LIMITS OF DEFINABLE FAMILIES AND DILATIONS IN NILMANIFOLDS

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Abstract. Let G be a unipotent group and $\mathcal{F} = \{F_t : t \in (0, \infty)\}$ a family of subsets of G, with $\mathcal{F}$ definable in an o-minimal expansion of the real field. Given a lattice $\Gamma \subseteq G$, we study the possible Hausdorff limits of $\pi(\mathcal{F})$ in G/Γ as t tends to ∞ (here $\pi : G \to G/\Gamma$ is the canonical projection). Towards a solution, we associate to $\mathcal{F}$ finitely many real algebraic subgroups $L \subseteq G$, and, uniformly in Γ, determine if the only Hausdorff limit at ∞ is G/Γ, depending on whether $L^\Gamma = G$ or not. The special case of polynomial dilations of a definable set is treated in detalis.

1. Introduction

Contents

Let G be $\langle \mathbb{R}^n, + \rangle$ or more generally a real unipotent group, and let $X \subseteq G$ be a definable set in some o-minimal structure over $\mathbb{R}$. In [?o-minflows] and [?nilpotent] we examined the following problem: *For a lattice $\Gamma \subseteq G$, and $\pi : G \to G/\Gamma$, what is the topological closure of $\pi(X)$ in G/Γ?*

Using model theoretic machinery, we described the frontier of $\mathrm{cl}(\pi(X))$ as the projection of finitely many definable families of cosets of positive dimensional subgroups associated to X. The answer can be seen, in a certain sense, as uniform in Γ.

Here we consider an extension of the problem:

For G as above, let $\{F_s : s \in S\}$ be a family of subsets of G that is definable in an o-minimal structure over the reals, let $\Gamma \subseteq G$ be a lattice and $\pi : G \to G/\Gamma$ the projection. What are the possible Hausdorff limits of the family $\{\pi(F_s) : s \in S\}$ in G/Γ? How does the answer vary with Γ?

Some results of this paper can be seen as an extension of work [?KSS] and [?fish] on polynomial dilations in nilmanifolds. But instead of

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considering equidistributionf of certain measures linked to these dilations, we focus here on topological properties (see Section **??** below).

Our precise setting is as follows: Let $\mathbb{R}_{\rm om}$ be an o-minimal expansion of the field of reals. Let $\mathcal{F}=\{F_t : t\in(0,\infty)\}$ be an $\mathbb{R}_{\rm om}$-definable family of subsets of G, and let Γ be a lattice in G. We study the possible Hausdorff limits of the family $\{\pi(F_t)\colon t\in(0,\infty)\}$, as t tends to ∞. Using model theory, we replace the Hausdroff limits question by a question on non-standard members of the family in an elementary extension. More precisely, we consider an elementary extension $\mathfrak{R}$ of $\langle\mathbb{R}_{\rm om},\Gamma\rangle$ where for every definable set Z in $\langle\mathbb{R},\Gamma\rangle$ we denote by $Z^\sharp$ its realization in $\mathfrak{R}$ (see Section **??** for details). Now every Hausdorff limit at ∞ of $\pi(\mathcal{F})$ is the standard part of $\pi(F^\sharp_\tau{\cdot}\Gamma^\sharp)$ for $\tau>\mathbb{R}$ a non-standard parameter in $\mathfrak{R}$ (see Section **??**). Thus, the problem reduces to the study of sets of the form $\mathrm{st}(F^\sharp_\tau{\cdot}\Gamma^\sharp)$.

Similarly to the answers to the closure problem, we associate to the family $\mathcal{F}$ finitely many normal co-commutative subgroups $L_i\subseteq G$, and then for every Γ, the answers depend on whether one of the L_i is a Γ-dense subgroup or not. More precisely, (see Section **??** for details), let L^Γ be the smallest Γ-rational real algebraic subgroup of G containing L. We prove: (Theorem **??**):

Theorem (see Theorem **??**). *Let G be a unipotent group, $\mathcal{F}=\{F_t : t\in(0,\infty)\}$ an $\mathbb{R}_{\rm om}$-definable family of subsets of G.*

Then, there exists a finite collection $\mathscr{L}(\mathcal{F})$ of normal co-commutative subgroups of G, such that for every lattice $\Gamma\subseteq G$ and $\pi:G\to G/\Gamma$, we have:

(1) $L^\Gamma=G$ for some $L\in\mathscr{L}(\mathcal{F})$ if and only if $\pi(\mathcal{F})$ converges strongly to G/Γ at ∞ (i.e. G/Γ is the only Hausdroff limit at ∞ of $\pi(\mathcal{F})$ and this remains true for every lattice in G commensurable with Γ).

(2) $L^\Gamma\neq G$ for all $L\in\mathscr{L}(\mathcal{F})$ if and only if there exists a subgroup $\Gamma_0\subseteq\Gamma$ of finite index such that all Hausdorff limits at ∞ of $\pi_0(\mathcal{F})$ are proper subsets of G/Γ_0 (here $\pi_0:G\to G/\Gamma_0$ is the quotient map).

Note that the theorem above does not identify all the possible Hausdorff limits of families $\pi(\mathcal{F})$ in G/Γ. However, we can do it when G is a abelian and $\mathcal{F}$ is a family of polynomial dilations with no constant term (see **??**). We prove:

Theorem (see Corollary **??**). *Let $\{\rho_t\colon\mathbb{R}^k\to\mathbb{R}^m\colon t\in(0,\infty)\}$ be a family of polynomial dilations with no constant term, and $X\subseteq\mathbb{R}^k$ an $\mathbb{R}_{\rm om}$-definable set.*

Then there are linear subspaces $L_1, \dots, L_n \subseteq \mathbb{R}^n$, and there is a coset of a linear space $\bar{c} + V \subseteq (\mathbb{R}^m)^n$ such that set of Hausdorff limits at ∞ of the family $\{\pi_\Gamma \circ \rho_t(X) \colon t \in (0, \infty)\}$ is exactly the family

$$\left\{ \pi_\Gamma\Big(\bigcup_{i=1}^{n} (d_i + L_i^\Gamma)\Big) : (d_1, \dots, d_n) \in \bar{c} + V^{\Gamma^n} \right\}.$$

In particular, it is the projection under π_Γ of a definable family of subsets of $\mathbb{R}^n$.

Our work on dilations was motivated by [?KSS], of Kra, Shah and Sun.

The structure of the paper. From a model theoretic point of view, the main complexity of this work over the closure theorems in [?nilpotent] is the fact that we study sets defined over $\mathbb{R}\langle\tau\rangle$, where τ is a non-standard parameter as above. This requires several adjustments to our previous work in [?o-minflows] and [?nilpotent]. In Section 2 we develop the notion of short and long types (which replace bounded and unbounded types over $\mathbb{R}$). In addition, we modify the theory of μ-stabilizers developed in [?mustab], so it fits our setting. In Section 3 we study types and their nearest co-commutative subgroups (again, the results in [?nilpotent] need adjustments since the types are over $\mathbb{R}\langle\tau\rangle$). In Section 4, lattices come in and we prove the main theorems about the Γ-closure of types. In Section 5, we study definable sets over $\mathbb{R}\langle\tau\rangle$ and formulate conditions under which such sets are Γ-dense. In Section 6 we translate the results obtained thus far back to the original problem of Hausdorff limits, and in Section 7 we study in more details families given by polynomial dilations.

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2. Long types and μ-stabilizers

2.1. Model theoretic preliminaries.

As background on model theory and o-minimality we refer to [?omin] and [?marker]. We follow the set-up from [?o-minflows, Section 2] and [?mustab, Section 2.3].

We fix an o-minimal structure $\mathbb{R}_{\mathrm{om}} = \langle \mathbb{R}, <, +, \cdot, \cdots \rangle$ expanding the real field and denote by $\mathcal{L}_{\mathrm{om}}$ its language. For convenience we add to $\mathcal{L}_{\mathrm{om}}$ a constant symbol for every real number.

We use $\mathcal{L}_{\mathrm{full}}$ for a language in which every subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, has a predicate symbol, and denote the corresponding structure on $\mathbb{R}$ by $\mathbb{R}_{\mathrm{full}}$. This will allow us to talk about lattices as definable sets.

We let $\mathfrak{R}_{\text{full}} = \langle \mathfrak{R}, < \ldots \rangle$ be an elementary extension of $\mathbb{R}_{\text{full}}$ which is $|\mathbb{R}|^+$-saturated and strongly-$|\mathbb{R}|^+$-homogeneous, and let $\mathfrak{R}_{\text{om}}$ be the reduct to $\mathcal{L}_{\text{om}}$. Clearly, $\mathfrak{R}_{\text{om}}$ is an elementary extensions of $\mathbb{R}_{\text{om}}$.

We use Roman letters X, Y, Z to denote subsets of $\mathbb{R}^n$ and let $X^\sharp, Y^\sharp, Z^\sharp$ denote their realizations in $\mathfrak{R}_{\text{full}}$. We use script $\mathcal{X}$ to denote subsets of $\mathfrak{R}^n$ which are not necessarily of the form $X^\sharp$. When we write $A \subseteq \mathfrak{R}$, for a parameter set over which definable sets and types are considered, we mean that $|A| \leq |\mathbb{R}|$.

For $\mathcal{L} = \mathcal{L}_{\text{om}}$ or $\mathcal{L} = \mathcal{L}_{\text{full}}$, as usual, a complete $\mathcal{L}$-type over A is an ultrafilter on sets which are $\mathcal{L}$-definable using parameters in A. For $A \subseteq \mathfrak{R}$ and $\mathcal{X} \subseteq \mathfrak{R}^n$ an $\mathbb{R}_{\text{om}}$-definable set over A, we let $S_{\mathcal{X}}(A)$ be the collection of all complete $\mathcal{L}_{\text{om}}$-types over A, containing the set $\mathcal{X}$. If $\mathcal{X} = X^\sharp$ for some $\mathcal{L}_{\text{om}}$ definable $X \subseteq \mathbb{R}^n$ then instead of $S_{X^\sharp}(A)$ we write $S_X(A)$. For $p \in S_{\mathcal{X}}(A)$ we let $p(\mathfrak{R})$ denote the set of its realizations in $\mathfrak{R}_{\text{om}}$.

