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Tame complex analysis and o-minimality

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Abstract. We describe here a theory of holomorphic functions and analytic manifolds, restricted to the category of definable objects in an o-minimal structure which expands a real closed field R. In this setting, the algebraic closure K of the field R, identified with R^2, plays the role of the complex field. Although the ordered field R may be non-Archimedean, o-minimality allows to develop many of the basic results of complex analysis for definable K-holomorphic functions even in this non-standard setting. In addition, o-minimality implies strong theorems on removal of singularities for definable manifolds and definable analytic sets, even when the field R is $\mathbb{R}$. We survey some of these results and several examples.

We also discuss the definability in o-minimal structures of several classical holomorphic maps, and some corollaries concerning definable families of abelian varieties.

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1. Introduction

Consider a real closed field R and its algebraic closure $K = R(\sqrt{-1})$. After fixing $\sqrt{-1}$, we can identify K with R^2, and then view subsets of K^n as subsets of R^{2n}. Under this identification polynomial functions from K^n into K become R-polynomial maps from R^{2n} into R^2.

When the fields are $\mathbb{R}$ and $\mathbb{C}$, the order topology of the reals endows the complex numbers, through the product topology, with the structure of a topological locally compact field. This is of course the setting of classical complex analysis, and local analytic theory is usually developed using convergent power series and integration (here and below, when we say "classical" we refer to the case $R = \mathbb{R}$ and $K = \mathbb{C}$). When R is an arbitrary real closed field then its order topology still endows K with the structure of a topological field but, since R could be non-Archimedean, this topology may be far from locally compact. In this case, the tools of integration and power series are often not available for the development of complex analysis over K.

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While analysis in a non-Archimedean setting is also tackled in rigid analytic geometry we present here a different approach. The main idea is to consider only a limited collection of sets and maps, namely those which are definable in an o-minimal expansion $\mathcal{R} = \langle R, <, +, \cdot, \cdots \rangle$ of the field R. Recall that $\mathcal{R}$ is called o-minimal if every definable (with parameters) subset of R is a finite union of R-intervals whose endpoints are in $R \cup \{\pm\infty\}$. Real closed fields are the standard example but we are going to consider below much richer o-minimal structures (see [6], [38], [8], [19] and [20]).

It turns out (see [7] and [10]) that almost all basic theorems of real differential calculus hold for functions definable in $\mathcal{R}$, even though the field R may be non-Archimedean and as a topological space could be totally disconnected. As we will show, the same is true for many of the basic theorems of complex analysis.

When the field R equals $\mathbb{R}$, the category of definable sets in an o-minimal structure can be viewed as a natural candidate for Grothendieck's vision of "tame topology" (see discussion in [36]). The exclusion of wild topological phenomena from the tame setting of o-minimality implies that definable holomorphic functions cannot have essential singularities. This is easy to see, for if f is a holomorphic function on the punctured unit disc and 0 is an essential singularity then there exist $c \in \mathbb{C}$ with $f^{-1}(c)$ an infinite discrete subset of $\mathbb{C}$. But then, either $\{Im(z) : f(z) = c\}$ or $\{Re(z) : f(z) = c\}$ is an infinite discrete subset of $\mathbb{R}$, so f cannot be definable in an o-minimal structure. At first sight, this seems to exclude too much of classical analytic theory, but as we will see, it is still possible to define in o-minimal structures many classical holomorphic functions on properly chosen domains in a way which permits rich mathematical constructions.

Thus, the theory of holomorphic functions in o-minimal structures allows on one hand to develop analytic-like theory for an arbitrary algebraically closed field of characteristic zero K with respect to a maximal real closed field $R \subseteq K$ and an o-minimal expansion of R. On the other hand, when we specialize the investigation to the classical setting of the complex and real fields, we obtain, in addition, new results on holomorphic functions, complex manifolds and analytic sets, when these are definable in some o-minimal expansion of the real field. The treatment of both of these settings is uniform and independent of the particular fields in questions.

Our goal here is to present the main definitions and a survey of results, accompanied with examples from both the standard and the nonstandard settings. The paper is structured as follows: In Section 2 we give the basic definition of a K-holomorphic function and discuss a variety of examples. In Section 3 we show how analogues of basic results from complex analysis can be obtained for definable K-holomorphic functions in arbitrary o-minimal structures. In Section 4 we discuss analogues of complex manifolds and analytic sets in o-minimal structures and in particular, in 4.2 and in the Appendix expand on how compact complex manifolds can be viewed within the o-minimal structure $\mathbb{R}_{an}$. In 4.3 we present a more general, o-minimal, version of Chow's theorem on analytic subvarieties of projective space (which in particular implies the classical version). We also consider definable families of manifolds, the particular case of complex tori and point out the connection between such families and non-standard tori. In this section we discuss how Riemann's Existence Theorem can fail (or hold) in the category of definable manifolds in an o-minimal structure. In Section 5 we present several results on what is probably the main feature of tame complex analysis: the theory on removal of singularities. Finally, in Section 6 we discuss theorems on the definability in o-minimal structures of certain classical holomorphic functions such as Schwarz-Christoffel maps, the Weierstrass $\wp$-functions and Riemann's theta functions. We also mention connections to arithmetical questions in algebraic geometry.

We assume here basic knowledge of definability, and o-minimality (see [7] and [10] for a presentation aimed at non-logicians).

Remark. Some work on complex analytic geometry restricted to semiagebraic and subanalytic sets can be found in [11] and [12]. In the non-standard setting of an arbitrary real closed field, such work was carried out, from a different point of view than ours, in [15].

2. K-holomorphic functions

We start with the basic definitions. Let $\mathcal{R} = \langle R, <, +, \cdot, \ldots \rangle$ be an o-minimal expansion of a real closed field, and $K = R(\sqrt{-1})$ the algebraic closure of R. After fixing $i = \sqrt{-1}$, we can identify K with R^2, as in the classical case, and view subsets of K^n and maps from K^n into K as subsets of R^{2n} and maps from R^{2n} into R^2, respectively. The field operations of K become definable in the ordered field R. We have the order topology on R, the product topology on R^k, and with respect to this topology the field K, identified with R^2, is a topological field. We therefore have a natural notion of $\lim_{x \to a} f(x)$ for functions $f : K^n \to K$, where the limit is taken with respect to the topologies of K^n and of K.

Definition 2.1. Let $U \subseteq K$ be an open set. A function $f\colon U \to K$ is *K-differentiable at $z_0 \in U$* if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists in } K.$$

The limit, if exists, is called the *K-derivative of f at z_0* and is denoted by $f'(z_0)$. If f is K-differentiable at every $z \in U$ then it is called *K-holomorphic on U*.

For $U \subseteq K^n$ an open set and $f : U \to K$ a continuous function, f is called *K-holomorphic on U* if it is K-differentiable in each of the variables separately.

The above definitions coincide with the classical definitions of holomorphic functions in one and several variables when $R = \mathbb{R}$ and $K = \mathbb{C}$. As pointed out above, in the general case the topology on R is not well-behaved and very far from locally compact or separable. Hence, although the definitions make sense for arbitrary functions, we are going to restrict our attention to K-holomorphic functions which are in addition definable in the o-minimal structure $\mathcal{R}$.

Note: Although every algebraically closed field of characteristic zero K contains a maximal real closed field R, the choice of R is far from being unique and even the field $\mathbb{C}$ contains maximal real closed subfields which are not isomorphic to $\mathbb{R}$ and are non-Archimedean.

Here are some examples of K-holomorphic functions which are definable in o-minimal structures.

Classical examples

• Let $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot \rangle$ (so $K = \mathbb{C}$). By Tarski's work, $\overline{\mathbb{R}}$ is o-minimal. Every complex polynomial is $\mathbb{C}$-holomorphic and definable in $\overline{\mathbb{R}}$.

• Consider the o-minimal structure

$$\mathbb{R}_{an} = \langle \mathbb{R}, <, +, \cdot, \{f|[-1,1]^n : f \text{ real analytic on open } U \supseteq [-1,1]^n\} \rangle$$

(see [6]). Using the real and imaginary parts, every power series convergent in a neighborhood of $0 \in \mathbb{C}^n$ can be represented by a definable $\mathbb{C}$-holomorphic function in $\mathbb{R}_{an}$.

