Complex geometry in tame model theoretic settings: definability, analyticity and algebraicity

Kobi Peterzil

Department of Mathematics
University of Haifa

The Riemann Conference, Münster, October 2016

Chow’s Theorem (1949)

Chow's Theorem (1949)

Theorem

If V is a complex analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ then V is a $\mathbb{C}$-algebraic.

Chow's Theorem (1949)

Theorem

If V is a complex analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ then V is a $\mathbb{C}$-algebraic.

$*$ An **analytic subvariety** of a complex manifold M is given locally in M as the zero set of finitely many holomorphic functions.

Chow's Theorem (1949)

Theorem

If V is a complex analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ then V is a $\mathbb{C}$-algebraic.

$*$ An **analytic subvariety** of a complex manifold M is given locally in M as the zero set of finitely many holomorphic functions.

$*$ A **$\mathbb{C}$-algebraic subset of $\mathbb{P}^n(\mathbb{C})$** is given by finitely many homogeneous polynomials in $n+1$ complex variables.

Chow's Theorem (1949)

Theorem

If V is a complex analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ then V is a $\mathbb{C}$-algebraic.

$*$ An **analytic subvariety** of a complex manifold M is given locally in M as the zero set of finitely many holomorphic functions.

$*$ A **$\mathbb{C}$-algebraic subset of** $\mathbb{P}^n(\mathbb{C})$ is given by finitely many homogeneous polynomials in $n+1$ complex variables.

GAGA- Serre, 1956

Chow's Theorem is an important feature of a wider phenomenon, GAGA (Géometrie Algébrique et Géométrie Analytique), relating the analytic and algebraic theories of complex projective spaces.

Chow's Theorem (1949)

Theorem

If V is a complex analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ then V is a $\mathbb{C}$-algebraic.

$*$ An **analytic subvariety** of a complex manifold M is given locally in M as the zero set of finitely many holomorphic functions.

$*$ A $\mathbb{C}$**-algebraic subset of** $\mathbb{P}^n(\mathbb{C})$ is given by finitely many homogeneous polynomials in $n+1$ complex variables.

GAGA- Serre, 1956

Chow's Theorem is an important feature of a wider phenomenon, GAGA (Géometrie Algébrique et Géométrie Analytique), relating the analytic and algebraic theories of complex projective spaces.

Failure in affine spaces

Clearly, the theorem fails in $\mathbb{C}^n$: $V = \{(z, e^z) : z \in \mathbb{C}\} \subseteq \mathbb{C}^2$.

An affine result

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

$*$ A **semi-algebraic subset of** $\mathbb{C}^n \sim \mathbb{R}^{2n}$ is a finite boolean combination of sets of the form $p(\bar{x}) = 0$, $p(\bar{x}) > 0$, for $p(\bar{x})$ **a real polynomial in** $2n$**-real variables**.

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

$*$ A **semi-algebraic subset of** $\mathbb{C}^n \sim \mathbb{R}^{2n}$ is a finite boolean combination of sets of the form $p(\bar{x}) = 0$, $p(\bar{x}) > 0$, for $p(\bar{x})$ **a real polynomial in** $2n$**-real variables.**

A model theoretic formulation

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

$*$ A **semi-algebraic subset of** $\mathbb{C}^n \sim \mathbb{R}^{2n}$ is a finite boolean combination of sets of the form $p(\bar{x}) = 0$, $p(\bar{x}) > 0$, for $p(\bar{x})$ **a real polynomial in** $2n$**-real variables.**

A model theoretic formulation

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. TFAE:

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

$*$ A **semi-algebraic subset of** $\mathbb{C}^n \sim \mathbb{R}^{2n}$ is a finite boolean combination of sets of the form $p(\bar{x}) = 0$, $p(\bar{x}) > 0$, for $p(\bar{x})$ **a real polynomial in** $2n$**-real variables.**

A model theoretic formulation

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. TFAE:
(1) V is a $\mathbb{C}$-algebraic set.

An affine result

Theorem (Fortuna and Lojasiewicz-1981)

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is semi-algebraic* (under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$).

$*$ A **semi-algebraic subset of** $\mathbb{C}^n \sim \mathbb{R}^{2n}$ is a finite boolean combination of sets of the form $p(\bar{x}) = 0$, $p(\bar{x}) > 0$, for $p(\bar{x})$ **a real polynomial in** $2n$**-real variables.**

A model theoretic formulation

Assume that V is a complex analytic subvariety of $\mathbb{C}^n$. TFAE:
(1) V is a $\mathbb{C}$-algebraic set.
(2) V is **definable** in the real field $\langle \mathbb{R}; <, +, \cdot \rangle$ (to be discussed soon).

A comment on the reals inside $\mathbb{C}$

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical.

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$.

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!.

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!. Call such an R **a maximal real closed subfield**.

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!. Call such an R **a maximal real closed subfield.**

In fact, we have:

Pe-Starchenko-Tressle

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!. Call such an R **a maximal real closed subfield**.

In fact, we have:

Pe-Starchenko-Tressle

Assume that V is a subset of $\mathbb{C}^n$. Then the following are equivalent:

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!. Call such an R **a maximal real closed subfield**.

In fact, we have:

Pe-Starchenko-Tressle

Assume that V is a subset of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is semi-algebraic with respect to two distinct maximal real closed subfields of $\mathbb{C}$ (namely, two identifications of $\mathbb{C}$ with R^2).

A comment on the reals inside $\mathbb{C}$

Caution

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ is not canonical. There are many subfields R of $\mathbb{C}$ with $\mathbb{C} = R(\sqrt{-1})$. R could be archimedean or non-archimedean!. Call such an R **a maximal real closed subfield**.