Unless otherwise stated, by "definable" we mean "$\mathcal{L}_{\text{om}}$-definable". In particular dcl denotes the definable closure in the structure $\mathfrak{R}_{\text{om}}$. Note that by our assumptions, $\mathcal{L}_{\text{om}}$ contains constant symbols for real numbers, hence, by definability of Skolem functions, for any set $A \subseteq \mathfrak{R}$, the definable closure $\text{dcl}(A)$ is an elementary substructure of $\mathfrak{R}_{\text{om}}$ which contains $\mathbb{R}_{\text{om}}$ as an elementary substructure.

A *type-definable subset of* $\mathfrak{R}^n$*, over* A*,* is the intersection of (possibly infinitely many) definable sets over A, which by our convention means $\mathcal{L}_{\text{om}}$-definable sets. The notion of a $\mathcal{L}_{\text{full}}$ type-definable set is similarly defined. Since $|A| \leq |\mathbb{R}|$, every collection of such definable sets is bounded in size. A subset of $\mathfrak{R}^n$ is said to be *type-definable* if it is type-definable over some $A \subseteq \mathfrak{R}$.

We let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in $\mathfrak{R}$, namely,

$$\mathcal{O} = \{\alpha \in \mathfrak{R} : \exists r \in \mathbb{R}^{>0}\, |\alpha| < r\},$$

It is a valuation ring of $\mathfrak{R}$, whose associated maximal ideal is

$$\mathbf{m} = \{\alpha \in \mathfrak{R} : \forall r \in \mathbb{R}^{>0}\, |\alpha| < r\}.$$

The ring homomorphism $\mathcal{O} \to \mathcal{O}/\mathbf{m}$ restricts to an isomorphism between $\mathbb{R}$ and $\mathcal{O}/\mathbf{m}$. The corresponding ring homomorphism $\text{st} \colon \mathcal{O} \to \mathbb{R} \simeq \mathcal{O}/\mathbf{m}$ is called *the standard part map*, and we extend it coordinate-wise to $\text{st} : \mathcal{O}^n \to \mathbb{R}^n$. For $\mathcal{X} \subseteq \mathfrak{R}^n$, we write $\text{st}(\mathcal{X})$ instead of $\text{st}(\mathcal{X} \cap \mathcal{O}^n)$.

For $a \in \mathbb{R}^n$ and $r > 0$, we let $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$.

We need the following lemma.

Lemma 2.1. *(1) If* $\mathcal{X} \subseteq \mathfrak{R}^n$ *is an* $\mathcal{L}_{\text{full}}$ *type-definable set then* $\text{st}(\mathcal{X})$ *is a closed subset of* $\mathbb{R}^n$*.*

(2) For a set $X \subseteq \mathbb{R}^n$, we have $\mathrm{cl}(X) = \mathrm{st}(X^\sharp)$ (where $\mathrm{cl}(X)$ is the topological closure of X).
(3) Let Σ be a collection of $\mathcal{L}_{\mathrm{full}}$-definable subsets of $\mathfrak{R}^n$ with $|\Sigma| \leq |\mathbb{R}|$. If Σ is closed under finite intersections, then

$$\mathrm{st}\Big(\bigcap \Sigma\Big) = \bigcap_{\mathcal{X} \in \Sigma} \mathrm{st}(\mathcal{X}).$$

In particular $\mathrm{st}(\bigcap \Sigma)$ is closed.

Proof. (1) If $a \in \mathrm{cl}(\mathrm{st}(\mathcal{X}))$ then for every $r \in \mathbb{R}^{>0}$, $B_r(a)^\sharp \cap \mathrm{st}(\mathcal{X}) \neq \emptyset$ and therefore also for every $r \in \mathbb{R}^{>0}$, $B_r(a)^\sharp \cap \mathcal{X} \neq \emptyset$. By saturation, there is $b \in \mathcal{X}$ such that $b \in a + \mathbf{m}$, so $a \in \mathrm{st}(\mathcal{X})$.

(2) is easy and (3) is just [?o-minflows, Claim 3.1] □

The following standard fact is easy to prove.

Fact 2.2. *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be closed subsets and $f\colon X \to Y$ a continuous function. Let $\mathrm{st}_m\colon \mathcal{O}^m \to \mathbb{R}^m$ and $\mathrm{st}_n\colon \mathcal{O}^n \to \mathbb{R}^n$ be the corresponding standard part maps. For every $\gamma \in \mathcal{O}^m \cap X^\sharp$ we have $f(\mathrm{st}_m(\gamma)) = \mathrm{st}_n(f(\gamma))$. In particular, for every $\mathcal{X} \subseteq X^\sharp$ we have $f(\mathrm{st}_m(\mathcal{X})) \subseteq \mathrm{st}_n(f(\mathcal{X}))$.*

We will need the following important result:

Fact 2.3 ([?lou-limit, Proposition 8.1])**.** *If $\mathcal{X}$ is definable in $\mathfrak{R}_{\mathrm{om}}$ then $\mathrm{st}(\mathcal{X})$ is definable in $\mathbb{R}_{\mathrm{om}}$.*

Let $G \subseteq \mathbb{R}^m$ be an $\mathbb{R}_{\mathrm{om}}$-definable group, namely the universe of G and the group operation are definable in $\mathbb{R}_{\mathrm{om}}$. For example, any real algebraic, or more generally semi-algebraic group is definable in $\mathbb{R}_{\mathrm{om}}$ (for more on definable groups in o-minimal structures see [?Otero]). The group G can be endowed with a group topology with a definable basis (see [?p]). This topology might disagree with the natural o-minimal topology, coming from the fact that G is a subset of $\mathbb{R}^m$. However, as we observed in [?mustab, Claim 3.1], we may embed G definably as a closed subset of $\mathbb{R}^n$ for some n, such that the above group topology agrees with the induce $\mathbb{R}^n$-topology. Thus, whenever $G \subseteq \mathbb{R}^n$ is an $\mathbb{R}_{\mathrm{om}}$-definable group we assume it to be closed in $\mathbb{R}^n$, and in addition assume that the Euclidean topology makes G a topological group.

We will use very often the following.

Fact 2.4 ([?p])**.** *Let G be an $\mathbb{R}_{\mathrm{om}}$-definable group.*
(1) If $H \subseteq G$ is an $\mathbb{R}_{\mathrm{om}}$-definable subgroup then H is closed in G.
(2) If H is an $\mathbb{R}_{\mathrm{om}}$-definable group and $f\colon G \to H$ an $\mathbb{R}_{\mathrm{om}}$-definable homomorphism then f is continuous.

For $G \subseteq \mathbb{R}^n$ as above, we consider two distinguished subgroups of $G^\sharp$. The first is the infinitesimal group μ_G, defined as follows:

$$\mu_G = \bigcap \{X^\sharp : X \subseteq G \text{ an } \mathbb{R}_{\mathrm{om}}\text{-definable open neighborhood of } e\}.$$

It is a type-definable subgroup and under our assumption on G it equals $e + \mathbf{m}^n \cap G^\sharp$, with $\mathbf{m} \subseteq \mathcal{O} \subseteq \mathfrak{R}$ the infinitesimal ideal defined above.

The second subgroup is $\mathcal{O}_G$, defined by:

$$\mathcal{O}_G = \bigcup \{X^\sharp : X \subseteq G \text{ an } \mathbb{R}_{\mathrm{om}}\text{-definable compact neighborhood of } e\}.$$

$\mathcal{O}_G$ is a $\bigvee$-definable (or Ind-definable) subgroup of $G^\sharp$, which equals, under our assumptions on G, to $\mathcal{O}^n \cap G^\sharp$. In particular, $G \subseteq \mathcal{O}_G$. The group μ_G is a normal subgroup of $\mathcal{O}_G$ and the latter can be written as the semi-direct product $\mathcal{O}_G = \mu_G \rtimes G$. We identify the quotient $\mathcal{O}_G/\mu_G$ with G and call the quotient map $\mathrm{st}_G : \mathcal{O}_G \to G$ *the standard part map.* As before, we extend it coordinate-wise to $\mathrm{st}_G \colon \mathcal{O}_G^n \to G^n$. By our assumptions on G, for every $a \in \mathcal{O}_G$, we have $\mathrm{st}(a) = \mathrm{st}_G(a) \in G$. In particular, Lemma **??** holds if one restricts to subsets of G and to st_G.

When the underlying group G is fixed we omit the subscript G and just use $\mathcal{O}$, μ and st.

As before, given an arbitrary set $\mathcal{X} \subseteq G^\sharp$, we let $\mathrm{st}(\mathcal{X})$ denote the set $\mathrm{st}(\mathcal{X} \cap \mathcal{O}_G) \subseteq G$.

Fact 2.5. *Assume that G_1, G_2 are definable in $\mathbb{R}_{\mathrm{om}}$. Then,*

(1) $\mu_{G_1 \times G_2} = \mu_{G_1} \times \mu_{G_2}$ and $\mathcal{O}_{G_1 \times G_2} = \mathcal{O}_{G_1} \times \mathcal{O}_{G_2}$.
(2) If $f \colon G_1 \to G_2$ is an $\mathbb{R}_{\mathrm{om}}$-definable surjective homomorphism then $f(\mu_{G_1}) = \mu_{G_2}$ and $f(\mathcal{O}_{G_1}) = \mathcal{O}_{G_2}$

Proof. (1) follows from the fact that the topology on $G_1 \times G_2$ is the product topology. For (2), see [?nilpotent, Lemma 3.10]. □

2.2. **Long and short sets.**

Let $G \subseteq \mathbb{R}^n$ be an $\mathbb{R}_{\mathrm{om}}$-definable group.