If $V \subseteq \mathbb{C}^n$ is an open bounded set, and $f : V \to \mathbb{C}$ is a holomorphic function, which can be holomorphically extended to an open set $U \supseteq Cl(V)$ (where $Cl(V)$ is the topological closure of V), then $f|V$ is definable in $\mathbb{R}_{an}$. Indeed, $Cl(V)$ can be covered by finitely many open sets on each of which f is definable, hence $f|V$ is definable.

• Let $\mathbb{R}_{an,exp}$ be the o-minimal expansion of $\mathbb{R}_{an}$ by the real exponential function (see [38], [10], [8]). The restriction of the complex exponential function e^z to any horizontal strip $\{a < Im(z) < b\}$, $a < b \in \mathbb{R}$, is definable in $\mathbb{R}_{an,exp}$, using the real exponential function and restricted $\sin, \cos$. It follows that every branch of $\ln z$ is definable in $\mathbb{R}_{an,exp}$. However, e^z is not definable on the whole of $\mathbb{C}$ because of its infinite discrete kernel, and in fact (see [23], Claim 2.1), if e^z is definable in some o-minimal structure on a set $U \subseteq \mathbb{C}$ then necessarily $Im(z)$ is bounded on U.

• Let $\mathbb{R}_{exp} = \langle \mathbb{R}, <, +, \cdot, e^x \rangle$. It follows from [3], that every germ of an n-variable holomorphic function which is definable in $\mathbb{R}_{exp}$ is already definable in $\overline{\mathbb{R}}$, namely semi-algebraic.

Non-standard examples

• Let $\overline{R} = \langle R, <, +, \cdot \rangle$, where R is a real closed field: Every polynomial over $K = R(\sqrt{-1})$ is K-holomorphic and definable in $\overline{R}$. In fact, in [25], Theorem 2.17, we prove a converse statement:

Theorem 2.2. *If $f : K^n \to K$ is definable and K-holomorphic then it is a polynomial over K.*

• Let $\mathcal{R}$ be a proper extension of $\mathbb{R}_{an,exp}$: If $\alpha \in R^{>0}$ is infinitesimally close to 0 (by that we mean that $0 < \alpha < 1/n$ for every $n \in \mathbb{N}$) then $e^{\alpha z}$ is K-holomorphic and definable on "infinitely wide" strip $-1/\alpha < Im(z) < 1/\alpha$.

These two non-standard examples of o-minimal structures are elementary extensions of structures over the field of reals. The example below does not arise from any structure over the reals (this is made precise in [14]):

• **Divergent power series as K-holomorphic functions.** Consider the real closed field of formal Puiseux series over $\mathbb{R}$, denoted by $R = \mathbb{R}((t^*))$, and its algebraic closure, $K = \mathbb{C}((t^*))$. The field R admits a natural valuation (with $v(t) = 1$) and the infinitesimal elements of R, denoted by μ, are all those of positive valuation. The valuation topology coincides in this case with the order topology of R.

Every formal power series $a(\bar{x}) \in \mathbb{R}[[x_1, \dots, x_n]]$ can be computed on μ^n and hence defines a function $\bar{a} : \mu^n \to R$. Clearly, if we expand the field R by such a function, the expanded structure will not be o-minimal because μ^n is not definable in any o-minimal structure. However, consider the interval $I = [-t, t]$ in R and the structure

$$\mathcal{R} = \langle R, <, +, \cdot, \bar{a}|I^n \rangle_{a(\bar{x}) \in \mathbb{R}[[\bar{x}]]}.$$

It is proved in [20] that $\mathcal{R}$ is o-minimal.

Now, every formal power series $a(\bar{z}) \in \mathbb{C}[[z_1, \dots, z_n]]$ (even if the series diverges in the complex field) determines a K-holomorphic function on the poly-disc of radius t in K^n, a map which is definable in $\mathcal{R}$.

3. Analogues of classical results in non-Archimedean fields

We assume here that $\mathcal{R}$ is an arbitrary o-minimal expansion of a real closed field R and $K = R(\sqrt{-1})$. All definability is assumed to take place in $\mathcal{R}$.

Although the classical tools of power series and integration are not available in this general setting, it is still possible to develop analogues of the classical theory for K-holomorphic functions which are definable in $\mathcal{R}$, by using methods of topological analysis, together with o-minimality. In the 1-variable case we followed the work of Whyburn from [37], and then extended it to functions of several variables. Almost all classical results go through in this case. When we specialize to the classical case, i.e. when R equals the field $\mathbb{R}$ and K equals $\mathbb{C}$, results of this type contribute no new information. However, even in this case model theory allows us to obtain new uniformity results for definable families of holomorphic functions.

3.1. The one-variable case.

All references are to [24].

Fact 3.1 (The Cauchy-Riemann equations). *If $U \subseteq K$ is an open definable set and $f : U \to K$ is a definable function then f is K-holomorphic if and only if, as a map $(x, y) \mapsto (v(x, y), w(x, y))$ from $U \subseteq R^2$ into R^2, it is R-differentiable* *and its R-derivatives satisfy*

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

(see Fact 2.27)

We let $D \subseteq K$ denote the closed unit disc and C its boundary. For $z = a + b\sqrt{-1} \in K$, we use $|z| = a^2 + b^2 \in R$.

Theorem 3.2. *1.* (**Maximum Principle**) *If $f : D \to K$ is a definable continuous function which is K-holomorphic on $Int(D)$ then $|f|$ attains its maximum on C (Theorem 2.31).*

2. (**Open mapping theorem**) *If $U \subseteq K$ is open, definable and $f : U \to K$ is a definable K-holomorphic, non-constant function then f is an open map (Corollary 2.34).*

3. (**Infinite differentiability**) *If $U \subseteq K$ is open, definable and $f : U \to K$ is a definable K-holomorphic map then $f'(z)$ is also K-holomorphic on U (Theorem 2.40).*

4. (**Identity Theorem**) *If $f : U \to K$ is definable and K-holomorphic in a neighborhood of $0 \in K$, and if $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ then f vanishes in a neighborhood of 0.*

We re-emphasize that (4) is true although there is no available theory of converging power series (indeed, if the underlying o-minimal structure is sufficiently saturated then there are no converging sequences in R other than the eventually constant ones). One corollary of (4) is that "raising to an infinite power" is not possible for elements in K. The situation is different in the case of R-variables: Consider $\mathcal{R}$ a nonstandard elementary extension of $\mathbb{R}_{exp}$ and let $\alpha > 0$ be an element greater than all $n \in \mathbb{N}$. The function

$$h_\alpha(x) = \begin{cases} x^\alpha & x \geq 0 \\ -\ x^\alpha & x < 0 \end{cases}$$

is definable in $\mathcal{R}$ by $x^\alpha = e^{\alpha \ln x}$ for all $x \in R$. It is infinitely R-differentiable at 0 and all of its R-derivatives are 0 there.

Given a definable K-holomorphic function $f : U \to K$ in a neighborhood $U \subseteq K$ of 0, we let $ord_0(f)$ be the minimal $k \geq 0$ such that $f^{(k)}(0) \neq 0$, or ∞ if there is no such k. The Identity Theorem implies that if f does not vanish in a neighborhood of 0 then $ord_0(f) < \infty$. Moreover, since the above result holds in arbitrary o-minimal structures, we get a uniform version which is interesting over $\mathbb{R}$ as well:

Given a definable open $0 \in U \subseteq K$, we say that *a family $\mathcal{F}$ of functions from U to K is definable in $\mathcal{R}$* if there are definable sets $T \subseteq R^n$ and $F \subseteq U \times K \times T$, such that for every $t \in T$, the set $\{(z, y) \in U \times K : (z, y, t) \in F\}$ is the graph of a function, call it f_t, and $\mathcal{F} = \{f_t : t \in T\}$. Assume now that every $f_t \in \mathcal{F}$ is K-holomorphic on U and does not vanish in a neighborhood of 0. Then, we claim that there is a bound k on $ord_0(f_t)$ as t varies in T. Indeed, if not then by logical compactness we would be able to realize (possibly in an elementary extension) a K-holomorphic non-vanishing f_{t_0} such that $f_{t_0}^{(k)}(0) \neq 0$ for all $k \in \mathbb{N}$. A contradiction. We therefore proved:

Theorem 3.3. *For $U \subseteq K$ a definable neighborhood of 0, let $\mathcal{F} = \{f_t : t \in T\}$ be a definable family of K-holomorphic maps $f_t : U \to K$. Then there is $k \in \mathbb{N}$ such that for every $t \in T$, if $f^{(i)}(0) = 0$ for all $i = 0, \ldots, k$ then f_t vanishes in a neighborhood of 0.*

3.2. Functions of several variables.

Definable K-holomorphic functions of several variables also share many common properties with classical holomorphic functions (see [25]). We limit ourselves here to several results about the ring of germs at 0 of definable K-holomorphic functions.