In fact, we have:

Pe-Starchenko-Tressle

Assume that V is a subset of $\mathbb{C}^n$. Then the following are equivalent:
(1) V is semi-algebraic with respect to two distinct maximal real closed subfields of $\mathbb{C}$ (namely, two identifications of $\mathbb{C}$ with R^2).
(2) V is a $\mathbb{C}$-constructible set (i.e. a finite boolean combination of $\mathbb{C}$-algebraic varieties).

Some “logic free” model theory

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's.

Some “logic free” model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's.

The structure on U spanned by $\tilde{S}$

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's.
The structure on U spanned by $\tilde{S}$ is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k : k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's. **The structure on U spanned by $\tilde{S}$** is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k \colon k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's.
The structure on U spanned by $\tilde{S}$ is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k : k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.
2. Each singleton $\{a\}$ is in $\mathcal{D}_1$; each diagonal $\{(x_1, \ldots, x_k) : x_1 = x_k\}$ is in $\mathcal{D}_k$.

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's. **The structure on U spanned by $\tilde{S}$** is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k : k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.
2. Each singleton $\{a\}$ is in $\mathcal{D}_1$; each diagonal $\{(x_1, \ldots, x_k) : x_1 = x_k\}$ is in $\mathcal{D}_k$.
3. $\mathcal{D}_k$ is closed under finite union, intersection, and complement in U^k.

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's. **The structure on U spanned by $\tilde{S}$** is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k \colon k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.
2. Each singleton $\{a\}$ is in $\mathcal{D}_1$; each diagonal $\{(x_1, \ldots, x_k) : x_1 = x_k\}$ is in $\mathcal{D}_k$.
3. $\mathcal{D}_k$ is closed under finite union, intersection, and complement in U^k.
4. **If $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_r$ then $A \times B \in \mathcal{D}_{k+r}$.**

Some “logic free” model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's. **The structure on U spanned by $\tilde{S}$** is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k \colon k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.
2. Each singleton $\{a\}$ is in $\mathcal{D}_1$; each diagonal $\{(x_1, \ldots, x_k) : x_1 = x_k\}$ is in $\mathcal{D}_k$.
3. $\mathcal{D}_k$ is closed under finite union, intersection, and complement in U^k.
4. **If $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_r$ then $A \times B \in \mathcal{D}_{k+r}$.**
5. **If $A \in \mathcal{D}_{k+1}$ and $\pi \colon U^{k+1} \to U^k$ is the projection then $\pi(A) \in \mathcal{D}_k$.**

Some "logic free" model theory

A structure on a set U

Let $U \neq \varnothing$, and $\tilde{S}$ a fixed collection of subsets of U^k, for various k's.
The structure on U spanned by $\tilde{S}$ is the smallest collection of sets $\mathfrak{D} = \bigcup\{\mathcal{D}_k \colon k \in \mathbb{N}\}$, each $\mathcal{D}_k$ consisting of subsets of U^k, satisfying:

1. $\mathfrak{D}$ contains $\tilde{S}$.
2. Each singleton $\{a\}$ is in $\mathcal{D}_1$; each diagonal $\{(x_1, \ldots, x_k) : x_1 = x_k\}$ is in $\mathcal{D}_k$.
3. $\mathcal{D}_k$ is closed under finite union, intersection, and complement in U^k.
4. **If $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_r$ then $A \times B \in \mathcal{D}_{k+r}$.**
5. **If $A \in \mathcal{D}_{k+1}$ and $\pi \colon U^{k+1} \to U^k$ is the projection then $\pi(A) \in \mathcal{D}_k$.**

Write $\mathfrak{D} = \langle U; \tilde{S} \rangle$ for the structure spanned by $\tilde{S}$.

The sets in $\mathfrak{D}$ are called $\mathfrak{D}$-definable, or (first-order) definable in $\mathfrak{D}$.

Examples of structures

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$.

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$. (A theorem of Chevalley).

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$. (A theorem of Chevalley).

• Take $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot \rangle$. Then each $\mathcal{D}_k$ is the collection of the semi-algebraic subsets of $\mathbb{R}^k$.

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$. (A theorem of Chevalley).

• Take $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot \rangle$. Then each $\mathcal{D}_k$ is the collection of the semi-algebraic subsets of $\mathbb{R}^k$. (Tarski).

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$. (A theorem of Chevalley).

• Take $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot \rangle$. Then each $\mathcal{D}_k$ is the collection of the semi-algebraic subsets of $\mathbb{R}^k$. (Tarski).

A non-tame example

Take $\mathfrak{D} = \langle \mathbb{N}; +, \cdot \rangle$. The $\mathfrak{D}$-definable sets are very complicated-*the arithmetical sets* (Gödel).

Examples of structures

Tame examples

• Take $\mathfrak{D} = \langle \mathbb{C}; +, \cdot \rangle$ (i.e. we take the graphs of $+$ and $\cdot$). Then each $\mathcal{D}_k$ is the collection of the constructible subsets of $\mathbb{C}^k$. (A theorem of Chevalley).

• Take $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot \rangle$. Then each $\mathcal{D}_k$ is the collection of the semi-algebraic subsets of $\mathbb{R}^k$. (Tarski).

A non-tame example

Take $\mathfrak{D} = \langle \mathbb{N}; +, \cdot \rangle$. The $\mathfrak{D}$-definable sets are very complicated-*the arithmetical sets* (Gödel).

A group as a structure

Every group $\langle G; \cdot \rangle$ can be viewed as a structure. The complexity of definable sets will depend on the specific group.