Definition 2.6. A subset $\mathcal{X} \subseteq G^\sharp$ is called *left-short in G* if there exists some compact set $K \subseteq G$ and $g \in G^\sharp$ such that $\mathcal{X} \subseteq K^\sharp \cdot g$ (since G can be written as an increasing union of relatively compact $\mathbb{R}_{\mathrm{om}}$-definable open sets, we may always take K to be $\mathbb{R}_{\mathrm{om}}$-definable).

Otherwise, $\mathcal{X}$ is called *left-long in G*. For $A \subseteq \mathfrak{R}$, we say that a type $p \in S_G(A)$ is left-short (left-long) in G if $p(\mathfrak{R})$ is left-short (left-long) in G.

The following are easy to verify:

Lemma 2.7. *Given $\mathcal{X} \subseteq G^\sharp$,*

(1) *$\mathcal{X}$ is left-short in G if and only if $\mathcal{X}\cdot\mathcal{X}^{-1} \subseteq K^\sharp$ for some compact set $K \subseteq G$.*
(2) *For every $g \in G^\sharp$, $\mathcal{X}$ is left-short in G if and only if $\mathcal{X}g$ is left-short in G.*
(3) *For every $g \in G$, $\mathcal{X}$ is left-short in G if and only if $g\mathcal{X}$ is left-short in G.*
(4) *If $\mathcal{X} = X^\sharp$, for $X \subseteq G$ an $\mathbb{R}_{\text{full}}$-definable set, then $\mathcal{X}$ is left-short in G if and only if X is bounded in $\mathbb{R}^n$.*

We may similarly define right-short and right-long in G and in general these notions are different. **However, for the rest of the paper we use *short* and *long* to refer only to left-short and left-long**.

If $H \subseteq G$ is an $\mathbb{R}_{\text{om}}$-definable subgroup and $\mathcal{X}$ a subset of $H^\sharp$ then, by Lemma **??**(1), $\mathcal{X}$ is short in H if and only if it is short in G, hence we omit the reference to the group when the context is clear.

Note that by saturation, an $\mathcal{L}_{\text{full}}$ type-definable set $\mathcal{X} \subseteq G^\sharp$ is short if and only if $\mathcal{X} \subseteq \mathcal{O}\cdot g$ for some $g \in G^\sharp$.

Lemma 2.8. *Let G_1, G_2, G be $\mathbb{R}_{\text{om}}$-definable groups.*

(1) *If $\mathcal{X} \subseteq G_1^\sharp$ is short and $f : G_1 \to G_2$ is an $\mathbb{R}_{\text{om}}$-definable homomorphism then $f(\mathcal{X})$ is short in G_2.*
(2) *If $\mathcal{X}_1 \subseteq G_1^\sharp$ is short and $\mathcal{X}_2 \subseteq G_2^\sharp$ is short then $\mathcal{X}_1 \times \mathcal{X}_2$ is short in $G_1 \times G_2$.*
(3) *If H_1, H_2 are two normal $\mathbb{R}_{\text{om}}$-definable subgroups of G and $\mathcal{X} \subseteq G^\sharp$ an arbitrary set then the image of $\mathcal{X}$ in $G^\sharp/(H_1^\sharp \cap H_2^\sharp)$ is short if and only if its images in $G^\sharp/H_1^\sharp$ and in $G^\sharp/H_2^\sharp$ are short.*

Proof. (1) and (2) are immediate since the image of a compact set under a $\mathbb{R}_{\text{om}}$-definable homomorphism is compact, and similarly the direct product of such sets is compact.

For (3), let $\pi_i : G \to G/H_i$, $i = 1, 2$, be the natural projections, and let $\pi : G \to G/H_1 \times G/H_2$ be the map $\pi(g) = (\pi_1(g), \pi_2(g))$. The kernel of π is $H_1 \cap H_2$ hence the image is isomorphic to $G/(H_1 \cap H_2)$.

If $\pi_1(\mathcal{X})$ and $\pi_2(\mathcal{X})$ are both short then by (2), so is $\pi_1(\mathcal{X}) \times \pi_2(\mathcal{X}) \subseteq G^\sharp/H_1^\sharp \times G^\sharp/H_2^\sharp$. But $\pi(\mathcal{X})$ is contained in $\pi_1(\mathcal{X}) \times \pi_2(\mathcal{X})$ so also short. The converse follows from (1) using the natural homomorphisms from $G/(H_1 \cap H_2)$ onto G/H_i, $i = 1, 2$. ☐

Lemma 2.9. *For $A \subseteq \mathfrak{R}$, and $p \in S_G(A)$, we have:*

(1) *p is short in G if and only if there exists $a \in \operatorname{dcl}(A) \cap G^\sharp$ such that $p \vdash \mu\cdot a$, namely $p(\mathfrak{R}) \subseteq \mu\cdot a$.*

(2) Let $H \subseteq G$ be a $\mathbb{R}_{\rm om}$-definable normal subgroup and $\pi : G \to G/H$ the quotient map. Then $\pi(p) \in S_{G/H}(A)$ is short in G/H if and only if there exists $a \in \mathrm{dcl}(A) \cap G^\sharp$ such that $p(\mathfrak{R}) \subseteq \mu{\cdot}aH^\sharp$.

Proof. (1) Assume that p is short. Hence $p(\mathfrak{R}){\cdot}p(\mathfrak{R})^{-1} \subseteq K^\sharp$ for some $\mathbb{R}_{\rm om}$-definable compact set $K \subseteq G$. By logical compactness, there exists an A-definable set $\mathcal{X}$ in p such that $\mathcal{X}{\cdot}\mathcal{X}^{-1} \subseteq K$. By definability of Skolem functions in o-minimal structures, the set $\mathcal{X}$ contains a point $a \in \mathrm{dcl}(A)$. Consider the complete $\mathbb{R}_{\rm om}$-type over A, $p{\cdot}a^{-1}$. We have $p(\mathfrak{R}){\cdot}a^{-1} \subseteq \mathcal{X}{\cdot}a^{-1} \subseteq K^\sharp$. Let $\beta \models p{\cdot}a^{-1}$ and $g = \mathrm{st}(\beta)$ (this is defined since $\beta \in \mathcal{O}$). Because $p{\cdot}a^{-1}$ is a complete A-type and $\beta \in \mu{\cdot}g$, we have $p{\cdot}a^{-1} \vdash \mu{\cdot}g$. Hence $p(\mathfrak{R}) \subseteq \mu{\cdot}g{\cdot}a$. Clearly, $g{\cdot}a \in \mathrm{dcl}(A)$.

The converse is clear.

For (2), notice that if p is short then by Lemma **??**(1), $\pi(p)$ is short and hence by (1) there exists $b \in \mathrm{dcl}(A) \cap (G/H)^\sharp$ such that $\pi(p(\mathfrak{R})) \in \mu_{G/H}{\cdot}b$.

We now take any $a \in \mathrm{dcl}(A)$ in the A-definable set $\pi^{-1}(b)$, and we have $p(\mathfrak{R}) \subseteq \pi^{-1}(\mu_{G/H}{\cdot}b) \subseteq \mu_G{\cdot}aH^\sharp$.

For the converse, notice that by Fact **??**, $\pi(\mu_G) = \mu_{G/H}$. Hence, $\pi(\mu_G{\cdot}a) = \mu_{G/H}{\cdot}\pi(a)$, so if $p(\mathfrak{R}) \subseteq \mu_G{\cdot}aH^\sharp$ then $\pi(p)(\mathfrak{R}) \subseteq \mu_{G/H}{\cdot}\pi(a)$ is short. □

2.3. **The μ-stabilizer of a type.** We fix an $\mathbb{R}_{\rm om}$-definable group G. In [?mustab] we developed a theory for μ-stabilizers of types over $\mathbb{R}$. Here we take a more general viewpoint which we now explain.

Consider the set $S_G(A)$. Given $p \in S_G(A)$ we let $\mu{\cdot}p$ (below written as μp) denote the partial type over A whose realization is the set $\mu{\cdot}p(\mathfrak{R})$.

Remark 2.10. We note that when we consider here types over arbitrary $A \subseteq \mathfrak{R}$, then, unlike [?mustab], we still keep $\mu = \mu_G$ fixed and not change it to a smaller infinitesimal group (namely, the intersection of all A-definable open neighborhoods of e).

Since our point of view here is slightly different from [?mustab], we go briefly through the results we need and explain how their proofs differ from the analogous results in [?mustab].

Given $p, q \in S_G(A)$, we say that p and q are μ-equivalent, $p \sim_\mu q$, if $\mu p = \mu q$, i.e. $\mu p(\mathfrak{R}) = \mu q(\mathfrak{R})$.

Fact 2.11. *For $p, q \in S_G(A)$, the following are equivalent:*

(1) $p \sim_\mu q$
(2) $\mu p(\mathfrak{R}) \cap \mu q(\mathfrak{R}) \neq \emptyset$.

(see [?mustab, Claim 2.7] for an identical argument).

It is easy to verify that if $p \sim_\mu q$ then p is a long if and only if q is long.

Let $S_G^\mu(A) = \{\mu p : p \in S_G(A)\}$. The group G acts from the left on $S_G^\mu(A)$ by $g{\cdot}\mu p = \mu(gp)$. The following subgroup plays a crucial role in our analysis:

Definition 2.12. Given $p \in S_G(A)$, the *left stabilizer of μp* is defined as:

$$\mathrm{Stab}^\mu(p) = \{g \in G : g{\cdot}\mu p = \mu p\}.$$

Since the definition of $\mathrm{Stab}^\mu(p)$ depends only on μp, if $p \sim_\mu q$ then $\mathrm{Stab}^\mu(p) = \mathrm{Stab}^\mu(q)$.

