ”. I.e., if X is “first-order definable” from \check{S} .

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- BUT $\int f dx$ is not definable in general (Riemann sums). E.g. $f(x) = 1/x$ is semi-algebraic but $\int 1/x = \log x$ is not.

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- BUT $\int f dx$ is not definable in general (Riemann sums). E.g. $f(x) = 1/x$ is semi-algebraic but $\int 1/x = \log x$ is not.
- If $f : \mathbb{C} \rightarrow \mathbb{C}$ is definable (its graph is a definable subset of \mathbb{R}^4) then the set $\{z \in \mathbb{C} : f \text{ is analytic at } z\}$ is definable.

Some more examples of first order definability

In groups

Let $\mathcal{D} = \langle G; \cdot \rangle$ for some group G . Then

$Z(G) = \{x \in G : \forall y \ x \cdot y = y \cdot x\}$ is \mathcal{D} -definable. **BUT**

The commutator $[G, G]$ is in general **not** \mathcal{D} -definable.

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- The real field $\mathcal{D}_{\mathbb{R}}$ is **NOT** definable in the complex field $\mathcal{D}_{\mathbb{C}}$. **WHY?**

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Strongly minimal structures are the building blocks of “stable” structures. They possess nice dimension theory, very much like Zariski dimension and transcendence degree.

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Topological tameness-o-minimal structures

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Assume that $\tilde{\mathcal{S}}$ (the initial collection) contains a linear ordering $<$ of U .

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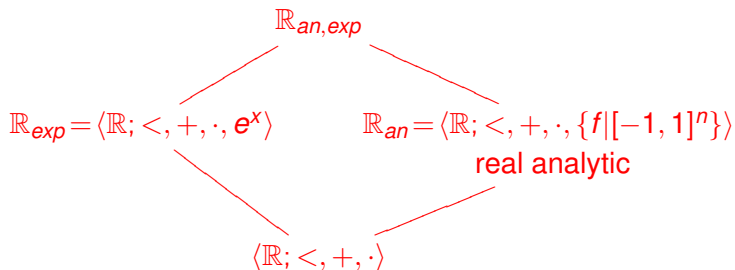
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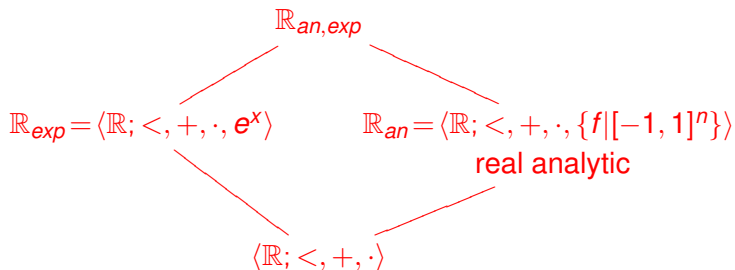


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A non-example: $\langle \mathbb{R}; <, +, \cdot, \sin x \rangle$ is not o-minimal.

Some tame features of o-minimality

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Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{\mathcal{S}} \rangle$ be an o-minimal structure expanding the real field.

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- ▶ The restriction of e^z to any strip $|Im(z)| < A$ is definable in $\mathbb{R}_{an,exp}$.

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In higher dimensions

A strong variant of the Remmert-Stein Theorem on removal of singularities, for \mathfrak{D} -definable analytic sets.

Some algebraicity results

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Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ is a complex-polynomial.

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- ▶ Hence, X is an algebraic subset of $\mathbb{P}^n(\mathbb{C})$.

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Corollary-Part Zil'ber's conjecture is true here

Assume that $V \subseteq \mathbb{C}^n$ is an \mathfrak{D} -definable set. If the structure $\langle \mathbb{C}; +, \cdot, V \rangle$ is strongly minimal then V is \mathbb{C} -constructible.

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Let \mathfrak{D} be an o-minimal structure over the reals.

Theorem (Hasson-Kowalski, 2008)

Assume that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a non-linear \mathfrak{D} -definable function such that $\langle \mathbb{C}; +, f \rangle$ is strongly minimal (i.e. **we added f to the additive group of \mathbb{C}**). Then up to conjugation by a matrix in $GL(2, \mathbb{R})$, the function f is \mathbb{C} -rational.

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Theorem* (Eleftherious-Hasson-Pe, 2016)

Let $\langle G; + \rangle$ be a \mathfrak{D} -definable 2-dimensional real Lie group and assume that $V \subseteq G^n$ is a “non-linear” \mathfrak{D} -definable set such that $\mathfrak{D}_G = \langle G; +, V \rangle$ is strongly minimal.

Then \mathfrak{D}_G , up to an isomorphism, is a one-dimensional complex algebraic group H endowed with all constructible subsets of H^n .

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- ▶ If $d_a f = 0$ then f is not locally injective near a .

Still open

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Assume that \mathcal{N} is a strongly minimal structure definable in an o-minimal one. Then \mathcal{N} is either “linear” (technically **locally modular**) or a finite cover of a complex algebraic curve over an \mathcal{N} -definable algebraically closed field.