Definability and logic

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Let $C \subseteq U^k \times U^r$ be a $\mathfrak{D}$-definable set.

1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is $\mathfrak{D}$-definable. The collection $\{C_b : b \in U^r\}$ is called **a definable family of sets**.

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Let $C \subseteq U^k \times U^r$ be a $\mathfrak{D}$-definable set.

1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is $\mathfrak{D}$-definable. The collection $\{C_b : b \in U^r\}$ is called **a definable family of sets**.
2. The set $\{a \in U^k : \exists b \in U^r \ (a, b) \in C\}$ is $\mathfrak{D}$-definable as well: it is the projection of C onto the first k coordinates.

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Let $C \subseteq U^k \times U^r$ be a $\mathfrak{D}$-definable set.

1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is $\mathfrak{D}$-definable. The collection $\{C_b : b \in U^r\}$ is called **a definable family of sets**.
2. The set $\{a \in U^k : \exists b \in U^r \ (a, b) \in C\}$ is $\mathfrak{D}$-definable as well: it is the projection of C onto the first k coordinates.
3. The set $\{a \in U^k : \forall b \in U^r \ (a, b) \in C\}$ is also $\mathfrak{D}$-definable:

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Let $C \subseteq U^k \times U^r$ be a $\mathfrak{D}$-definable set.

1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is $\mathfrak{D}$-definable. The collection $\{C_b : b \in U^r\}$ is called **a definable family of sets**.
2. The set $\{a \in U^k : \exists b \in U^r \ (a, b) \in C\}$ is $\mathfrak{D}$-definable as well: it is the projection of C onto the first k coordinates.
3. The set $\{a \in U^k : \forall b \in U^r \ (a, b) \in C\}$ is also $\mathfrak{D}$-definable: It is the complement in U^k of $\pi(\neg C)$.

Definability and logic

Let $\mathfrak{D} = (\mathcal{D}_k)$ be the structure on U spanned by some $\tilde{S}$.

By definition, each $\mathcal{D}_k$ is closed under finite Boolean combinations.

Let $C \subseteq U^k \times U^r$ be a $\mathfrak{D}$-definable set.

1. Given $a \in U^k$, each fiber $C_a = \{b \in U^r : (a, b) \in C\}$ is $\mathfrak{D}$-definable. The collection $\{C_b : b \in U^r\}$ is called **a definable family of sets**.
2. The set $\{a \in U^k : \exists b \in U^r \ (a, b) \in C\}$ is $\mathfrak{D}$-definable as well: it is the projection of C onto the first k coordinates.
3. The set $\{a \in U^k : \forall b \in U^r \ (a, b) \in C\}$ is also $\mathfrak{D}$-definable: It is the complement in U^k of $\pi(\neg C)$.

Fact (bringing back Logic)

A set $X \subseteq U^k$ is $\mathfrak{D}$-definable **iff** it can be obtained from the sets in $\tilde{S}$ via a finite sequence of Boolean operations and quantifiers *"there exist $x \in U$ ..."*, *"for all $x \in U$ ..."*. I.e., if X is "first-order definable" from $\tilde{S}$.

Some more examples of first order definability

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then $Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable.

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

- If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior.

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

• If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior. (since the ϵ, δ definitions are first-order).

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

• If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior. (since the ϵ, δ definitions are first-order).
• If $f : \mathbb{R} \to \mathbb{R}$ is definable (i.e. its graph is definable) and differentiable then its derivative is also definable (ϵ, δ definition).

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

• If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior. (since the ϵ, δ definitions are first-order).
• If $f : \mathbb{R} \to \mathbb{R}$ is definable (i.e. its graph is definable) and differentiable then its derivative is also definable (ϵ, δ definition).
• BUT $\int f dx$ is not definable in general (Riemann sums). E.g.
$f(x) = 1/x$ is semi-algebraic but $\int 1/x = log x$ is not.

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y \; x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

- If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior. (since the ϵ, δ definitions are first-order).
- If $f : \mathbb{R} \to \mathbb{R}$ is definable (i.e. its graph is definable) and differentiable then its derivative is also definable (ϵ, δ definition).
- BUT $\int fdx$ is not definable in general (Riemann sums). E.g. $f(x) = 1/x$ is semi-algebraic but $\int 1/x = logx$ is not.
- If $f : \mathbb{C} \to \mathbb{C}$ is definable (its graph is a definable subset of $\mathbb{R}^4$) then the set $\{z \in \mathbb{C} : f \text{ is analytic at } z\}$ is definable.

Some more examples of first order definability

In groups

Let $\mathfrak{D} = \langle G; \cdot \rangle$ for some group G. Then
$Z(G) = \{x \in G : \forall y\ x \cdot y = y \cdot x\}$ is $\mathfrak{D}$-definable. **BUT**
The commutator $[G, G]$ is in general **not** $\mathfrak{D}$-definable.

In $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$

• If $X \subseteq \mathbb{R}^n$ is definable then so is its closure and interior. (since the ϵ, δ definitions are first-order).
• If $f : \mathbb{R} \to \mathbb{R}$ is definable (i.e. its graph is definable) and differentiable then its derivative is also definable (ϵ, δ definition).
• BUT $\int f dx$ is not definable in general (Riemann sums). E.g. $f(x) = 1/x$ is semi-algebraic but $\int 1/x = log x$ is not.
• If $f : \mathbb{C} \to \mathbb{C}$ is definable (its graph is a definable subset of $\mathbb{R}^4$) then the set $\{z \in \mathbb{C} : f \text{ is analytic at } z\}$ is definable. Since the Cauchy-Riemann conditions are first-order expressible.