Definition 3.4. For definable functions f, g in a neighborhood of $0 \in K^n$, we say that f *and* g *have the same germ at* 0 if there is an open neighborhood $U \ni 0$ such that $f(z) = g(z)$ for all $z \in U$. Let $\mathcal{O}_n(\mathcal{R})$ be the ring of germs at $0 \in K^n$ of all K-holomorphic functions near $0 \in K^n$ which are definable in $\mathcal{R}$.

Here are some results about $\mathcal{O}_n(\mathcal{R})$ (see [25] for all references).

Theorem 3.5.

1. *The map from $\mathcal{O}_n$ into $K[[\bar{z}]]$, which sends a germ $f \in \mathcal{O}_n$ to its formal Taylor expansion at 0, is injective. Said differently, if all derivatives of a definable K-holomorphic f vanish at $0 \in K^n$ then f itself vanishes in a neighborhood of 0 (Theorem 2.30 (2)).*
2. *$\mathcal{O}_n$ is a local ring.*
3. *The ring $\mathcal{O}_n$ satisfies the Weierstrass preparation and division theorems (see Theorem 2.20 and Theorem 2.23).*
4. *The ring $\mathcal{O}_n$ is Noetherian, (Theorem 2.30).*

4. Definable K-manifolds and K-analytic sets

4.1. Basic definitions.

Once we have the notion of a K-holomorphic function in several variables we may define the notions of a manifold and an analytic set, with respect to the field K. We restrict our attention only to definable functions and definable sets in a fixed o-minimal expansion $\mathcal{R}$ of a real closed field R, with $K = R(\sqrt{-1})$.

Definition 4.1. *A definable n-dimensional K-manifold* is a definable set M (living in some R^k), equipped with a finite cover of definable sets $M = \bigcup_i U_i$, each of which is in definable bijection $\phi_i : U_i \to V_i$ with a definable open set $V_i \subseteq K^n$, such that the transition maps

$$\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_j)$$

are K-holomorphic (as maps between open subsets of K^n). The collection $\{\langle U_i, \phi_i \rangle : i \in I\}$ is called a definable atlas for M.

Let M be a definable n-dimensional K-manifold. A definable $N \subseteq M$ is called *a d-dimensional K-submanifold of M* if every $a \in N$ has a definable open neighborhood $U \subseteq M$ and a definable K-holomorphic $f : U \to K^{n-d}$ such that $N \cap U = f^{-1}(0)$ and such that the K-differential of f at a (which is defined exactly as in the classical case) has K-rank $n - d$.

In [28], Lemma 3.3, we show that every definable K-submanifold of a definable manifold is itself a definable K-manifold, namely has a definable *finite* atlas.

If M and N are definable K-manifolds then a definable map $f : M \to N$ is called *K-holomorphic* if, when read through the charts of M and N, becomes a (definable) K-holomorphic map.

Definition 4.2. A definable $A \subseteq M$ is called *a K-analytic subset of M* if at every $z \in M$, the set A is given, locally near z, as the zero set of finitely many definable K-holomorphic functions. The set $A \subseteq M$ is called *a locally K-analytic subset of M* if the same is true for every $z \in A$.

The *K-dimension of a K-analytic set* A is defined to be the maximal d such that A contains a d-dimensional K-submanifold of M.

We use $\dim_K A$ to denote the dimension of A as a K-analytic set and $\dim_R A$ to denote its o-minimal dimension. As we show in [28], $\dim_R A = 2 \dim_K A$.

When the underlying real closed field is the field of real numbers then definable $\mathbb{C}$-manifolds and definable $\mathbb{C}$-analytic subsets are just complex manifolds and complex analytic subsets, respectively, which are in addition definable in the underlying o-minimal structure $\mathcal{R}$.

We now review several examples of definable K-manifolds and K-analytic sets in o-minimal structures.

4.2. Compact complex manifolds.

An important collection of definable manifolds in o-minimal structures is that of compact complex manifolds.

Every compact complex analytic manifold is isomorphic, as a complex manifold, to a definable $\mathbb{C}$-manifold in the structure $\mathbb{R}_{an}$. More explicitly, assume that $\{\langle U_i, \phi_i \rangle : i \in I\}$ is a finite atlas for an n-dimensional real analytic compact manifold M. Then, as we show in the Appendix, the atlas can be replaced by a new finite atlas $\{\langle B_x, \phi_{i(x)}|B_x \rangle : x \in X\}$, with each B_x an open subset of $U_{i(x)}$ for some $i(x) \in I$, and such that: (i) each $\phi_{i_x}(B_x)$ is a definable subset of $\mathbb{R}^n$ in $\mathbb{R}_{an}$, and (ii) for all $x, y \in X$, the transition maps $\phi_{y,x} := \phi_{i(y)} \phi_{i(x)}^{-1}$ are definable on $\phi_{i(x)}(B_x \cap B_y)$ in $\mathbb{R}_{an}$. It is not hard now to realize M as a definable quotient and, using definable choice in o-minimal expansions of fields (see 6.1.2 in [7]), as a definable set, with a definable atlas.

If M is a compact complex manifold then we use the same process as above. Since the transition maps we obtain are just restrictions of the original maps, we get in this manner a complex manifold which is definable in $\mathbb{R}_{an}$.

If M is a compact complex manifold which is already definable in $\mathbb{R}_{an}$ then every complex analytic subset of M is definable in $\mathbb{R}_{an}$.

Compact complex manifolds were studied elsewhere in model theory after Zil'ber ([39]) proved that, when endowed with all analytic subsets, they admit quantifier elimination and produce a stable structure of finite Morley rank. One may then study the many-sorted structure, denoted sometimes by CCM, given by the category of all compact complex manifolds (up to an isomorphism), with all their analytic subsets and with the analytic maps between them. For a survey of this work see [21] (see also in [13] and in [34]).

Our above discussion implies that the category CCM is interpretable in the o-minimal $\mathbb{R}_{an}$. However, this hides a subtlety that we wish to address here. Note that a compact complex manifold can be realized in many different ways, depending on the underlying set and the choice of atlas. Since we sometimes wish to examine the definability of a particular presentation of a manifold, or the definability of a particular holomorphic function on this manifold, it is often not sufficient to study the manifold "up to an isomorphism". As the following claim shows, it is possible that the underlying topological space of compact complex manifold is semialgebraic, and yet a complex atlas for the manifold is only definable in $\mathbb{R}_{exp}$.

Claim 4.3. *There is an $\mathbb{R}_{exp}$-definable complex manifold structure $\mathcal{S}$ on the unit sphere S_2 in $\mathbb{R}^3$ such that $\mathcal{S}$ does not have an atlas definable in $\mathbb{R}_{an}$.*

Proof. Let $S_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$, $p_n = (0, 0, 1), p_s = (0, 0, -1)$, and $S_2^* = S_2 \setminus \{p_n, p_s\}$. It is easy to see that S_2^* is semalgebraically homeomorphic to the cylinder $S_1 \times \mathbb{R}$, where $S_1 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$, and we fix such a homeomorphism $h \colon S_2^* \to S_1 \times \mathbb{R}$. Let $\varphi \colon S_1 \times R \to \mathbb{C}^*$ be the map $\varphi : ((x, y), r) \mapsto (x + iy)e^r$. It is not difficult to see that φ is a homeomorphism definable in $\mathbb{R}_{exp}$. The map $\varphi \circ h \colon S_2^* \to \mathbb{C}^*$ extends to a homeomorphism $\Phi \colon S_2 \to \mathbb{P}_1(\mathbb{C})$ by mapping p_s to 0 and p_N to ∞. Obviously, Φ is definable in $\mathbb{R}_{exp}$.