Our main focus in [?mustab] was on unbounded definable types. For $\mathbb{R}_{\mathrm{om}}$-definable groups, and $\mathcal{L}_{\mathrm{om}}$-types over $\mathbb{R}$ these are types which do not contain any formula over $\mathbb{R}$ defining a compact subset of G. Since we are considering here types which are not only over $\mathbb{R}$ our focus is shifted to long types.

Recall that for p an $\mathcal{L}_{\mathrm{om}}$-type we let $\dim(p)$ be the smallest o-minimal dimension of the formulas in p.

Definition 2.13. We say that a type $p \in S_G(A)$ is *μ-reduced* if for all $q \in S_G(A)$, if $p \sim_\mu q$ then $\dim(p) \leq \dim(q)$.

Clearly, every $p \in S_G(A)$ is μ-equivalent to a μ-reduced type in $S_G(A)$: just take a μ-equivalent type of minimal dimension (but there might be more than one such). Notice that by Lemma **??**, if p is short and μ-reduced then $\dim p = 0$ and $p = \mathrm{tp}(a/A)$ for some $a \in \mathrm{dcl}(A)$.

Our main goal in this section is to prove:

Proposition 2.14. *Let $p \in S_G(A)$. Then*

(1) $\mathrm{Stab}^\mu(p)$ *is $\mathbb{R}_{\mathrm{om}}$-definable and can be written as* $\mathrm{st}(\mathcal{S}{\cdot}\alpha^{-1})$*, for some definable $\mathcal{S}$ in p and $\alpha \models p$.*

(2) If p is a long type then $\dim(\mathrm{Stab}^\mu(p)) > 0$*. Moreover, in this case* $\mathrm{Stab}^\mu(p)$ *is a torsion-free solvable group.*

The proof is very similar to the proof of [?mustab, Theorem 3.10] so we only point out the differences. As we noted above, we may assume that p is μ-reduced and if p is short that $\mu p = \mu{\cdot}a$ for some $a \in \mathrm{dcl}(A)$ so $\mathrm{Stab}^\mu(p)$ is trivial. Thus, we fix a long μ-reduced type $p \in S_G(A)$ and $\alpha \in p(\mathfrak{R})$.

We start with an analogue of [?mustab, Claim 3.12]:

Claim 2.15. *If $Y \subseteq G^\sharp$ is A-definable and* $\dim Y < \dim p$ *then* $\mathcal{O}{\cdot}\alpha \cap Y = \emptyset$*.*

Proof. Assume towards contradiction that $\beta \in Y \cap \mathcal{O}\cdot\alpha$. Then $\beta \in \mu\cdot r\alpha$, for some $r \in G$, hence $r^{-1}\beta \in \mu p$. But $\dim(r^{-1}\beta/A) \leq \dim(Y) < \dim p$, contradicting the fact that p is μ-reduced. □

Next, we note, just like [?mustab, Claim 3.8], that for every A-definable set $\mathcal{S}$ in p, $\mathrm{Stab}^{\mu}(p) \subseteq \mathrm{st}(\mathcal{S}\cdot\alpha^{-1})$. Indeed, if $g \in \mathrm{Stab}^{\mu}(p)$ then there exists $\beta \models p$ and $\epsilon \in \mu$ such that $g\alpha = \epsilon\beta$. It follows that $\beta \in \mathcal{S}$ and $\beta\alpha^{-1} \in \mathcal{O}$, thus $g = \mathrm{st}(\beta\alpha^{-1}) \in \mathrm{st}(\mathcal{S}\cdot\alpha^{-1})$.

The next claim is similar to [?mustab, Claim 3.13].

Claim 2.16. *There exists an A-definable set $\mathcal{S}$ in p such that every element in $\mathcal{S} \cap \mathcal{O}\cdot\alpha$ realizes p.*

Let us explain the proof: As in [?mustab], for every A-definable set $\mathcal{S}$ in p, the set $\mathcal{S}\cdot\alpha^{-1} \cap \mathcal{O}$ is a relatively definable subset of $\mathcal{O}$. Hence, by [?mustab, Theorem B.2] it has finitely many connected components (see precise definition of connectedness there). We choose an A-definable such cell $\mathcal{S}$ in p with $\dim S = \dim(p)$, for which the number of components of $\mathcal{S}\cdot\alpha^{-1} \cap \mathcal{O}$ is minimal. Using Claim **??**, we can prove, just as in [?mustab, Claim 3.13], that any $\beta \in \mathcal{S} \cap \mathcal{O}\cdot\alpha$ must realize p.

Finally, we prove an analogue of [?mustab, Claim 3.14]:

Claim 2.17. *For $\mathcal{S}$ as in Claim* **??**, *we have* $\mathrm{Stab}^{\mu}(p) = \mathrm{st}(\mathcal{S}\cdot\alpha^{-1})$.

Proof. It is sufficient to show that $\mathrm{st}(\mathcal{S}\cdot\alpha^{-1}) \subseteq \mathrm{Stab}^{\mu}(p)$, so we take $g \in \mathrm{st}(\mathcal{S}\cdot\alpha^{-1})$ and note that for some $\epsilon \in \mu$, we have $\epsilon g\alpha \in \mathcal{S} \cap \mathcal{O}\cdot\alpha$, so by our choice of $\mathcal{S}$, $\epsilon g\alpha \models p$. It follows that $\mu g p(\mathfrak{R}) \cap p(\mathfrak{R}) \neq \emptyset$, so by Fact **??**, $g\cdot\mu p = \mu p$. □

Thus, by Fact **??**, $\mathrm{Stab}^{\mu}(p) = \mathrm{st}(\mathcal{S}\cdot\alpha^{-1})$ is definable.

Since p is long the set $\mathcal{S}\cdot\alpha^{-1}$ is not contained in $\mathcal{O}$, thus $\mathrm{Stab}^{\mu}(p)$ is unbounded in G.

To see that $\mathrm{Stab}^{\mu}(p)$ is solvable, torsion-free we repeat the argument from [?mustab, Theorem 3.6]: By [?mustab, Fact 3.25], G can be written as a product of two sets $G = C\cdot H$, with $C \subseteq G$ a $\mathcal{L}_{\mathrm{om}}$-definable compact set and H a $\mathcal{L}_{\mathrm{om}}$-definable torsion-free solvable group. Thus, $\alpha \in G^{\sharp}$ as above can be written as $\alpha = \epsilon\cdot g\cdot h^*$ for $\epsilon \in \mu_G, g \in C, h^* \in H^{\sharp}$. It follows that $\alpha \in \mu\cdot(H^g)^{\sharp}\cdot g$, so $\mathrm{tp}(\alpha/A)$ is μ-equivalent to a type $q \vdash (H^g)^{\sharp}\cdot g$. But then $\mathrm{Stab}^{\mu}(p) = \mathrm{Stab}^{\mu}(pg) \subseteq H^g$ so $\mathrm{Stab}^{\mu}(p)$ is a torsion-free solvable group.

This ends the proof of Proposition **??**. □

Remark 2.18. In fact, the remainder of the proof of [?mustab, Theorem 3.12] goes through identically and thus we could have proved the stronger result, saying that for p a μ-reduced type over A, the dimension of $\mathrm{Stab}^{\mu}(p)$ equals to $\dim(p)$. However, this will not be needed here.

3. Nearest cosets

We now assume again that G is a definable group in $\mathbb{R}_{\mathrm{om}}$.

3.1. Nearest co-commutative cosets.

Definition 3.1. Given a type $p \in S_G(A)$, an $\mathbb{R}_{\mathrm{om}}$-definable subgroup $H \subseteq G$ and $a \in \mathrm{dcl}(A) \cap G^\sharp$, we say that the coset $aH^\sharp$ *is near* p if $p(\mathfrak{R}) \subseteq \mu{\cdot}aH^\sharp$.

Sometimes we omit $\sharp$, write $p \vdash \mu aH$, and say that aH is near p.

Notice that in the above definition the subgroup H is defined over $\mathbb{R}$, but the element a is taken from $\mathrm{dcl}(A) \subseteq \mathfrak{R}$.

Remark 3.2. By Lemma **??**, a type $p \in S_G(A)$ is short if and only if, for the trivial subgroup $\{e\}$, a coset $a{\cdot}e$, is near p.

Also, for a normal $\mathbb{R}_{\mathrm{om}}$-definable subgroup $H \subseteq G$, some coset aH is near p if and only if the image of p in G/H is short.

Lemma 3.3. *Let $p \in S_G(A)$, $H_1, H_2 \subseteq G$ be two $\mathbb{R}_{\mathrm{om}}$-definable normal subgroups, $a_1, a_2 \in \mathrm{dcl}(A)$, and assume both aH_1 and aH_2 are near p. Then there exists $d \in \mathrm{dcl}(A)$ such that the coset $d(H_1 \cap H_2)$ is near p.*

Proof. Let $G_i = G/H_i$, $i = 1, 2$, and $\pi_i \colon G \to G_i$ the natural projection. Let $f \colon G \to G_1 \times G_2$ be the definable homomorphism $f(g) = (\pi_1(g), \pi_2(g))$. We have $\ker(f) = H_1 \cap H_2$.

By Lemma **??**(2), the images of $p(\mathfrak{R})$ in both G/H_1 and in G/H_2 are short. Hence, by Lemma **??**(3), its image in $G/(H_1 \cap H_2)$ is also short. By Lemma **??** (2), there exists $d \in \mathrm{dcl}(A)$ such that $d(H_1 \cap H_2)$ is near p. □

In the case of a unipotent group G and $A = \mathbb{R}$, the above lemma holds without assuming normality of H_1 and H_2 (see [?nilpotent, Theorem 3.7]). Unfortunately, in general, this fails for an arbitrary A:

Example 3.4. We consider the Heisenberg group, identified with $\mathbb{R}^3$, as

$$[a, b, c]{\cdot}[d, e, f] = [a + d, b + e, ae + c + f].$$

We let $H_1 = Z(G) = \{[0, 0, x] : x \in \mathbb{R}\}$, and $H_2 = \{[0, t, t] : t \in \mathbb{R}\}$. We now consider $H_1^\sharp$ and $H_2^\sharp$ in $G^\sharp$. Fix $\tau \in \mathfrak{R}$ with $\tau > \mathbb{R}$ and let $A = \mathrm{dcl}(\tau)$.