Definability of one structure in another

Definability of one structure in another

Definition

Given a structure $\mathfrak{D}$, we say that another structure $\mathcal{E} = \langle V; \tilde{T} \rangle$ is **definable in** $\mathfrak{D}$ if the set V and each set in $\tilde{T}$ are definable in $\mathfrak{D}$.

Definability of one structure in another

Definition

Given a structure $\mathfrak{D}$, we say that another structure $\mathcal{E} = \langle V; \tilde{T} \rangle$ is **definable in** $\mathfrak{D}$ if the set V and each set in $\tilde{T}$ are definable in $\mathfrak{D}$.

Examples

• If $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ then every complex algebraic group $\langle G; \cdot \rangle$ is definable in $\mathfrak{D}_{\mathbb{C}}$.

Definability of one structure in another

Definition

Given a structure $\mathfrak{D}$, we say that another structure $\mathcal{E} = \langle V; \tilde{T} \rangle$ is **definable in** $\mathfrak{D}$ if the set V and each set in $\tilde{T}$ are definable in $\mathfrak{D}$.

Examples

• If $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ then every complex algebraic group $\langle G; \cdot \rangle$ is definable in $\mathfrak{D}_{\mathbb{C}}$.
• If G is any simple algebraic group over $\mathbb{C}$ then $\langle \mathbb{C}; +, \cdot \rangle$ is definable in $\langle G; \cdot \rangle$.

Definability of one structure in another

Definition

Given a structure $\mathfrak{D}$, we say that another structure $\mathcal{E} = \langle V; \tilde{T}\rangle$ is **definable in** $\mathfrak{D}$ if the set V and each set in $\tilde{T}$ are definable in $\mathfrak{D}$.

Examples

• If $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot\rangle$ then every complex algebraic group $\langle G; \cdot\rangle$ is definable in $\mathfrak{D}_{\mathbb{C}}$.
• If G is any simple algebraic group over $\mathbb{C}$ then $\langle \mathbb{C}; +, \cdot\rangle$ is definable in $\langle G; \cdot\rangle$.
• If $\mathfrak{D}_{\mathbb{R}} = \langle \mathbb{R}; <, +, \cdot\rangle$ then the complex field $\langle \mathbb{C}, +, \cdot\rangle$ is definable in $\mathfrak{D}_{\mathbb{R}}$ (because $\mathbb{C} = \mathbb{R}^2$ and $+$, $\cdot$ are polynomial in the real and imaginary parts).

Definability of one structure in another

Definition

Given a structure $\mathfrak{D}$, we say that another structure $\mathcal{E} = \langle V; \tilde{T} \rangle$ is **definable in** $\mathfrak{D}$ if the set V and each set in $\tilde{T}$ are definable in $\mathfrak{D}$.

Examples

• If $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ then every complex algebraic group $\langle G; \cdot \rangle$ is definable in $\mathfrak{D}_{\mathbb{C}}$.

• If G is any simple algebraic group over $\mathbb{C}$ then $\langle \mathbb{C}; +, \cdot \rangle$ is definable in $\langle G; \cdot \rangle$.

• If $\mathfrak{D}_{\mathbb{R}} = \langle \mathbb{R}; <, +, \cdot \rangle$ then the complex field $\langle \mathbb{C}, +, \cdot \rangle$ is definable in $\mathfrak{D}_{\mathbb{R}}$ (because $\mathbb{C} = \mathbb{R}^2$ and $+, \cdot$ are polynomial in the real and imaginary parts).

• The real field $\mathfrak{D}_{\mathbb{R}}$ is **NOT** definable in the complex field $\mathfrak{D}_{\mathbb{C}}$. **WHY?**

Two criteria of tameness

Two criteria of tameness

Combinatorial tameness

A structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **strongly minimal** if every definable subset of U is finite or co-finite.

Two criteria of tameness

Combinatorial tameness

A structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **strongly minimal** if every definable subset of U is finite or co-finite. Moreover, if $C \subseteq U^{n+1}$ is definable then there is a number K such that for all $a \in U^n$, either $|C_a| < K$ or $|U \setminus C_a| < K$.

Two criteria of tameness

Combinatorial tameness

A structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **strongly minimal** if every definable subset of U is finite or co-finite. Moreover, if $C \subseteq U^{n+1}$ is definable then there is a number K such that for all $a \in U^n$, either $|C_a| < K$ or $|U \setminus C_a| < K$.

The prototype for strongly minimal structures

$\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ is strongly minimal: constructible subsets of $\mathbb{C}$ are finite or co-finite. Also true for every algebraically closed field.

Two criteria of tameness

Combinatorial tameness

A structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **strongly minimal** if every definable subset of U is finite or co-finite. Moreover, if $C \subseteq U^{n+1}$ is definable then there is a number K such that for all $a \in U^n$, either $|C_a| < K$ or $|U \setminus C_a| < K$.

The prototype for strongly minimal structures

$\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ is strongly minimal: constructible subsets of $\mathbb{C}$ are finite or co-finite. Also true for every algebraically closed field.

A theorem (Macintyre)

If $\mathfrak{D}$ is strongly minimal then any $\mathfrak{D}$-definable field is algebraically closed. So, $\langle \mathbb{R}; +, \cdot \rangle$ cannot be definable in $\langle \mathbb{C}; +, \cdot \rangle$.

Two criteria of tameness

Combinatorial tameness

A structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **strongly minimal** if every definable subset of U is finite or co-finite. Moreover, if $C \subseteq U^{n+1}$ is definable then there is a number K such that for all $a \in U^n$, either $|C_a| < K$ or $|U \setminus C_a| < K$.