We use Φ to pull-back the complex structure from $\mathbb{P}_1(\mathbb{C})$ to S_2, and obtain a complex manifold structure $\mathcal{S}$ on S_2. With respect to this structure the map Φ is a biholomorphism. Since $\mathbb{P}^1(\mathbb{C})$ has a semialgebraic atlas, the complex manifold $\mathcal{S}$ has an atlas definable in $\mathbb{R}_{exp}$.

We claim that $\mathcal{S}$ does not have a complex manifold atlas definable in $\mathbb{R}_{an}$. Indeed, if $\mathcal{S}$ admits a definable complex atlas in $\mathbb{R}_{an}$ then Φ should be definable in $\mathbb{R}_{an}$ as well the map $\varphi \colon S_1 \times R \to \mathbb{C}^*$, contradicting the fact that the real exponential function is not definable in $\mathbb{R}_{an}$. □

Remark 4.4. In the above example, since S_2^* is an open subset of S_2, it has an induced complex manifold structure, call is $\mathcal{S}^*$. It is not hard to see that $\mathcal{S}^*$ has an atlas definable in the structure $\mathbb{R}_{an}$(but, as we saw above, this atlas cannot be extended definably in $\mathbb{R}_{an}$ to an atlas for $\mathcal{S}$)

4.3. K-algebraic and K-analytic sets. For every real closed field R and its algebraic closure K, the sets K^n and $\mathbb{P}^n(K)$ are naturally K-manifolds definable in $\langle R, <, +, \cdot \rangle$. More generally, every non-singular algebraic subvariety of K^n or $\mathbb{P}^n(K)$ can be naturally endowed with a semialgebraic K-manifold structure. Algebraic subvarieties of K^n or $\mathbb{P}^n(K)$ are K-analytic subsets of K^n or $\mathbb{P}^n(K)$, respectively.

In fact, using Theorem 2.2 above, we also have the converse (see [28], Theorem 5.1):

Theorem 4.5. *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field R, with K its algebraic closure. If V is a definable K-analytic subset of K^n or of $\mathbb{P}^n(K)$ then V is an algebraic variety over K.*

When we specialize the above theorem to the o-minimal structure $\mathbb{R}_{an}$, we obtain that every definable analytic subset of $\mathbb{P}^n(\mathbb{C})$ is an algebraic variety. However, as we pointed out earlier, every analytic subset of a compact complex manifold is definable in $\mathbb{R}_{an}$ so we obtain the classical theorem of Chow: *Every analytic subset of $\mathbb{P}^n(\mathbb{C})$ is algebraic.*

Similar results for semialgebraic complex analytic sets can be found in [11], and in the "isoalgebraic" setting in [15].

4.4. Definable families of K-manifolds. If X, Y, F are sets with $F \subseteq X \times Y$ then for $x \in X$, we will denote by F_x the fiber $F_x = \{y \in Y : (x, y) \in F\}$. We say that a family $\mathcal{F} = \{F_x : x \in X\}$ of subsets of Y is definable if X, Y and $F \subseteq X \times Y$ are definable sets.

If $\mathcal{R}$ is an o-minimal expansion of $\overline{\mathbb{R}}$ and $\mathcal{R}^*$ is an elementary extension of $\mathcal{R}$ then every K-manifold M which is definable in $\mathcal{R}^*$ is obtained as a fiber in a definable family $\mathcal{F}$ of complex manifolds in the structure $\mathcal{R}$ (by that we mean that the underlying sets of the manifolds as well as their atlases are given by definable families in $\mathcal{R}$). Thus, first order properties of the manifold M reflect uniform properties of manifolds in the family $\mathcal{F}$. Let us consider one such property:

As we know by Riemann's work, every one-dimensional compact complex manifold M is biholomorphic with an algebraic nonsingular projective curve $\mathcal{C}$. If M is definable in $\mathbb{R}_{an}$ then the graph of this biholomorphism is an analytic subset of the definable compact manifold $M \times \mathcal{C}$ and therefore is itself definable in $\mathbb{R}_{an}$.

Assume now that we are given a definable family of compact one-dimensional complex-manifolds $\{M_t : t \in T\}$ in some o-minimal structure $\mathcal{R}$ over $\mathbb{R}$. Is there a definable family of biholomorphisms of these manifolds with projective varieties? Or, equivalently, consider an elementary extension $\mathcal{R}^*$ of $\mathcal{R}$ and a member M_{t_0} of the family, for a parameter t_0 from $\mathcal{R}^*$. Is the K-manifold M_{t_0} definably K-biholomorphic with a projective algebraic variety over K? Our original motivation for asking this question was an analogous theorem of Moosa (see [22]) stating that if the family $\mathcal{F}$ is definable in CCM then indeed there is in CCM a definable family of such biholomorphisms with projective algebraic varieties. It turns out that in the o-minimal setting the answer is negative, as we now describe.

4.5. The family of complex tori. For $\bar{\omega} = (\omega_1, \ldots, \omega_{2n})$ a tuple of $2n$ vectors in $\mathbb{C}^n$ which are linearly independent over $\mathbb{R}$, let $\Lambda_{\bar{\omega}} \subset \mathbb{C}^n$ be the lattice $\mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}$. Since $\Lambda_{\bar{\omega}}$ is a discrete subgroup of $(\mathbb{C}^n, +)$, the quotient group $\mathcal{E}_{\bar{\omega}} = (\mathbb{C}^n, +)/(\Lambda_{\bar{\omega}}, +)$ inherits a complex-analytic structure, and with respect to this structure, $\mathcal{E}_{\bar{\omega}}$ is a connected compact complex Lie group of dimension n, i.e. *an n-dimensional complex torus.*

Although the lattice $\Lambda_{\bar{\omega}}$ is an infinite discrete set and thus is not definable in any o-minimal structure, we are going to view these tori definably as follows: The underlying set of $\mathcal{E}_{\bar{\omega}}$ is identified with the definable parallelogram

$$E_{\bar{\omega}} = \{t_1\omega_1 + \cdots + t_{2n}\omega_{2n} : \bigwedge_{i=1}^{2n} 0 \le t_i < 1\}, \tag{1}$$

and then it is not hard to produce a semialgebraic atlas on $E_{\bar{\omega}}$ with semialgebraic transition maps, corresponding to the complex analytic structure of $\mathcal{E}_{\bar{\omega}}$. Therefore each $\mathcal{E}_{\bar{\omega}}$ can be viewed as a $\mathbb{C}$-manifold definable in the field $\overline{\mathbb{R}}$, and moreover these definable charts and transition maps can be constructed uniformly in $\bar{\omega}$, thus obtaining a semi-algebraic family of all n-dimensional complex tori. It follows that in every real closed field R, if we take a tuple $\bar{\omega}$ of $2n$ vectors in K^n ($K = R(\sqrt{-1})$) which are linearly independent over R, we have a corresponding definable K-manifold $\mathcal{E}_{\bar{\omega}}$, which we call a K-torus.

In [26] we considered the family, call it $\mathcal{F}$, of all one-dimensional complex tori in various o-minimal expansions of $\overline{\mathbb{R}}$. Each member of $\mathcal{F}$ is an elliptic curve, i.e. biholomorphic with a smooth projective cubic curve. The biholomorphism between these two compact complex manifolds is definable in $\mathbb{R}_{an}$. However, as we show in [26], Corollary 5.6, a full family of such biholomorphisms is not definable in an o-minimal structure. Formulated in the language of non-standard o-minimal structures we have:

Theorem 4.6. *Let $\mathcal{R}$ be an arbitrary o-minimal expansion of $\mathbb{R}_{an,exp}$ and let $\mathcal{R}^* = \langle R, <, +, \cdot, \cdots \rangle$ be a non-Archimedean elementary extension of $\mathcal{R}$, with $K = R(\sqrt{-1})$. If $\tau \in K$ is such that $Im(\tau) > 0$ and $Re(\tau)$ greater than all standard $n \in \mathbb{N}$, then the K-torus $\mathcal{E}_{1,\tau}$ is not definably K-biholomorphic, in the structure $\mathcal{R}^*$, with any algebraic curve.*

We thus showed the failure of the definable analogue to Riemann's Existence Theorem, for definably compact one-dimensional K-manifolds in o-minimal structure (a "definably compact manifold" here can be taken to mean a $\mathcal{R}^*$-fiber in an $\mathcal{R}$-definable family of compact real manifolds).