Consider the 1-type over $\mathrm{dcl}(A)$:

$$q(t) = \{r < t < c : r \in \mathbb{R}\,, c \in \mathrm{dcl}(A) \text{ with } c > \mathbb{R}\}.$$

Let $p(t)$ be the type over $\mathrm{dcl}(A)$ given by $\{[\tau, 0, t] : t \models q\}$. For $\alpha = [\tau, 0, 0]$, the realizations of p are contained in the coset αH_1. It is easy to see that p is a long type and we claim that αH_2 is near p.

Indeed, consider the type $q_0 = (1/\tau)q$. It is also a 1-type over $\mathrm{dcl}(A)$, whose realizations are contained in $\mu \subseteq \mathfrak{R}$, and, for every $\beta \models q_0$, the element $g_\beta = [0, \beta, \beta]$ is in $H_2^\sharp$. Now, for $\varepsilon_\beta = [0, -\beta, -\beta] \in \mu_G$, we have

$$\varepsilon_\beta \cdot \alpha \cdot g_\beta = [0, -\beta, -\beta] \cdot [\tau, \beta, \tau\beta + \beta] = [\tau, 0, \tau\beta] \models p,$$

and also $\varepsilon_\beta \cdot \alpha \cdot g_\beta \in \mu_G \cdot \alpha \cdot H_2^\sharp$. Hence the coset αH_2 is near p.

Thus both αH_1 and αH_2 are near p. However, since p is long, a coset of $H_1 \cap H_2 = \{e\}$ can not be near p.

The main part of this paper deals with unipotent groups, and, in the unipotent case, instead of nearest cosets, as in [?nilpotent], it is more convenient to work with nearest co-commutative cosets.

Definition 3.5. We say that a subgroup $H \subseteq G$ is *co-commutative* if it is normal and the quotient G/H is abelian (equivalently H contains $[G, G]$).

Since the intersection of two co-commutative subgroups is co-commutative, using Lemma **??**, we may conclude:

Corollary 3.6. *Given $p \in S_G(A)$ there exists a smallest (by inclusion) $\mathbb{R}_{\mathrm{om}}$-definable co-commutative subgroup $L \subseteq G$ such that for some $a \in \mathrm{dcl}(A)$ the coset aL is near p.*

We can now define:

Definition 3.7. Given $p \in S_G(A)$, *a nearest co-commutative coset to* p is a coset of the form aL, where $a \in \mathrm{dcl}(A)$ and $L \subseteq G$ is an $\mathbb{R}_{\mathrm{om}}$-definable co-commutative subgroup as in Corollary **??**. It is unique up to μ_G, namely if $a_1 L_1$ and $a_1 L_1$, are both nearest co-commutative cosets to p then $L_1 = L_2$ and $\mu_G \cdot a_1 L_1^\sharp = \mu_G \cdot a_2 L_2^\sharp$.

We will denote this subgroup L as L_p.

We now prove some basic properties of nearest co-commutative cosets.

Lemma 3.8. *Let $p \in S_G(A)$. If $K \subseteq G$ is a compact definable set, $a \in \mathrm{dcl}(A)$ and $p(\mathfrak{R}) \subseteq K^\sharp \cdot a \cdot L^\sharp$ for some $\mathbb{R}_{\mathrm{om}}$-definable co-commutative $L \subseteq G$ then $L_p \subseteq L$.*

Proof. Clearly $\pi(p)$ is short in G/L, where $\pi\colon G \to G/L$ is the quotient map. By Lemma **??**(2), there exists $a' \in \mathrm{dcl}(A)$ such that $a' \cdot L$ is near p, hence $L_p \subseteq L$. □

Lemma 3.9. *Let $f\colon G \to H$ be an $\mathbb{R}_{\mathrm{om}}$-definable surjective homomorphism of definable groups and $A \subseteq \mathfrak{R}$. For a type $p \in S_G(A)$ and $q = f(p)$, if $D_p = aL_p$ is a nearest co-commutative coset to p then $f(D_p)$ is a nearest co-commutative coset to q, and in particular, $L_q = f(L_p)$.*

Proof. Since f is surjective, it maps a co-commutative subgroup onto a co-commutative subgroup.

Let D_q be a nearest co-commutative coset to q. It is sufficient to see that $\mu_H D_q = \mu_H f(D_p)$. We have $p \vdash \mu_G D_p$, so by Fact **??**, $q \vdash \mu_H f(D_p)$, hence $D_q \subseteq \mu_H f(D_p)$. Conversely, since $q \vdash \mu_H D_q$ then $p \vdash \mu_G f^{-1}(D_q)$, so $D_p \subseteq \mu_G f^{-1}(D_q)$, hence $f(D_p) \subseteq \mu_H D_q$. ☐

Lemma 3.10. *For $p \in S_G(A)$, let $H \subseteq G$ be the μ-stabilizer of p, and let aL_p be a nearest co-commutative coset to p. Then $H \subseteq L_p$.*

Proof. Fix $\beta \models p$. Then by assumption, there exists $\varepsilon \in \mu$ and $\ell \in L_p^\sharp$ such that $\beta = \varepsilon a \ell$. Given $h \in H$, we have $h\beta \in \mu p(\mathfrak{R})$, hence $h\beta = \varepsilon' a \ell'$, with $\varepsilon' \in \mu$ and $\ell' \in L_p^\sharp$. Thus

$$h = h\beta\beta^{-1} = \varepsilon' a \ell' \ell^{-1} a^{-1} \varepsilon^{-1}.$$

Since L_p is normal in G (so $L_p^\sharp$ normal in $G^\sharp$), it follows that $h \in \mu \cdot L_p^\sharp$. However, h is in G and L_p is closed in G, therefore $h \in L_p$. ☐

3.2. **The set $\mathscr{L}_{\max}(\mathcal{X})$.** Again we fix a group G definable in $\mathbb{R}_{\mathrm{om}}$.

Recall that by our assumption, $G \subseteq \mathbb{R}^n$ is a closed subset. For $r > 0$, we will denote by $\overline{B}_r \subseteq G$ the set $\overline{B}_r(e) \cap G$, where $\overline{B}_r(e)$ is the closed ball of radius r centered at e. Clearly, each $\overline{B}_r$ is a compact subset of G, definable in $\mathbb{R}_{\mathrm{om}}$, with $\mu_G = \bigcap_{r \in \mathbb{R}^{>0}} \overline{B}_r^\sharp$.

Lemma 3.11. *For $A \subseteq B \subseteq \mathfrak{R}$, let $\mathcal{X} \subseteq G$ be a set $\mathcal{L}_{\mathrm{om}}$-definable over A, and $p \in S_{\mathcal{X}}(A)$.*

(1) If $q \in S_{\mathcal{X}}(B)$ is an extension of p then $L_q \subseteq L_p$.
(2) There is $q \in S_{\mathcal{X}}(B)$ extending p such that $L_q = L_p$.

Proof. (1). Choose $a_p \in \mathrm{dcl}(A)$ such that $p(\mathfrak{R}) \subseteq \mu \cdot a_p \cdot L_p^\sharp$. We have $q(\mathfrak{R}) \subseteq p(\mathfrak{R}) \subseteq \mu \cdot a_p \cdot L_p^\sharp$. Hence the coset $a_p L_p$ is near q and $L_q \subseteq L_p$.

(2). Let $\mathcal{Q}$ be the set of all $q \in S_{\mathcal{X}}(B)$ extending p. For each $q \in \mathcal{Q}$ we choose $b_q \in \mathrm{dcl}(B)$ such that $q(\mathfrak{R}) \subseteq \mu \cdot b_q \cdot L_q^\sharp$. We have

$$p(\mathfrak{R}) \subseteq \bigcup_{q \in \mathcal{Q}} q(\mathfrak{R}) \subseteq \bigcup_{q \in \mathcal{Q}} \mu \cdot b_q \cdot L_q^\sharp \subseteq \bigcup_{q \in \mathcal{Q}} \overline{B}_1^\sharp \cdot b_q \cdot L_q^\sharp.$$

Thus the type definable set $p(\mathfrak{R})$ is covered by a bounded family of definable sets of the form $\overline{B}_1^\sharp \cdot b_q \cdot L_q^\sharp$. Hence, by logical compactness, we can find a set $\mathcal{X}_0 \in p$, definable over A, and a finite subset $\mathcal{Q}_0 \subseteq \mathcal{Q}$ such that

$$\mathcal{X}_0 \subseteq \bigcup_{q \in \mathcal{Q}_0} \overline{B}_1^\sharp \cdot b_q \cdot L_q^\sharp.$$

Since $\mathrm{dcl}(A)$ is an elementary substructure of $\mathfrak{R}_{\mathrm{om}}$, we can find $a_q \in \mathrm{dcl}(A)$, for each $q \in \mathcal{Q}_0$, such that $\mathcal{X}_0 \subseteq \bigcup_{q\in\mathcal{Q}_0} \overline{B}_1^\sharp \cdot a_q \cdot L_q^\sharp$. Since p is a complete over A, there is $q \in \mathcal{Q}_0$ with $p(\mathfrak{R}) \subseteq \overline{B}_1^\sharp \cdot a_q \cdot L_q^\sharp$. By Lemma **??**, $L_p \subseteq L_q$, hence by (1), we have $L_p = L_q$. □

For $A \subseteq \mathfrak{R}$ and a set $\mathcal{X} \subseteq \mathfrak{R}^n$ definable over A, we denote by $\mathscr{L}_A(\mathcal{X})$ the set

$$\mathscr{L}_A(\mathcal{X}) = \{L_p \colon p \in S_{\mathcal{X}}(A)\}.$$

Corollary 3.12. *For $A \subseteq B \subseteq \mathfrak{R}$, let $\mathcal{X} \subseteq G^\sharp$ be definable over A. Then,*

(1) $\mathscr{L}_A(\mathcal{X}) \subseteq \mathscr{L}_B(\mathcal{X})$.