The prototype for strongly minimal structures

$\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ is strongly minimal: constructible subsets of $\mathbb{C}$ are finite or co-finite. Also true for every algebraically closed field.

A theorem (Macintyre)

If $\mathfrak{D}$ is strongly minimal then any $\mathfrak{D}$-definable field is algebraically closed. So, $\langle \mathbb{R}; +, \cdot \rangle$ cannot be definable in $\langle \mathbb{C}; +, \cdot \rangle$.

Strongly minimal structure are the building blocks of "stable" structures. They posses nice dimension theory, very much like Zariski dimension and transcendence degree.

Zil’ber’s conjecture, its failure and resurrection

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

There are functions $\oplus, \odot$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \oplus, \odot \rangle$ is an algebraically closed field of characteristic 2 (or any p), and yet $\langle \mathbb{C}; +, \cdot, \oplus, \odot \rangle$ is still strongly minimal.

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

There are functions $\oplus, \odot$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \oplus, \odot \rangle$ is an algebraically closed field of characteristic 2 (or any p), and yet $\langle \mathbb{C}; +, \cdot, \oplus, \odot \rangle$ is still strongly minimal. Clearly, $\oplus$ and $\odot$ are **not** constructible.

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

There are functions $\oplus, \odot$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \oplus, \odot \rangle$ is an algebraically closed field of characteristic 2 (or any p), and yet $\langle \mathbb{C}; +, \cdot, \oplus, \odot \rangle$ is still strongly minimal. Clearly, $\oplus$ and $\odot$ are **not** constructible.

Zariski geometries (Hrushovski-Zil'ber, 1993)

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

There are functions $\oplus, \odot$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \oplus, \odot \rangle$ is an algebraically closed field of characteristic 2 (or any p), and yet $\langle \mathbb{C}; +, \cdot, \oplus, \odot \rangle$ is still strongly minimal. Clearly, $\oplus$ and $\odot$ are **not** constructible.

Zariski geometries (Hrushovski-Zil'ber, 1993)

By endowing a strongly minimal structure $\mathfrak{D}$ with a Neotherian topology, and listing some axioms, one can prove that the structure $\mathfrak{D}$ "**is**" an algebraically closed field.

Zil'ber's conjecture, its failure and resurrection

In the late 1970's Boris Zil'ber conjectured that "the only **rich** strongly minimal structures are algebraically closed fields". For example:

An optimistic conjecture

Consider $\mathfrak{D}_{\mathbb{C}} = \langle \mathbb{C}; +, \cdot \rangle$ and let $V \subseteq \mathbb{C}^n$ be an arbitrary set. If $\langle \mathbb{C}; +, \cdot, V \rangle$ is still strongly minimal then V is necessarily constructible.

Hrushovski's refutation(s)-1991

There are functions $\oplus, \odot$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \oplus, \odot \rangle$ is an algebraically closed field of characteristic 2 (or any p), and yet $\langle \mathbb{C}; +, \cdot, \oplus, \odot \rangle$ is still strongly minimal. Clearly, $\oplus$ and $\odot$ are **not** constructible.

Zariski geometries (Hrushovski-Zil'ber, 1993)

By endowing a strongly minimal structure $\mathfrak{D}$ with a Neotherian topology, and listing some axioms, one can prove that the structure $\mathfrak{D}$ "**is**" an algebraically closed field. First applications by Hrushovski of model theory to Diophantine geometry used **Zariski geometries**.

Topological tameness-o-minimal structures

Definition (Pillay-Steinhorn)

Assume that $\tilde{S}$ (the initial collection) contains a linear ordering $<$ of U.

Topological tameness-o-minimal structures

Definition (Pillay-Steinhorn)

Assume that $\tilde{S}$ (the initial collection) contains a linear ordering $<$ of U. The structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **o-minimal** if every $\mathfrak{D}$-definable subset of U is a finite union of points and $<$-intervals.

Topological tameness-o-minimal structures

Definition (Pillay-Steinhorn)

Assume that $\tilde{S}$ (the initial collection) contains a linear ordering $<$ of U. The structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **o-minimal** if every $\mathfrak{D}$-definable subset of U is a finite union of points and $<$-intervals.

Important examples

Topological tameness-o-minimal structures

Definition (Pillay-Steinhorn)

Assume that $\tilde{S}$ (the initial collection) contains a linear ordering $<$ of U. The structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **o-minimal** if every $\mathfrak{D}$-definable subset of U is a finite union of points and $<$-intervals.

Important examples

$$\mathbb{R}_{an,exp}$$

$$\mathbb{R}_{exp} = \langle \mathbb{R}; <, +, \cdot, e^x \rangle \qquad \mathbb{R}_{an} = \langle \mathbb{R}; <, +, \cdot, \{f | [-1, 1]^n\} \rangle$$

real analytic

$$\langle \mathbb{R}; <, +, \cdot \rangle$$

Topological tameness-o-minimal structures

Definition (Pillay-Steinhorn)

Assume that $\tilde{S}$ (the initial collection) contains a linear ordering $<$ of U. The structure $\mathfrak{D} = \langle U; \tilde{S} \rangle$ is **o-minimal** if every $\mathfrak{D}$-definable subset of U is a finite union of points and $<$-intervals.

Important examples

$$\mathbb{R}_{an,exp}$$

$$\mathbb{R}_{exp} = \langle \mathbb{R}; <, +, \cdot, e^x \rangle \qquad \mathbb{R}_{an} = \langle \mathbb{R}; <, +, \cdot, \{f|[-1,1]^n\} \rangle$$

real analytic

$$\langle \mathbb{R}; <, +, \cdot \rangle$$

A non-example: $\langle \mathbb{R}; <, +, \cdot, sinx \rangle$ is not o-minimal.