In Section 4.6 below and Section 6.2 we discuss some positive cases of Riemann's theorem.

4.6. Mild manifolds. We let $\mathcal{R}$ be an o-minimal expansion of a real closed field R and $K = R(\sqrt{-1})$

Let M be a definable K-manifold, and let $\mathcal{A}(M)$ be the structure whose universe is M and its atomic relation are all the definable K-analytic subsets of M^n,

$n \in \mathbb{N}$. In [27] we called M *a mild manifold* if $\mathcal{A}(M)$ admits quantifier elimination. Examples are compact complex manifolds (by Zil'ber's work [39]), definably compact K-manifolds (see Theorem 8.3 in [28]), the set of K-regular points of an algebraic variety over K (projective or affine). On the other hand, the open unit disc in $\mathbb{C}$ is a definable complex-manifold which is not mild in any o-minimal structure.

In an attempt to understand better the previous example of a non-algebraic one-dimensional K-torus we proved the following result (see Theorem 6.0.1, and Theorem 4.4.3 [27]), which can be seen as a conditional Riemann Existence Theorem.

Theorem 4.7. *Let M be a definable K-manifold which is mild and also strongly minimal (namely, in the structure $\mathcal{A}(M)$ every definable subset of M is finite or co-finite). Then the following are equivalent:*

1. *$\mathcal{A}(M)$ is non locally modular.*

2. *There is a finite $F \subseteq M$ and a definable non-constant K-holomorphic function $\phi : M \setminus F \to K$ (we call ϕ* a K-mermomorphic function *on M).*

3. *There is a definable K-biholomorphism between M and a non-singular algebraic curve over K.*

In particular, the non algebraic one-dimensional K-torus $\mathcal{E}_{1,\tau}$ of Theorem 4.6 admits no definable nonconstant K-meromorphic map into K and $\mathcal{A}(\mathcal{E})$ is locally modular. If we translate the above theorem to definable families of compact complex one-dimensional manifolds (which are all mild and strongly minimal) then we get some uniform version of Riemann's theorem:

Corollary 4.8. *Let $\mathcal{R}$ be an o-minimal structure over $\mathbb{R}$ and let $\mathcal{F} = \{M_t : t \in T\}$ be a a definable family of one-dimensional compact complex manifolds, given together with a definable family $\phi_t : M_t \to \mathbb{C}$ of nonconstant meromorphic maps.*

Then, there is in $\mathcal{R}$ a definable family of complex algebraic curves $\{C_t : t \in T\}$ and a definable family of complex biholomorphisms $\sigma_t : M_t \to C_t$.

5. Theorems on removal of singularities

One of the most useful features of working with analytic objects which are definable in o-minimal structures is the theory of removal of singularities: Start with a complex manifold M, an open set $U \subseteq M$ and consider an analytic subset A of U. In general, the topological closure of A in M is not analytic in M. A great deal of attention has been given classically to conditions under which $Cl(A)$ is analytic in M. Assuming that M, U and A are definable in an o-minimal structure one obtains strong results in both the standard and non-standard settings.

5.1. Characterizing K-analytic sets.

Definition 5.1. Given a definable K-manifold M, and a definable $A \subseteq M$, we define *the set of K-regular points of A*, denoted by $Reg_K(A)$, as the set of all points $a \in A$ such that in some neighborhood of a, the set A is a K-submanifold of M. We let $Sing_K(A) = A \setminus Reg_K(A)$.

We call a definable $A \subseteq M$ *a finitely K-analytic subset of M* if M can be covered by finitely many definable open sets $M = \bigcup_j W_j$ and for each j there is a definable K-holomorphic map $\psi_j : W_j \to K^{m_j}$, such that $A \cap W_j = \psi^{-1}(0)$.

Clearly, every finitely K-analytic set in K-analytic. As for the converse, note that if M is a compact complex manifold, definable in $\mathbb{R}_{an}$, then every $\mathbb{C}$-analytic subset of M is $\mathbb{C}$-finitely analytic. It turns out that o-minimality can replace the role of compactness and that in the o-minimal setting this converse is always true. Here is one of the main theorems characterizing definable K-analytic sets (see[28], Corollary 4.14):

Theorem 5.2. *Let M be a definable K-manifold and $A \subseteq M$ a definable closed set. Then the following are equivalent:*

1. *A is a K-analytic subset of M.*
2. *A is a finitely K-analytic subset of M.*
3. *For every open $W \subseteq K^n$, $\dim_R(Sing_K(A \cap W)) \leq \dim_R(A \cap W) - 2$.*

Another strong variant of Remmert-Stein's Theorem is (see [28], Theorem 4.1.3):

Theorem 5.3. *Let M be a definable K-manifold and E a definable K-analytic subset of M. If A is a definable K-analytic subset of $M \setminus E$ then $Cl(A)$ is K-analytic in M.*

If we specialize to complex manifolds then the above theorem follows from Remmert-Stein when we assume that A is of pure dimension and $\dim_{\mathbb{C}} E < \dim_{\mathbb{C}} A$.

Remark 5.4. 1. Note that the implication (3) $\Rightarrow$ (1) in Theorem 5.2 fails without the definability assumption: Take $M = \mathbb{C}^3$ and let

$$A = \{(x, e^{1/x}, 1) \in \mathbb{C}^3 : x \neq 0\} \cup \{(0, y, z) \in \mathbb{C}^3\}.$$

The set A is a closed subset of $\mathbb{C}^3$ and its set of singular points is $\{(0, y, 1) \in \mathbb{C}^3\}$. For every open $W \subseteq \mathbb{C}^3$, either $W \cap \{(0, y, 1)\} = \emptyset$, in which case $Sing_{\mathbb{C}}(W \cap A) = \emptyset$ or $Sing_{\mathbb{C}}(W \cap A) = W \cap \{(0, y, 1) : y \in \mathbb{C}\}$, in which case the real dimension of this set is 2 while the real dimension of $W \cap A$ is 4. However, A is not an analytic subset of $\mathbb{C}^3$.

2. Clause *(3)* of Theorem 5.2 can be expressed in a first-order way, after showing that $Reg_K(A)$ is definable, uniformly in families, for $A \subseteq K^n$. Working in the charts of M, it then follows from Theorem 5.2 that if $\{A_t : t \in T\}$ is a definable family of subsets of M, then the collection

$$\{t \in T : A_t \text{ is an analytic subset of } M\}$$

is definable.

Putting this last observation together with Theorem 4.5, we obtain the following interesting result:

Theorem 5.5. *Let $\{X_t : t \in T\}$ be a definable family of subsets of K^n. Then the set of all $t \in T$ such that X_t is an algebraic subset of K^n is definable.*

5.2. Definable K-holomorphic maps.

Let us now consider the implications of the above results on definable K-holomorphic maps. The main results here are (see [29], Corollary 6.3, and [28], Theorem 7.3)

Theorem 5.6. *Let $f : M \to N$ be a definable K-holomorphic map between definable K-manifolds, and $A \subseteq M$ a definable K-analytic subset of M. Then*

1. *There is a closed definable set $E \subseteq N$, with $\dim_R(E) \leq \dim_R f(A) - 2$, and with $\dim_R(f^{-1}(E) \cap A) \leq \dim_R(A) - 2$, such that $f(A) \setminus E$ is a locally K-analytic subset of N.*
2. *If $f(A)$ is a closed subset of N then it is a K-analytic subset of N.*

Clause *(2)* is a strong variant of Remmert's proper mapping theorem. Again, it fails without the definability assumptions. Indeed, the projection of the analytic set $\{(n, 1/n) \in \mathbb{C} \times \mathbb{C} : n \geq 1\} \cup \{(0, 0)\}$ on its first coordinate is the closed set $\{1/n : n \geq 1\} \cup \{0\}$ which is clearly not an analytic subset of $\mathbb{C}$.