(2) An $\mathbb{R}_{\mathrm{om}}$-definable co-commutative subgroup L of G is maximal (by inclusion) in $\mathscr{L}_A(\mathcal{X})$ if an only if it is maximal in $\mathscr{L}_B(\mathcal{X})$

Proof. Follows from Lemma **??**. □

Remark 3.13. In general, for $A \subseteq B$ we do not have equality of sets, $\mathscr{L}_A(\mathcal{X}) = \mathscr{L}_B(\mathcal{X})$. As an example, consider the group $G = (\mathbb{R}^2, +)$ with $\mathcal{X} = \{(x, y) \in \mathfrak{R}^2 \colon x \geq 0, y = x^2\}$. For $A = \mathbb{R}$ there is only one unbounded type in $S_{\mathcal{X}}(A)$, whose a nearest co-commutative coset is the whole $\mathbb{R}^2$. Thus $\mathcal{L}_A(\mathcal{X}) = \{\{0\}, \mathbb{R}^2\}$. However it is not hard to see that in any proper elementary extension B of $\mathbb{R}$ there are types in $S_{\mathcal{X}}(B)$ whose nearest co-commutative cosets are translates of $L = \{0\} \times \mathbb{R}$, and $\mathscr{L}_B(\mathcal{X}) = \{\{0\}, L, \mathbb{R}^2\}$.

Definition 3.14. For $\mathcal{X} \subseteq G^\sharp$ an $\mathcal{L}_{\mathrm{om}}$-definable set over A, we denote by $\mathscr{L}_{\max}(\mathcal{X})$ the set of maximal subgroups, by inclusion, in $\mathscr{L}_A(\mathcal{X})$. By Corollary **??**, it does not depend on A.

We now have:

Theorem 3.15. *Let G be an $\mathbb{R}_{\mathrm{om}}$-definable group, $A \subseteq \mathfrak{R}$, and let $\mathcal{X} \subseteq G^\sharp$ be $\mathcal{L}_{\mathrm{om}}$-definable over A.*

For every $r \in \mathbb{R}^{>0}$, there are definable co-commutative subgroups $L_1, \ldots, L_k \subseteq G$, possibly with repetitions, and $a_1, \ldots, a_k \in \mathrm{dcl}(A)$ such that each $a_i L_i$ is a nearest co-commutative coset to some $p_i \in S_{\mathcal{X}}(A)$, and

$$\mathcal{X} \subseteq \overline{B}_r^\sharp \cdot \bigcup_{i=1}^{k} a_i \cdot L_i^\sharp.$$

In addition, every $\mathscr{L}_{\max}(\mathcal{X})$ appears at least once among $L_1, \ldots, L_k$.

Proof. For each $p \in S_{\mathcal{X}}(A)$, we choose $a_p \in \mathrm{dcl}(A)$ such that $p(\mathfrak{R}) \subseteq \mu \cdot a_p \cdot L_p^\sharp$.

We have

$$\mathcal{X} \subseteq \bigcup_{p \in S_{\mathcal{X}}(A)} p(\mathfrak{R}) \subseteq \bigcup_{p \in S_{\mathcal{X}}(A)} \mu \cdot a_p \cdot L_p^\sharp \subseteq \bigcup_{p \in S_{\mathcal{X}}(A)} \overline{B}_r^\sharp \cdot a_p \cdot L_p^\sharp.$$

Using logical compactness, we obtain finitely many $p_1, \ldots, p_k \in S_{\mathcal{X}}(A)$ such that

$$\mathcal{X} \subseteq \bigcup_{i=1}^{k} \overline{B}_r^\sharp \cdot a_{p_i} \cdot L_{p_i}^\sharp = \overline{B}_r^\sharp \cdot \bigcup_{i=1}^{k} a_{p_i} \cdot L_{p_i}^\sharp.$$

This proves the main part.

In addition, let $L \in \mathscr{L}_{\max}$. Choose $p \in S_{\mathcal{X}}(A)$ such that $L = L_p$ and also choose $a \in \operatorname{dcl}(A)$ such that $p(\mathfrak{R}) \subseteq \mu \cdot a \cdot L^\sharp$. We have

$$p(\mathfrak{R}) \subseteq \mathcal{X} \subseteq \bigcup_{i=1}^{k} \overline{B}_r^\sharp \cdot a_{p_i} \cdot L_{p_i}^\sharp.$$

Since p is a complete type over A, there is $1 \leq j \leq k$ such that $p(\mathfrak{R}) \subseteq \overline{B}_r^\sharp \cdot a_{p_j} \cdot L_{p_j}^\sharp$. Since aL is a nearest co-commutative coset to p, by Lemma **??**, we conclude $L \subseteq L_{p_j}$. By maximality of L we get $L = L_{p_j}$. □

4. Γ-dense types in unipotent groups

4.1. **Preliminaries on unipotent groups.** As in [?nilpotent], by *a unipotent group* we mean a real algebraic subgroup of the group of real $n \times n$ upper triangular matrices with 1 on the diagonal.

We list below some properties of unipotent groups that we need and refer to [?nilpotent] and [?nilpotent-book] for more details.

We fix a unipotent group G.

Fact 4.1. *For a subgroup H of G, the following are equivalent.*

(1) H is a closed connected subgroup of G.
(2) H is a real algebraic subgroup of G.
(3) H is definable in $\mathbb{R}_{\mathrm{om}}$.

A lattice in G is a discrete subgroup Γ such that G/Γ is compact.

Let $\Gamma \subseteq G$ be a lattice. A real algebraic subgroup H of G is called Γ-*rational* if $\Gamma \cap H$ is a lattice in H.

Fact 4.2. *Let Γ be a lattice in G.*

(1) The center $Z(G)$ is Γ-rational.
(2) The commutator subgroup $[G, G]$ is closed and Γ-rational.

(3) *If H is a Γ-rational normal subgroup of G and $\pi\colon G \to G/H$ is the quotient map then $\pi(\Gamma)$ is a lattice in G/H. In addition, for every $\pi(\Gamma)$-rational subgroup $K \subseteq G/H$, the preimage $\pi^{-1}(K)$ is Γ-rational.*

(4) *If H_1 and H_2 are Γ-rational subgroups of G then $H_1 \cap H_2$ is Γ-rational as well.*

It follows from the above fact that for any real algebraic subgroup H of G there is the smallest Γ-rational subgroup containing H. We call it *the Γ-rational closure of H* and denote by H^Γ.

The next fact easily follows from Fact **??**(3).

Fact 4.3. *Assume that H is a Γ-rational normal subgroup of G, $\pi\colon G \to G/H$ the quotient map and $\Gamma_0 = \pi(\Gamma)$. Then for every real algebraic subgroup $L \subseteq G$, $\pi(L^\Gamma) = \pi(L)^{\Gamma_0}$.*

We will need the following fact.

Fact 4.4. *Let Γ be a lattice in G and H be a real algebraic subgroup of G. If H is a normal subgroup then H^Γ is normal as well.*

The following is a restatement of Ratner's Orbit Closure Theorem in the case of unipotent groups.

Fact 4.5. [?ratner] *Let Γ be a lattice in G and H be a real algebraic subgroup of G. The topological closure of $H{\cdot}\Gamma$ in G is $H^\Gamma{\cdot}\Gamma$.*

We will be using the following well-known fact.

Fact 4.6. *Let H be a real algebraic subgroup of G and Γ_1, Γ_2 be lattices in G. If Γ_1 and Γ_2 are commensurable, i.e. $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2, then $H^{\Gamma_1} = H^{\Gamma_2}$.*

4.2. **Γ-dense sets in unipotent groups.** Let G be a unipotent group and $\Gamma \subseteq G$ be a lattice. We say that a subset $X \subseteq G$ is Γ-*dense in G* if the set $X{\cdot}\Gamma$ is dense in G, i.e. $\mathrm{cl}(X{\cdot}\Gamma) = G$. Using Lemma **??**(2), we conclude that a subset $X \subseteq G$ is Γ-dense in G if and only if $\mathrm{st}(X^\sharp{\cdot}\Gamma^\sharp) = G$. We use this fact to extend the notion of Γ-density to arbitrary subsets of $G^\sharp$.

Definition 4.7. Let G be a unipotent group, $\Gamma \subseteq G$ a lattice and $\mathcal{X} \subseteq G^\sharp$ be an arbitrary set.

(1) We say that $\mathcal{X}$ is Γ-*dense in G* if $\mathrm{st}(\mathcal{X}{\cdot}\Gamma^\sharp) = G$.
(2) We say that $\mathcal{X}$ is *strongly Γ-dense in G* if $\mathrm{st}(\mathcal{X}{\cdot}\Gamma_1^\sharp) = G$ for every lattice Γ_1 commensurable with Γ.
(3) We say that a type $p \in S_G(A)$ is *(strongly) Γ-dense in G* if the set $p(\mathfrak{R})$ is (strongly) Γ-dense in G.

Remark 4.8. Let G is be a unipotent group, $\Gamma \subseteq G$ a lattice and $\mathcal{X} \subseteq G^\sharp$. It is easy to see that $\mathcal{X}$ is strongly Γ-dense in G if and only if it is Γ_0-dense for every subgroup $\Gamma_0 \subseteq \Gamma$ of finite index.

Example 4.9. Let $G = (\mathbb{R}, +)$, $\Gamma = \mathbb{Z}$ and let X be the closed interval $[0, 1]$. The set $X^\sharp$ is Γ-dense in G, but not strongly Γ-dense.

The following fact follows from Facts **??** and **??**.

Fact 4.10. *Let G be a unipotent group and $L \subseteq G$ a real algebraic subgroup. For a lattice $\Gamma \subseteq G$ the following are equivalent.*
(1) L is Γ-dense in G.
(2) $L^\Gamma = G$
(3) L is strongly Γ-dense in G.