Some tame features of o-minimality

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ *be an o-minimal structure expanding the real field.*

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ *be an o-minimal structure expanding the real field.*

1. *Every* $\mathfrak{D}$*-definable subset of* $\mathbb{R}^n$ *has finitely many connected components*

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ *be an o-minimal structure expanding the real field.*

1. *Every* $\mathfrak{D}$*-definable subset of* $\mathbb{R}^n$ *has finitely many connected components (so, no definable infinite discrete sets).*

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ be an o-minimal structure expanding the real field.

1. *Every $\mathfrak{D}$-definable subset of $\mathbb{R}^n$ has finitely many connected components (so, no definable infinite discrete sets).*
2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathfrak{D}$-definable and $r \in \mathbb{N}$, then $\mathbb{R}^n$ can be partition into finitely many $\mathfrak{D}$-definable sets so that f is C^r on each of them.*

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ *be an o-minimal structure expanding the real field.*

1. *Every* $\mathfrak{D}$*-definable subset of* $\mathbb{R}^n$ *has finitely many connected components (so, no definable infinite discrete sets).*
2. *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $\mathfrak{D}$*-definable and* $r \in \mathbb{N}$*, then* $\mathbb{R}^n$ *can be partition into finitely many* $\mathfrak{D}$*-definable sets so that* f *is* C^r *on each of them.*
3. *If* A *is* $\mathfrak{D}$*-definable then* $\dim(cl(A) \setminus A) < \dim(A)$*.*

Some tame features of o-minimality

Theorem

Let $\mathfrak{D} = \langle \mathbb{R}; <, +, \cdot, \tilde{S} \rangle$ be an o-minimal structure expanding the real field.

1. *Every $\mathfrak{D}$-definable subset of $\mathbb{R}^n$ has finitely many connected components (so, no definable infinite discrete sets).*
2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathfrak{D}$-definable and $r \in \mathbb{N}$, then $\mathbb{R}^n$ can be partition into finitely many $\mathfrak{D}$-definable sets so that f is C^r on each of them.*
3. *If A is $\mathfrak{D}$-definable then $\dim(cl(A) \setminus A) < \dim(A)$.*
4. *Every $\mathfrak{D}$-definable field is either real closed or algebraically closed of characteristic zero.*

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable.

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

- Every complex polynomial is definable in $\langle \mathbb{R}; <, +, \cdot \rangle$.

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

- Every complex polynomial is definable in $\langle \mathbb{R}; <, +, \cdot \rangle$.
- Every germ of a holomorphic map is definable (on some neighborhood) in $\mathbb{R}_{an}$ (its real and imaginary parts are real analytic).

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

- Every complex polynomial is definable in $\langle \mathbb{R}; <, +, \cdot \rangle$.
- Every germ of a holomorphic map is definable (on some neighborhood) in $\mathbb{R}_{an}$ (its real and imaginary parts are real analytic).
- The map $z \mapsto e^z$ on $\mathbb{C}$ cannot be definable in any o-minimal structure (the pre-image of every point is infinite discrete).

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

- Every complex polynomial is definable in $\langle \mathbb{R}; <, +, \cdot \rangle$.
- Every germ of a holomorphic map is definable (on some neighborhood) in $\mathbb{R}_{an}$ (its real and imaginary parts are real analytic).
- The map $z \mapsto e^z$ on $\mathbb{C}$ cannot be definable in any o-minimal structure (the pre-image of every point is infinite discrete). **BUT**

Complex analysis and o-minimality

Fix $\mathfrak{D}$ an o-minimal structure expanding the real field.

Remark

Topological tameness implies that any definable $f : \mathbb{R}^n \to \mathbb{R}$ is piece-wise **real** differentiable. It cannot imply that any definable $f : \mathbb{C}^n \to \mathbb{C}$ is piecewise holomorphic (e.g. the $\mathbb{R}$-linear maps).

Examples of definable holomorphic maps

- Every complex polynomial is definable in $\langle \mathbb{R}; <, +, \cdot \rangle$.
- Every germ of a holomorphic map is definable (on some neighborhood) in $\mathbb{R}_{an}$ (its real and imaginary parts are real analytic).
- The map $z \mapsto e^z$ on $\mathbb{C}$ cannot be definable in any o-minimal structure (the pre-image of every point is infinite discrete). **BUT**
- The restriction of e^z to any strip $|Im(z)| < A$ is definable in $\mathbb{R}_{an,exp}$.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Let $G_f = graph(f)$.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Let $G_f = graph(f)$. If $f : U \setminus \{0\} \to \mathbb{C}$ has an essential singularity then $\{0\} \times \mathbb{C} \subseteq Cl(G_f)$.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Let $G_f = graph(f)$. If $f : U \setminus \{0\} \to \mathbb{C}$ has an essential singularity then $\{0\} \times \mathbb{C} \subseteq Cl(G_f)$.
But then $dim(Cl(G_f) \setminus G_f) = 2 = \dim(G_f)$. Contradiction.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Let $G_f = graph(f)$. If $f : U \setminus \{0\} \to \mathbb{C}$ has an essential singularity then $\{0\} \times \mathbb{C} \subseteq Cl(G_f)$.
But then $dim(Cl(G_f) \setminus G_f) = 2 = \dim(G_f)$. Contradiction.

Tame complex geometry

We still assume that $\mathfrak{D}$ is some o-minimal structure expanding $\mathfrak{D}_{\mathbb{R}}$.