5.3. Compactification of analytic spaces.

Consider an action of an infinite discrete group Γ on a complex manifold M. Under various assumptions one can endow the quotient $\Gamma \backslash M$ with the structure of a complex analytic space or even that of quasi-affine or quasi-projective variety (see the seminal work [2] on arithmetic quotients). We note here how one may apply the theory on removal of singularities in order to prove results of similar flavor, assuming the existence of a partially definable holomorphic Γ-periodic map ϕ from M into another manifold N. Note that even if M and N are definable in some o-minimal structure the map ϕ is generally not definable there, because of the infinite period Γ. However, as we demonstrate in sections 6.2 and 6.3, we can sometimes prove the definability of ϕ on a definable $U \subseteq M$, with $\phi(U) = \phi(M)$ and, as the following result shows, for certain purposes this is sufficient (for a proof, see Appendix).

Theorem 5.7. *Let $\mathcal{R}$ be an o-minimal expansion of the real field. Let $\phi : U \to N$ be a definable finite-to-one holomorphic map from an open $U \subseteq \mathbb{C}^n$ into a definable complex manifold N. Assume that there is a set $D \subseteq U$ (not necessarily definable) which is closed in $\mathbb{C}^n$, such that $\phi(U) = \phi(D)$. Then the topological closure of $\phi(U)$ in N, call it A, is a complex analytic subset of N, and $\dim_{\mathbb{R}}(A \setminus \phi(U)) \leq 2n - 2$.*

6. Classical holomorphic functions in an o-minimal setting

Although all germs of holomorphic maps are definable in $\mathbb{R}_{an}$, if one wishes to apply o-minimal techniques to classical mathematical questions, it is necessary to consider certain holomorphic functions on their natural domains, or on sufficiently large sub-domains, and prove their definability in some o-minimal structure. In this section we consider several such cases.

6.1. The Riemann mapping. The Riemann mapping theorem says that if $\Omega \subseteq \mathbb{C}$ is a non-empty simply connected open set which is not equal to $\mathbb{C}$ then there is a biholomorphism $f : \Omega \to D$ with the open unit disc in $\mathbb{C}$. The map is unique up to a biholomorphism of D. What can be said about the definability of f in some o-minimal structure, assuming that Ω is definable there?

In [17] Kaiser shows that when Ω is a polygon (in which case f is known as the Shwarz-Chirstoffel map), the map f is indeed definable in the o-minimal structure $\mathbb{R}_{an}^{\mathbb{R}}$, the expansion of $\mathbb{R}_{an}$ by all power functions x^α, $\alpha \in \mathbb{R}$. In [18] he also shows:

Theorem 6.1. *There is an o-minimal structure $\mathcal{R}$ with the following property. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain that is definable in $\mathbb{R}_{an}$ and assume that for every x which is a singular boundary point of Ω, the angle of the boundary at x is an irrational multiple of π. Then the biholomorphic map $f : \Omega \to D$ which is given by Riemann's theorem is definable in $\mathcal{R}$.*

The o-minimal structure in the theorem is constructed in [19].

6.2. The Weierstrass $\wp$-function and elliptic curves. We return to the family of one dimensional tori discussed in Section 4.5. For the classical facts mentioned here, see [35].

Every one dimensional torus is bi-holomorphic with a torus $\mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and τ in the upper half plain $\mathcal{H} = \{\tau \in \mathbb{C}\colon \mathrm{Im}(\tau) > 0\}$. We denote the corresponding torus by $\mathcal{E}_\tau$, and its underlying set defined in Section 4.5 (1) by E_τ.

The group of $SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d},$$

and two tori $\mathcal{E}_\tau, \mathcal{E}_{\tau'}$ are biholomorphic if and only if $\tau = A\tau'$ for some $A \in SL(2,\mathbb{Z})$.

Recall that the Weierstrass $\wp$-function is a meromorphic map $\wp(\tau, z)$ from $\mathcal{H} \times \mathbb{C}$ into $\mathbb{C}$, so that for each $\tau \in \mathcal{H}$ the map $\wp_\tau\colon z \mapsto \wp(z,\tau)$ is a Λ_τ-periodic meromorphic map on the whole of $\mathbb{C}$, and the map $g_\tau\colon z \mapsto (1\colon \wp_\tau(z)\colon \wp'_\tau(z))$ induces an embedding of $\mathcal{E}_\tau$ into $\mathbb{P}_2(\mathbb{C})$. We also have $\wp(\tau, z) = \wp(A\tau, z)$ for any $A \in SL_2(2,\mathbb{Z}), \tau \in \mathcal{H}, z \in \mathbb{C}$.

The function $\wp(z,\tau)$ cannot be definable in an o-minimal structure on all of $\mathbb{C} \times \mathcal{H}$ because of the periodicity in z and in τ. We consider, instead of the whole of $\mathcal{H}$, the set

$$\mathfrak{F} = \{\tau \in \mathcal{H} : -1/2 \le \mathrm{Re}(\tau) < 1/2 \quad \text{and} \quad |\tau| \ge 1\},$$

and the family of tori $\mathcal{E}^{\mathfrak{F}} = \{\mathcal{E}_\tau : \tau \in \mathfrak{F}\}$. The choice of the subfamily $\mathcal{E}^{\mathfrak{F}}$ is quite standard, since $\mathfrak{F}$ contains a representative of every orbit of $SL(2,\mathbb{Z})$, and therefore every one-dimensional torus is biholomorphic with some $\mathcal{E}_\tau$ for $\tau \in \mathfrak{F}$.

We have (see [26], Theorem 4.1):

Theorem 6.2. *The restriction of $\wp(\tau, z)$ to the set*

$$\{(z,\tau) \in \mathbb{C} \times \mathcal{H} : \tau \in \mathfrak{F} \text{ and } z \in E_\tau\}$$

is definable in the structure $\mathbb{R}_{an,exp}$.

Since $\mathcal{H}$ and $\mathfrak{F}$ are semialgebraic sets, they can be interpreted in any real closed field R. We denote these by $\mathcal{H}(R)$ and $\mathfrak{F}(R)$. As a corollary to the theorem above we have (see [26], theorem 5.4):

Theorem 6.3. *Let $\mathcal{R} = \langle R, <, +, \cdots \rangle$ be an arbitrary model of $\mathbb{R}_{an,exp}$, $K = R(\sqrt{-1})$.*
(i) If $\tau \in \mathfrak{F}(R)$ then $\mathcal{E}_\tau$ is definably K-biholomorphic to a nonsingular cubic curve in $\mathbb{P}^2(K)$.
(ii) If $\mathcal{C} \subseteq \mathbb{P}^n(K)$ is a nonsingular algebraic curve of genus one then there is a $\tau \in \mathfrak{F}(K)$ and a K-biholomorphism of $\mathcal{C}$ and $\mathcal{E}_\tau$ which is definable in $\mathcal{R}$.

Thus in all models of $\mathbb{R}_{an,exp}$, every projective curve of genus one over K is definably K-biholomorphic to a one-dimensional K-torus $\mathcal{E}_\tau$ with $\tau \in \mathfrak{F}(R)$. But as we showed in Section 4.5, it is not true that every one dimensional K-torus is definably K-biholomorphic to an algebraic curve.

As before, the last theorem can be stated in the language of $\mathbb{R}_{an,exp}$-definable families of complex curves and holomorphic maps.

6.2.1. O-minimality and arithmetic. Several articles in recent years make connections between o-minimality and arithmetical questions in complex algebraic geometry. The starting point of this analysis is a theorem of Pila and Wilkie concerning the distribution of rational points on subsets of $\mathbb{R}^n$ which are definable in o-minimal structures (see [33]). Given a complex algebraic variety, Pila and Zannier, [32], used transcendental holomorphic functions on bounded sets, definable in $\mathbb{R}_{an}$, to translate questions about torsion points in complex abelian varieties into questions on rational points of $\mathbb{R}_{an}$-definable subsets of $\mathbb{C}^n$. Having that, they use the Pila-Wilkie result, together with number theoretic considerations and o-minimality to give a new proof for the Manin-Mumford conjecture.