We observe:

Lemma 4.11. *Let Γ be a lattice in a unipotent group G. A subset $\mathcal{X} \subseteq G^\sharp$ is Γ-dense in G if and only if $\mu \cdot \mathcal{X} \cdot \Gamma^\sharp = G^\sharp$.*

Proof. The "if" part is clear.

For "the only if" part, since G/Γ is compact, given $g \in G^\sharp$ there is $\gamma \in \Gamma^\sharp$ such that $g\gamma \in \mathcal{O}$. Thus, since $\mathcal{X}$ is Γ-dense in G, there is $a \in \mathcal{X}$ such that $\mathrm{st}(g\gamma) = \mathrm{st}(a)$. It follows that $g \in \mu{\cdot}a{\cdot}\Gamma^\sharp$. □

We will need the following fact.

Fact 4.12 ([?nilpotent, Lemma 5.1])**.** *Let $\pi \colon G \to H$ be a real algebraic surjective homomorphism of unipotent groups, and $\mathcal{X}$ a subset of $G^\sharp$. Then, for every lattice $\Gamma \subseteq G$, if $\pi(\Gamma)$ is closed in H then*

$$\pi(\mathrm{st}(\mathcal{X}{\cdot}\Gamma^\sharp)) = \mathrm{st}(\pi(\mathcal{X}){\cdot}\pi(\Gamma^\sharp)).$$

Our main goal is to describe Γ-dense types. We will consider the abelian case first.

4.3. **Γ-dense types in abelian groups.** Since every abelian unipotent group is algebraically isomorphic to $(\mathbb{R}^m, +)$ for some m, we often identify an abelian unipotent group with an $\mathbb{R}$-vector space $(\mathbb{R}^m, +)$.

In the abelian case, every subgroup is co-commutative, hence for a set $A \subseteq \mathfrak{R}$ and a type $p(x) \in S(A)$ on $\mathbb{R}^m$, instead of a nearest co-commutative coset to p we say *a nearest coset to p*.

Our first goal of this section is to prove the following:

Proposition 4.13. *Let G be an abelian unipotent group, $A \subseteq \mathfrak{R}$, $p \in S_G(A)$, and $a_p \in \mathrm{dcl}(A)$ be such that $a_p + L_p$ is a nearest coset to p. For a lattice $\Gamma \subseteq G$, the following are equivalent.*
(1) The type p is Γ-dense in G.

(2) $L_p^\Gamma = G$.
(3) The type p *is strongly* Γ*-dense in* G.

Proof. Since, by Fact **??**, $L^\Gamma = L^{\Gamma_0}$ for any subgroup $\Gamma_0 \subseteq \Gamma$ of finite index, it is sufficient to show (1) $\Leftrightarrow$ (2).

For simplicity we denote L_p^Γ by L, and assume $A = \mathrm{dcl}(A)$.

(1) $\Rightarrow$ (2). Assume that $L \neq G$, hence, by Fact **??**, the set $L + \Gamma$ is a closed proper subgroup of G. By Lemma **??**(2), $\mu + L^\sharp + \Gamma^\sharp$ is a proper subgroup of $G^\sharp$, hence the coset $a_p + \mu + L^\sharp + \Gamma^\sharp$ is a proper subset of $G^\sharp$.

Since $\mu + p(\mathfrak{R}) + \Gamma^\sharp \subseteq a_p + \mu + L^\sharp + \Gamma^\sharp$, the set $\mu + p(\mathfrak{R}) + \Gamma^\sharp$ is a proper subset of $G^\sharp$ and, by Fact **??**, $p(\mathfrak{R})$ is not Γ-dense, so (1) fails.

(2) $\Rightarrow$ (1). Assume $L = G$ and we prove that $\mathrm{st}(p(\mathfrak{R}) + \Gamma^\sharp) = G$, The proof is similar to [?nilpotent, Proposition 5.3].

We use induction on $\dim G$.

If $\dim(G) = 0$ then there is nothing to prove.

Assume $\dim(G) = n > 0$ and the result holds for all abelian unipotent groups of dimension less than n.

We have $\dim(L) > 0$, hence, by Remark **??**, p is a long type. Let P be the μ-stabilizer of p. By Proposition **??**, P is a real algebraic subgroup of G of positive dimension, and by Lemma **??**, $P \subseteq L_p$, hence $P^\Gamma \subseteq L_p^\Gamma = L$.

Let $\pi : G \to G_0 := G/P^\Gamma$ be the quotient map, $\Gamma_0 = \pi(\Gamma)$, and $q = \pi(p)$. By Fact **??**, Γ_0 is a lattice in G_0. Notice that $\dim(G_0) < \dim(G)$.

Let $a_q + L_q$ be a nearest coset to q. It follows from Lemma **??** that $L_q = \pi(L_p)$. Since $G = L_p^\Gamma$, by Fact **??**(3), $G_0 = L_q^{\Gamma_0}$, hence, by induction hypothesis, the type q is Γ_0-dense in $G_0^\sharp$, and

$$\mathrm{st}(q(\mathfrak{R}) + \Gamma_0^\sharp) = G_0.$$

Applying Fact **??**, we obtain

$$\pi(\mathrm{st}(p(\mathfrak{R}) + \Gamma^\sharp)) = \mathrm{st}(q(\mathfrak{R}) + \Gamma_0^\sharp) = G_0.$$

Let $D = \mathrm{st}(q(\mathfrak{R}) + \Gamma^\sharp)$. By Lemma **??**(3), it is a closed subset of G. It is not hard to see that D is invariant ander the action of both P and Γ, hence it is invarint under $P + \Gamma$. Since D is closed, it is invariant under the action of the topological closure of $P + \Gamma$. By Fact **??**, P^Γ is contained in $\mathrm{cl}(P+\Gamma)$, hence D is invariant under P^Γ. Since $\pi(D) = G_0$ and $\ker(\pi) = P^\Gamma$, it follows then $D = G$, hence p is Γ-dense in G.

This finishes the proof of Proposition **??**. □

As a corollary we obtain the following theorem.

Theorem 4.14. *Let $A \subseteq \mathfrak{R}$, $G = (\mathbb{R}^n, +)$, $p \in S_G(A)$, and let $a_p + L_p$ be a nearest coset to p (so $a_p \in \mathrm{dcl}(A)$). Then for every lattice $\Gamma \subseteq \mathbb{R}^n$, we have*

$$\mu + p(\mathfrak{R}) + \Gamma^\sharp = \mu + a_p + L_p^\sharp + \Gamma^\sharp.$$

Proof. Consider the type $p_1 = -a_p + p$. It is a complete $\mathcal{L}_{\mathrm{om}}$-type over A. Clearly L_p is a nearest coset to p_1, hence there exists a type $p_2 \in S_{L_p}(A)$ which is μ-equivalent to p_1, and therefore L_p is also a nearest coset to p_2. Let $G_0 = L_p^\Gamma$ and $\Gamma_0 = \Gamma \cap G_0$, a lattice in G_0.

Working in G_0 we have that $L_p^{\Gamma_0} = G_0$, hence by Proposition **??**, the type p_2 is Γ_0-dense in G_0, so, by Lemma **??**,

$$(\mu \cap G_0^\sharp) + p_2(\mathcal{R}) + \Gamma_0^\sharp = G_0^\sharp.$$

Obviously, L_p is also Γ_0-dense in G_0, hence $G_0^\sharp = (\mu \cap G_0^\sharp) + L_p^\sharp + \Gamma_0^\sharp$. We conclude

$$\mu + p(\mathcal{R}) + \Gamma^\sharp = \mu + a_p + \mu + p_2(\mathcal{R}) + \Gamma^\sharp = \mu + a_p + L_p^\sharp + \Gamma^\sharp.$$

□

4.4. **Abelianization and density.** For a unipotent group G we will denote by G_{ab} the abelianization of G, i.e. the group $G_{\mathrm{ab}} = G/[G,G]$, and by π_{ab} the quotient map $\pi_{\mathrm{ab}} \colon G \to G_{\mathrm{ab}}$. The group G_{ab} is also unipotent and $\dim G > 0$ if and only if $\dim G_{\mathrm{ab}} > 0$.

If $\Gamma \subseteq G$ is a lattice then we denote by Γ_{ab} the group $\Gamma_{\mathrm{ab}} = \pi_{\mathrm{ab}}(\Gamma)$. By Fact **??**, Γ_{ab} is a lattice in G_{ab}. Our main goal in this section to show that a type $p \in S_G(A)$ is Γ-dense in G if and only if its abelianization $\pi_{\mathrm{ab}}(p)$ is Γ_{ab}-dense in G_{ab}.

The next proposition is a key.

Proposition 4.15. *Let G be a unipotent group, $A \subseteq \mathfrak{R}$, Γ a lattice in G and $p \in S_G(A)$. Assume p is not Γ-dense in G. Then there is a co-commutative Γ-rational subgroup $H \subseteq G$ such that for the projection $\pi \colon G \to G/H$ the type $\pi(p)$ is not $\pi(\Gamma)$-dense in G/H.*

Proof. By induction on $\dim(G)$.

If $\dim(G) = 0$ then there is nothing to prove.

Assume $\dim(G) = n > 0$ and the proposition holds for all unipotent groups of dimension less than n.

If the type p is short then, by Lemma **??**(1), the type $\pi_{\mathrm{ab}}(p)$ is short as well, and it is easy too see, e.g. using Lemma **??**(1), that a short type is not Γ_{ab}-dense in G_{ab}. We can take $H = [G,G]$ that is Γ-rational by Fact **??**(2).

Thus we may assume that p is a long type. Let P be the μ-stabilizer of p. By Proposition **??**, P is an $\mathbb{R}_{\rm om}$-definable subgroup of G of positive dimension.