Key feature

If $f : U \setminus \{0\}$ is $\mathfrak{D}$-definable and holomorphic then 0 is not an essential singularity.
Moreover, given a parameterized family of such functions in $\mathfrak{D}$ there is a finite bound on the order of all the functions at 0.

Proof

Let $G_f = graph(f)$. If $f : U \setminus \{0\} \to \mathbb{C}$ has an essential singularity then $\{0\} \times \mathbb{C} \subseteq Cl(G_f)$.
But then $dim(Cl(G_f) \setminus G_f) = 2 = \dim(G_f)$. Contradiction.

In higher dimensions

A strong variant of the Remmmert-Stein Theorem on removal of singularities, for $\mathfrak{D}$-definable analytic sets.

Some algebraicity results

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a complex-polynomial.

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a complex-polynomial. Moreover, given a definable family $f_t : \mathbb{C} \to \mathbb{C}$ of holomorphic functions, there is a bound d so that every f_t is a polynomial of degree at most d.

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a complex-polynomial. Moreover, given a definable family $f_t : \mathbb{C} \to \mathbb{C}$ of holomorphic functions, there is a bound d so that every f_t is a polynomial of degree at most d.
2. Every definable holomorphic $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a complex-polynomial. Moreover, given a definable family $f_t : \mathbb{C} \to \mathbb{C}$ of holomorphic functions, there is a bound d so that every f_t is a polynomial of degree at most d.
2. Every definable holomorphic $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.

A generalization of Fortuna-Lojasiewicz:

Some algebraicity results

$\mathfrak{D}$ is an o-minimal expansion of the real field.

Theorem (Pe-Starchenko)

1. Every definable holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a complex-polynomial. Moreover, given a definable family $f_t : \mathbb{C} \to \mathbb{C}$ of holomorphic functions, there is a bound d so that every f_t is a polynomial of degree at most d.
2. Every definable holomorphic $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.

   A generalization of Fortuna-Lojasiewicz:
3. Every definable analytic subset of $\mathbb{C}^n$ is algebraic.

A proof of Chow's Theorem in this setting

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

- $\mathbb{P}^n(\mathbb{C})$ can be endowed with a sufficiently small finite atlas $\{U_i : i \in I\}$, such that the transition maps and the functions which define X on each U_i are all definable in the o-minimal structure $\mathbb{R}_{an}$ (since we may use **restricted** real analytic maps).

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

- $\mathbb{P}^n(\mathbb{C})$ can be endowed with a sufficiently small finite atlas $\{U_i : i \in I\}$, such that the transition maps and the functions which define X on each U_i are all definable in the o-minimal structure $\mathbb{R}_{an}$ (since we may use **restricted** real analytic maps).
- Thus, X is definable in $\mathbb{R}_{an}$.

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

- $\mathbb{P}^n(\mathbb{C})$ can be endowed with a sufficiently small finite atlas $\{U_i : i \in I\}$, such that the transition maps and the functions which define X on each U_i are all definable in the o-minimal structure $\mathbb{R}_{an}$ (since we may use **restricted** real analytic maps).
- Thus, X is definable in $\mathbb{R}_{an}$.
- It follows that $X \cap \mathbb{C}^n$ is definable in $\mathbb{R}_{an}$ and still analytic in $\mathbb{C}^n$.

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

- $\mathbb{P}^n(\mathbb{C})$ can be endowed with a sufficiently small finite atlas $\{U_i : i \in I\}$, such that the transition maps and the functions which define X on each U_i are all definable in the o-minimal structure $\mathbb{R}_{an}$ (since we may use **restricted** real analytic maps).
- Thus, X is definable in $\mathbb{R}_{an}$.
- It follows that $X \cap \mathbb{C}^n$ is definable in $\mathbb{R}_{an}$ and still analytic in $\mathbb{C}^n$.
- By the algebraicity result $X \cap \mathbb{C}^n$ is an algebraic subset of $\mathbb{C}^n$.

A proof of Chow's Theorem in this setting

Consider the complex manifold $\mathbb{P}^n(\mathbb{C})$ and an analytic subset $X \subseteq \mathbb{P}^n(\mathbb{C})$.

- $\mathbb{P}^n(\mathbb{C})$ can be endowed with a sufficiently small finite atlas $\{U_i : i \in I\}$, such that the transition maps and the functions which define X on each U_i are all definable in the o-minimal structure $\mathbb{R}_{an}$ (since we may use **restricted** real analytic maps).
- Thus, X is definable in $\mathbb{R}_{an}$.
- It follows that $X \cap \mathbb{C}^n$ is definable in $\mathbb{R}_{an}$ and still analytic in $\mathbb{C}^n$.
- By the algebraicity result $X \cap \mathbb{C}^n$ is an algebraic subset of $\mathbb{C}^n$.
- Hence, X is an algebraic subset of $\mathbb{P}^n(\mathbb{C})$.

Topological + combinatorial tameness imply Zil'ber's dream?

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one

Topological $+$ combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).

Let $\mathfrak{D}$ be an o-minimal structure (over the reals).

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle\mathbb{C};+,\cdot\rangle$ is definable in the real field $\langle\mathbb{R};+,\cdot\rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).

Let $\mathfrak{D}$ be an o-minimal structure (over the reals).

Theorem (Pe-Starchenko, 2001)

Assume that $f:\mathbb{C}\to\mathbb{C}$ is definable in $\mathfrak{D}$.

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).

Let $\mathfrak{D}$ be an o-minimal structure (over the reals).

Theorem (Pe-Starchenko, 2001)

Assume that $f : \mathbb{C} \to \mathbb{C}$ is definable in $\mathfrak{D}$. If the structure $\langle \mathbb{C}; +, \cdot, f \rangle$ is still strongly minimal then f is a $\mathbb{C}$-rational function.