More recently, Pila, [30], [31], used Theorem 6.2 above to translate questions about special points in the moduli space of elliptic curves into questions on quadratic points in $\mathbb{R}_{an,exp}$-definable subsets of $\mathbb{C}^n$. Using a variant of his theorem with Wilkie, together with number theoretic results and o-minimality, he was able to prove certain open cases of the André-Oort conjecture.

6.3. The theta functions and abelian varieties.

In this section we describe a recent, still unpublished work.

As was pointed out in Section 4.5, the family of n-dimensional complex tori $\mathcal{E}_{\bar{\omega}}$ can be viewed as a definable family of complex manifolds in the structure $\overline{\mathbb{R}}$. A torus $\mathcal{E}_{\bar{\omega}}$ is called *an abelian variety* if it is biholomorphic to a projective algebraic variety. We first review briefly the relevant information regarding abelian varieties (see [5] [16]).

We already discussed the fact that every 1-torus is an abelian variety. When $n > 1$, the family of abelian varieties is a proper sub-collection of the family of all n-tori, given as countable union of definable subfamilies $\mathcal{F}_D$, where D runs over all $n \times n$ diagonal matrices

$$D = Diag(d_1, d_2, \dots, d_n),$$

with $d_1|d_2|\dots|d_n$ positive integers. Each $\mathcal{F}_D$ is defined as follows:

We denote by $\mathcal{H}_n$ the Siegel half space of all $n\times n$ complex symmetric matrices with a positive definite imaginary part. We now fix D as above (called the polarization type). For $\tau \in \mathcal{H}_n$ we denote by $\Lambda_{\tau,D}$ the lattice which is generated by the columns of the $n \times 2n$ complex matrix (τ, D). We let $\mathcal{E}_{\tau,D}$ denote the corresponding torus. Let $\mathcal{F}_D = \{\mathcal{E}_{\tau,D} : \tau \in \mathcal{H}_n\}$ be the family of all polarized tori with polarization type D. It is known that a complex n-torus is an abelian variety if and only if it is biholomorphic to a torus from one of the families $\mathcal{F}_D$ (but each abelian variety appears in more than one such family).

Let $Sp(D, \mathbb{Z})$ be the group of $2n\times 2n$ integral matrices preserving the alternating form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

The group $Sp(D, \mathbb{Z})$ acts on $\mathcal{H}_n$ and any two polarized varieties $\mathcal{E}_{\tau_1,D}$ and $\mathcal{E}_{\tau_2,D}$ in $\mathcal{F}_D$ are isomorphic (as polarized varieties) if and only if they are in the same orbit of $Sp(D, \mathbb{Z})$.

There is a natural number k such that every $\mathcal{E}_{\tau,D}$ can be embedded, via a map which we denote by $\Theta_{\tau,D}$, into $\mathbb{P}^k(\mathbb{C})$. We are interested in uniform definability of these embeddings.

As in the case $n = 1$, although each $\Theta_{\tau,D}$ is definable in $\mathbb{R}_{an}$, the whole family $\Theta_{\tau,D}(z), \tau \in \mathcal{H}_n$, can not be defined in any o-minimal structure because of periodicity in τ, and we need to choose an appropriate subfamily. It follows from Siegel's reduction theory (see [16], p. 189-197), that there is a a semi-algebraic set $\mathfrak{F}_n^D \subseteq \mathcal{H}_n$ containing finitely many representatives for each orbit of $Sp(D, \mathbb{Z})$.

Theorem 6.4. *For every polarization type D the family of embeddings* $\{\Theta_{\tau,D} : \tau \in \mathfrak{F}_n^D\}$ *is definable in the structure* $\mathbb{R}_{an,exp}$.

The above theorem is equivalent to definability of certain theta functions which we now describe.

We use $\widetilde{\Theta}_{\tau,D} : \mathbb{C}^n \to \mathbb{P}^k(\mathbb{C})$ to denote the pullback of $\Theta_{\tau,D}$, i.e. $\widetilde{\Theta}_{\tau,D}$ is a $\Lambda_{\tau,D}$-periodic map which induces $\Theta_{\tau,D}$, and let $\widetilde{\Theta}_D : \mathcal{H}_n \times \mathbb{C}^n \to \mathbb{P}^k(\mathbb{C})$ be $\widetilde{\Theta}_D(\tau, z) = \widetilde{\Theta}_{\tau,D}(z)$. The map $\widetilde{\Theta}_D$ can be obtained as the composition $\pi \circ \widetilde{\vartheta}_D$, where $\pi : \mathbb{C}^{k+1} \to \mathbb{P}^k(\mathbb{C})$ is the canonical projection, and $\widetilde{\vartheta}_D : \mathcal{H}_n \times \mathbb{C}^n \to \mathbb{C}^{k+1}$ is a map whose coordinate functions are given by theta functions $\vartheta_{a,b}(z,\tau)$, for various $a, b \in \mathbb{Q}^n$. The theta functions are given explicitly by the following formula: For $a, b \in \mathbb{R}^n$, $z \in \mathbb{C}^n$ column vectors and a matrix $\tau \in \mathcal{H}_n$ (we use ${}^t z$ to denote the transpose of a column vector z),

$$\vartheta_{a,b}(\tau, z) = \sum_{m\in\mathbb{Z}^n} e^{i\pi({}^t(m+a)\tau(m+a)+2\,{}^t(m+a)(z+b))}.$$

We define

$$\Omega_n = \{(\tau, z) \in \mathcal{H}_n \times \mathbb{C}^n \colon \tau \in \mathfrak{F}_n^{I_n} \text{ and } z \in E_{\tau.I_n}\}.$$

Theorem 6.4 can be deduced from the following result.

Theorem 6.5. *For every $a, b \in \mathbb{R}^n$, the map $(\tau, z) \mapsto \vartheta_{a,b}(\tau, z)$ restricted to Ω_n is definable in the o-minimal structure $\mathbb{R}_{an,exp}$.*

We end this section by observing how o-minimality can be be used in the construction of moduli spaces of polarized abelian varieties. We assume that D is a polarization type with d_1 divisible by 4.

We need the following fact (see Theorem V.4 in [16])

Fact 6.6. *There is a subgroup $\Gamma < Sp(D,\mathbb{Z})$ of finite index and a holomorphic map $\varphi\colon \mathcal{H}_n \to \mathbb{P}^N(\mathcal{C})$, whose coordinates are given by maps $\tau \mapsto \vartheta_{a,b}(\tau, 0)$ such that $\varphi(\tau) = \varphi(\tau')$ if and only if τ and τ' are in the same orbit of Γ.*

The map φ from the above fact induces a map from $\Gamma \setminus \mathcal{H}_n$ into $\mathbb{P}^N(\mathbb{C})$, and an important issue in the theory of moduli spaces is the nature of the image of this map. The main result is that this image is dense inside some algebraic subvariety of $\mathbb{P}^N(\mathbb{C})$ (see Theorem V.8 in [16]). Let us see how o-minimality yields an alternative proof of this fact.

Since Γ has finite index in $Sp(D,\mathbb{Z})$, we can choose a semi-algebraic F consisting of finitely many translates of $\mathfrak{F}_n^D$ such that $\varphi(\mathcal{H}_n) = \varphi(F)$.

Using Theorem 6.5 and transformation formulas for theta functions, we can get φ to be definable on an open set $U \subseteq \mathcal{H}_n$ containing the closure of F. We now view $\varphi\colon U \to \mathbb{P}^N(\mathbb{C})$ as a definable holomorphic map from an open subset of $\mathbb{C}^\ell$ (with $\ell = dim(\mathcal{H}_n)$) into the definable manifold $\mathbb{P}^N(\mathbb{C})$. We can therefore apply Theorem 5.7 and deduce that the closure of $\varphi(F)$ is an analytic subvariety of $\mathbb{P}^N(\mathbb{C})$, so by Chow's Theorem must be algebraic. It immediately follows that the closure of image of $\Gamma \setminus \mathcal{H}_n$ under φ is algebraic as well.