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).

Let $\mathfrak{D}$ be an o-minimal structure (over the reals).

Theorem (Pe-Starchenko, 2001)

Assume that $f : \mathbb{C} \to \mathbb{C}$ is definable in $\mathfrak{D}$. If the structure $\langle \mathbb{C}; +, \cdot, f \rangle$ is still strongly minimal then f is a $\mathbb{C}$-rational function.(**We replaced the geometric assumption on f by combinatorial tameness!**).

Topological + combinatorial tameness imply Zil'ber's dream?

As we observed, the complex field $\langle \mathbb{C}; +, \cdot \rangle$ is definable in the real field $\langle \mathbb{R}; +, \cdot \rangle$. So we have a strongly minimal structure definable in an o-minimal one (the converse is impossible!).
Let $\mathfrak{D}$ be an o-minimal structure (over the reals).

Theorem (Pe-Starchenko, 2001)

Assume that $f : \mathbb{C} \to \mathbb{C}$ is definable in $\mathfrak{D}$. If the structure $\langle \mathbb{C}; +, \cdot, f \rangle$ is still strongly minimal then f is a $\mathbb{C}$-rational function.(**We replaced the geometric assumption on f by combinatorial tameness!**).

Corollary-Part Zil'ber's conjecture is true here

Assume that $V \subseteq \mathbb{C}^n$ is an $\mathfrak{D}$-definable set. If the structure $\langle \mathbb{C}; +, \cdot, V \rangle$ is strongly minimal then V is $\mathbb{C}$-constructible.

Pursuing further Zil'ber's conjecture

Pursuing further Zil'ber's conjecture

Let $\mathfrak{D}$ be an o-minimal structure over the reals.

Theorem (Hasson-Kowalski, 2008)

Assume that $f : \mathbb{C} \to \mathbb{C}$ is a non-linear $\mathfrak{D}$-definable function such that $\langle \mathbb{C}; +, f \rangle$ is strongly minimal (i.e. **we added f to the additive group of** $\mathbb{C}$). Then up to conjugation by a matrix in $GL(2, \mathbb{R})$, the function f is $\mathbb{C}$-rational.

Pursuing further Zil'ber's conjecture

Let $\mathfrak{D}$ be an o-minimal structure over the reals.

Theorem (Hasson-Kowalski, 2008)

Assume that $f : \mathbb{C} \to \mathbb{C}$ is a non-linear $\mathfrak{D}$-definable function such that $\langle \mathbb{C}; +, f \rangle$ is strongly minimal (i.e. **we added f to the additive group of** $\mathbb{C}$). Then up to conjugation by a matrix in $GL(2, \mathbb{R})$, the function f is $\mathbb{C}$-rational.

Theorem* (Eleftherious-Hasson-Pe, 2016)

Let $\langle G; + \rangle$ be a $\mathfrak{D}$-definable 2-dimensional real Lie group and assume that $V \subseteq G^n$ is a "non-linear" $\mathfrak{D}$-definable set such that $\mathfrak{D}_G = \langle G; +, V \rangle$ is strongly minimal.
Then $\mathfrak{D}_G$, up to an isomorphism, is a one-dimensional complex algebraic group H endowed with all constructible subsets of H^n.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of ”analyticity”

Want to show:

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of "analyticity"

Want to show:

- If $X \subseteq G^2$ is $\mathcal{N}$-definable then either $Cl(X) = G^2$ or $Cl(X) \setminus X$ is finite.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of "analyticity"

Want to show:

- If $X \subseteq G^2$ is $\mathcal{N}$-definable then either $Cl(X) = G^2$ or $Cl(X) \setminus X$ is finite.

  Assume that $f : G \to G$ is an $\mathcal{N}$-definable (local) function.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of "analyticity"

Want to show:

- If $X \subseteq G^2$ is $\mathcal{N}$-definable then either $Cl(X) = G^2$ or $Cl(X) \setminus X$ is finite.

  Assume that $f : G \to G$ is an $\mathcal{N}$-definable (local) function.

- Show: outside finitely many points the map f is continuous and open.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of "analyticity"

Want to show:

- If $X \subseteq G^2$ is $\mathcal{N}$-definable then either $Cl(X) = G^2$ or $Cl(X) \setminus X$ is finite.

  Assume that $f : G \to G$ is an $\mathcal{N}$-definable (local) function.

- Show: outside finitely many points the map f is continuous and open.

- If $d_a f$ is non-invertible then $d_a f = 0$.

How does it work?

Fix a strongly minimal structure $\mathcal{N} = (G; +, V)$ definable in an o-minimal $\mathfrak{D}$.

Goal: Identify traces of "analyticity"

Want to show:

- If $X \subseteq G^2$ is $\mathcal{N}$-definable then either $Cl(X) = G^2$ or $Cl(X) \setminus X$ is finite.

  Assume that $f : G \to G$ is an $\mathcal{N}$-definable (local) function.

- Show: outside finitely many points the map f is continuous and open.
- If $d_a f$ is non-invertible then $d_a f = 0$.
- If $d_a f = 0$ then f is not locally injective near a.

Still open

Still open

The full Zil'ber's Conjecture in this setting

Assume that $\mathcal{N}$ is a strongly minimal structure definable in an o-minimal one.

Still open

The full Zil'ber's Conjecture in this setting

Assume that $\mathcal{N}$ is a strongly minimal structure definable in an o-minimal one. Then $\mathcal{N}$ is either "linear" (technically **locally modular**) or a finite cover of a complex algebraic curve over an $\mathcal{N}$-definable algebraically closed field.