7. Appendix

7.1. Definability of compact real analytic manifolds in $\mathbb{R}_{an}$.

We prove here the result claimed in Section 4.2.

Proposition 7.1. *Let M be a n-dimensional compact real analytic manifold with a given finite atlas $\{\langle U_i, \phi_i\rangle : i \in I\}$. Then there is a finite open cover $M = \bigcup_{x\in X} B_x$ with the properties:*
(i) For each $x \in X$ there is an $i(x) \in I$ with $B_x \subseteq U_{i(x)}$, such that $\phi_{i(x)}(B_x)$ is a subset of $\mathbb{R}^n$ which is definable in $\mathbb{R}_{an}$.
(ii) For all $x, y \in X$, the sets $\phi_{i(x)}(B_x \cap B_y)$ and the restriction of the transition map $\phi_{i(y)}\phi_{i(x)}^{-1}$ to this set are definable in $\mathbb{R}_{an}$.

Proof. Without loss of generality each $\phi_i(U_i)$ is a bounded subset of $\mathbb{R}^n$. We denote by ϕ_{ij} the real analytic transition map

$$\phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j).$$

As was pointed out in the examples of Section 2, if $B \subseteq \phi_j(U_i\cap U_j)$ is a definable set whose closure is contained $\phi_j(U_i \cap U_j)$ then the restriction of ϕ_{ij} to B is definable in $\mathbb{R}_{an}$.

By compactness, for each $i \in I$ there is an open $V_i \subseteq Cl(V_i) \subseteq U_i$ such that $M = \bigcup_{i\in I} V_i$. Now, for every $x \in M$, we choose a neighborhood B_x of x such that

$$B_x \subseteq Cl(B_x) \subseteq \bigcap_{x\in V_i} V_i \cap \bigcap_{x\in U_j} U_j \cap \bigcap_{x\in Cl(V_t)^c} Cl(V_t)^c, \tag{2}$$

and

$$\text{for every } i \in I \text{ for which } x \in U_i, \text{ the set } \phi_i(B_x) \text{ is definable in } \mathbb{R}_{an}. \tag{3}$$

Indeed, this is possible to do since we only we need to choose B_x small enough to satisfy (2) and in addition require that for some fixed $U_j \ni x$, the set $\phi_j(B_x)$ is an open rectangular box in $\mathbb{R}^n$. To verify (2), by our choice of B_x, if $x \in U_i$ then $Cl(B_x) \subseteq U_i \cap U_j$ and hence $\phi_j(Cl(B_x))$ is a closed rectangular box inside $\phi_j(U_i\cap U_j)$. Therefore, as we observed already, the restriction of ϕ_{ij} to $Cl(\phi_j(B_x))$ is definable in $\mathbb{R}_{an}$. But then, $\phi_i(B_x) = \phi_{ij}(\phi_j(B_x))$ is definable as well, as required.

Claim Given $i, j \in I$, assume that $x \in V_i$, $y \in V_j$ and $B_x \cap B_y \neq \emptyset$. Then $Cl(B_x\cup B_y) \subseteq U_i\cap U_j$ and $\phi_i(B_x), \phi_i(B_y), \phi_j(B_x), \phi_j(B_y)$ are all definable in $\mathbb{R}_{an}$.

Indeed, since $B_x \cap B_y \neq \emptyset$ and $B_x \subseteq V_i$ we have $B_y \cap V_i \neq \emptyset$ and therefore $y \in Cl(V_i)$ (for otherwise, by the choice of B_y, we would have $B_y \subseteq Cl(V_i)^c$, a contradiction). By our choice the V_i's, it follows that $y \in U_i$ and therefore $Cl(B_y) \subseteq U_i$. We also have $Cl(B_x) \subseteq Cl(V_i) \subseteq U_i$ and therefore $Cl(B_x\cup B_y) \subseteq U_i$. Similarly, we have $Cl(B_x \cup B_y) \subseteq U_j$. By our definition of B_x, B_y we have $\phi_i(B_x), \phi_i(B_y), \phi_j(B_x), \phi_j(B_y)$ all definable, proving the claim.

By compactness, there is a finite set $X \subseteq M$, such that $M = \bigcup_{x\in X} B_x$. For each $x \in X$ we choose $i(x) \in I$ such that $x \in V_{i(x)}$. By the claim, if $B_x \cap B_y \neq \emptyset$ then $\phi_{i(x)}(B_x \cap B_y) = \phi_{i(x)}(B_x) \cap \phi_{i(x)}(B_y)$ is definable in $\mathbb{R}_{an}$ and furthermore, the closure of $\phi_{i(x)}(B_x \cap B_y)$ is contained in $\phi_{i(x)}(U_{i(x)} \cap U_{i(y)})$. It follows that the restriction of $\phi_{i(y)}\phi_{i(x)}^{-1}$ to this set is definable. □

7.2. The proof of Theorem 5.7.

Proof. Let $Fr(\phi(U)) = A \setminus \phi(U)$ be the frontier of $\phi(U)$. We first prove that $\dim_{\mathbb{R}}(Fr(\phi(U))) \le 2n-2$.

Consider $\mathbb{C}^n$ as a subset $\mathbb{P}^n(\mathbb{C})$, namely we write $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup H$ for H a hyperplane at ∞. Let G be the closure in $\mathbb{P}^n(\mathbb{C}) \times N$ of the graph of ϕ and let $\pi : \mathbb{P}^n(\mathbb{C}) \times N \to N$ be the projection onto the second coordinate. We claim that $Fr(\phi(U)) = \pi(G \cap (H \times N))$. The right-to-left inclusion is immediate. For the converse, if $y \in Fr(\phi(U))$ then there is a sequence $x_n \in U$ such that $\phi(x_n)$ tends to y. Since $\phi(U) = \phi(D)$ we may assume that $x_n \in D$. Because D is closed in $\mathbb{C}^n$ and $y \notin \phi(U)$, the sequence x_n does not have any converging subsequence in $\mathbb{C}^n$ and therefore it is unbounded in $\mathbb{C}^n$. But then, viewed in $\mathbb{P}^n(\mathbb{C})$, the sequence has a converging subsequence to an element $z \in H$, and then $(z, y) \in G \cap (H \times N)$, hence $y \in \pi(G \cap (H \times N))$

Next, consider the set B_{inf} of all $(z, y) \in G \cap (H \times N)$ such that there are infinitely many $y' \in N$ with $(z, y') \in G$ and let $B_{fin} = G \cap (H \times N) \setminus B_{\text{inf}}$. By [29], Lemma 6.7 (ii), $\dim_{\mathbb{R}}(B_{\text{inf}}) \le 2n-2$, and since $\dim_{\mathbb{R}} H = 2n-2$ we also have $\dim_{\mathbb{R}} B_{fin} \le 2n-2$. It follows that $\dim_{\mathbb{R}}(G \cap (H \times N)) \le 2n-2$. We now have,

$$\dim_{\mathbb{R}}(Fr(\phi(U))) = \dim_{\mathbb{R}}(\pi(G \cap (H \times N))) \le 2n-2,$$

as claimed.

By Theorem 5.6 (1), there is a definable closed set $E \subseteq N$, with $\dim_{\mathbb{R}}(E) \le \dim_{\mathbb{R}} \phi(U) - 2$, such that $\phi(U) \setminus E$ is locally analytic in N. Because $A = (\phi(U) \setminus E) \cup E \cup Fr(\phi(U))$, we have

$$Sing_{\mathbb{C}}(A) \subseteq E \cup Fr(\phi(U)) \cup Sing_{\mathbb{C}}(\phi(U) \setminus E).$$

Since ϕ is finite-to-one, the real dimension of $\phi(U)$ is $2n$ everywhere and therefore the real dimension of $Sing_{\mathbb{C}}(\phi(U) \setminus E)$ is at most $2n-2$ everywhere (see 5.2(3)). We thus have for all open $W \subseteq N$,

$$\dim_{\mathbb{R}}(Sing_{\mathbb{C}}(A \cap W)) \le 2n-2 = \dim_{\mathbb{R}}(W \cap A) - 2.$$

We can now apply Theorem 5.2(3) once more and conclude that A is an analytic subset of N. □

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