As in [?nilpotent, Propositioni 5.3], we consider the smallest $\mathbb{R}_{\rm om}$-definable, normal Γ-rational subgroup of G containing P and denote it by $N(P)^\Gamma$. Let N_0 be the intersection of $N(P)^\Gamma$ with the center of G. Since G is unipotent and $N(P)^\Gamma$ has positive dimension, the group N_0 is also of positive dimension (see, for example, [?stroppel, Proposition 7.13]), and, by Fact **??**, it is Γ-rational.

Let $\pi\colon G \to G_0 := G/N_0$ be the quotient map, $\Gamma_0 = \pi_0(\Gamma)$, and $q = \pi(p)$. By Fact **??**, Γ_0 is a lattice in G_0.

We claim that the type q is not Γ_0-dense in G_0.

Indeed, assume towards contradiction that q is Γ_0-dense in G_0. Then, by Fact **??**,

$$\pi_0(\mathrm{st}(p(\mathfrak{R}){\cdot}\Gamma^\sharp)) = G_0.$$

Let $D_{p,\Gamma} = \mathrm{st}(p(\mathfrak{R}){\cdot}\Gamma^\sharp)$ It follows from the above equation that

$$D_{p,\Gamma}{\cdot}N_0 = G. \tag{4.1}$$

Our aim is to show that $D_{p,\Gamma} = G$, contradicting the fact that p is not Γ-dense in G.

Since, by Lemma **??**(2), the set $D_{p,\Gamma}$ is a closed subset of G, it is sufficient to show that it is dense in G.

Claim A. *The set $D_{p,\Gamma}$ is left invariant under the μ-stabilizer P of p.*

Proof. Note that $D_{p,\Gamma} = \mathrm{st}(\mu{\cdot}p(\mathfrak{R}){\cdot}\Gamma^\sharp)$, and μp is left-invariant under P. Thus, for $g \in P$,

$$g \cdot D_{p,\Gamma} = g{\cdot}\,\mathrm{st}(\mu{\cdot}p(\mathfrak{R}){\cdot}\Gamma^\sharp) = \mathrm{st}(g{\cdot}\mu{\cdot}p(\mathfrak{R}){\cdot}\Gamma^\sharp) = \mathrm{st}(\mu{\cdot}p(\mathfrak{R}){\cdot}\Gamma^\sharp) = D_{p,\Gamma}.$$

□

Clearly $D_{p,\Gamma}$ is right-invariant under action of Γ. Thus, $P{\cdot}D_{p,\Gamma}{\cdot}\Gamma = D_{p,\Gamma}$, and, in addition, by equation (**??**), $D_{p,\Gamma}N_0 = G$.

Because $N_0 = N(P)^\Gamma \cap Z(G)$, our goal, $D_{p,\Gamma} = G$, follows from the following general result:

Claim B. *For a unipotent group G, assume that $D \subseteq G$ is a closed set, left invariant under a real algebraic subgroup $P \subseteq G$ and right invariant under a lattice $\Gamma \subseteq G$. Let $N_0 \subseteq N(P)^\Gamma \cap N_G(P)$ be a subgroup of G. If $DN_0 = G$ then $D = G$.*

Proof. Let $Y = \{g \in G : (P^g)^\Gamma = N(P)^\Gamma\}$. This is not, in general, a definable set, but, by [?nilpotent, Proposition 4.3], it is dense in G. Thus, it is sufficient to prove that $Y \subseteq D$.

First note that Y is left invariant under N_0. Indeed, assume that $a \in N_0 b$. Since $ab^{-1} \in N_0 \subseteq N_G(P)$, then $P^a = P^b$, implying that $b \in Y$ if and only if $a \in Y$.

Let $b \in Y$. Since $DN_0 = G$, there is $a \in D$ such that $b \in aN_0$, and therefore $a \in Y$. Using the definition of Y and the fact that $aP^a = Pa$, we obtain

$$b \in a{\cdot}N_0 \subseteq a{\cdot}N(P)^\Gamma = a{\cdot}(P^a)^\Gamma = a{\cdot}\mathrm{cl}(P^a{\cdot}\Gamma) \subseteq \mathrm{cl}(a{\cdot}P^a{\cdot}\Gamma) = \mathrm{cl}(P{\cdot}a{\cdot}\Gamma).$$

Since $a \in D$, by the invariance properties of D, also $b \in D$. Hence $Y \subseteq D$, so $D = G$, a contradiction.

This ends the proof of Claim B, and thus we conclude that q is not Γ_0-dense in G_0. □

Applying the induction hypothesis to G_0, Γ_0 and q, we obtain a co-commutative Γ_0-rational subgroup $H_0 \subseteq G_0$ such that the image of q in G_0/H_0 is not Γ_0/H_0-dense. It is not hard to see that $H = \pi_0^{-1}(H_0)$ is a co-commutative Γ-rational subgroup of G satisfying the conclusion of the proposition.

This finishes the proof of Proposition **??**

□

We can now prove the main theorem of this section.

Theorem 4.16. *Let G be a unipotent group, $A \subseteq \mathfrak{R}$ and Γ a lattice in G. A type $p \in S_G(A)$ is Γ-dense in G if and only if the type $\pi_{\mathrm{ab}}(p)$ is Γ_{ab}-dense in G_{ab}.*

Proof. Let $q = \pi_{\mathrm{ab}}(p)$. We write additively the group operation in G_{ab}. By Fact **??**,

$$\pi_{\mathrm{ab}}(\mathrm{st}(p(\mathfrak{R}) \cdot \Gamma^\sharp)) = \mathrm{st}(q(\mathfrak{R}) + \Gamma^\sharp_{\mathrm{ab}}).$$

This implies the "only if" part.

We prove the "if part" part by contraposition, using Proposition **??**. Indeed, assume the type p is not Γ-dense in G, and we derive that $\pi_{\mathrm{ab}}(p)$ is not G_{ab}-dense in G_{ab}.

Let H and $\pi\colon G \to G/H$ be as in Proposition **??**. Since H contains $[G, G]$, the map π factors through G_{ab}, i.e. there is $\pi'\colon G_{\mathrm{ab}} \to G/H$ with $\pi = \pi' \circ \pi_{\mathrm{ab}}$. By Fact **??**

$$\pi'\Big(\mathrm{st}\big(\pi_{\mathrm{ab}}(p)(\mathfrak{R}) + \Gamma^\sharp_{\mathrm{ab}}\big)\Big) = \mathrm{st}\Big(\pi(p)(\mathfrak{R}) + \pi(\Gamma)^\sharp\Big).$$

Since $\pi(p)$ is not $\pi(\Gamma)$-dense in G/H, the type $\pi_{\mathrm{ab}}(p)$ is not Γ_{ab}-dense in G_{ab}. □

The following summarizes main results of this section.

Theorem 4.17. *Let G be a unipotent group, $A \subseteq \mathfrak{R}$ and $p \in S_G(A)$.*
For a lattice $\Gamma \subseteq G$, the following are equivalent.

(1) The type p is Γ-dense in G.
(2) The type p is strongly Γ-dense in G.
(3) $L_p^\Gamma = G$.
(4) The type $\pi_{\mathrm{ab}}(p)$ is Γ_{ab}-dense in G_{ab}.
(5) The type $\pi_{\mathrm{ab}}(p)$ is strongly Γ_{ab}-dense in G_{ab}.
(6) $\pi_{\mathrm{ab}}(L_p)^{\Gamma_{\mathrm{ab}}} = G_{\mathrm{ab}}$.

5. Γ-dense definable subsets of unipotent groups

We fix a unipotent group G.

In this section we obtain a description of Γ-dense definable subsets of G similar to that of Theorem **??**.

First an elementary lemma.

Lemma 5.1. *For $A \subseteq \mathfrak{R}$, let $\mathcal{X} \subseteq G^\sharp$ be $\mathcal{L}_{\mathrm{om}}$-definable over A. For a lattice $\Gamma \subseteq G$, if some type $p \in S_{\mathcal{X}}(A)$ is Γ-dense in G then $\mathcal{X}$ is strongly Γ-dense G.*

Proof. Assume a type $p \in S_{\mathcal{X}}(A)$ is Γ-dense, hence, by Theorem **??**, it is strongly Γ-dense. Since $p(\mathfrak{R}) \subseteq \mathcal{X}$, obviously $\mathcal{X}$ is strongly Γ-dense. □

The main goal of this section (Theorem **??**) is to show that an appropriate converse of the above lemma holds. Namely, an $\mathcal{L}_{\mathrm{om}}$-definable subset $\mathcal{X} \subseteq G^\sharp$ is strongly Γ-dense in G if and only if some type on $\mathcal{X}$ is Γ-dense.

We need two lemmas and a proposition.

Lemma 5.2. *Let $L \subseteq G$ be a normal real algebraic subgroup and $\alpha \in G^\sharp$.*

(1) For any lattice $\Gamma \subseteq G$ the set $\mathcal{O} \cap \alpha\Gamma^\sharp$ is nonempty, and for every $g \in \mathrm{st}(\alpha\Gamma^\sharp)$ we have

$$\mathrm{st}(\alpha{\cdot}L^\sharp{\cdot}\Gamma^\sharp) = g{\cdot}L^\Gamma{\cdot}\Gamma.$$

(2) If L is co-commutative then αL is a nearest co-commutative coset to some type $p \in S(\alpha)$ on $\alpha L^\sharp$, hence $\mathscr{L}_{\max}(\alpha L^\sharp) = \{L\}$.

Proof. (1) Because Γ is co-compact, the set $\mathcal{O} \cap \alpha\Gamma^\sharp$ is not empty. Let $g \in \mathrm{st}(\alpha\Gamma^\sharp)$, and we choose $\gamma \in \Gamma^\sharp$ with $\alpha \cdot \gamma \in \mu g$. Since L is normal, we have

$$\begin{aligned}\mathrm{st}(\alpha{\cdot}L^\sharp{\cdot}\Gamma^\sharp) &= \mathrm{st}(\alpha{\cdot}\Gamma^\sharp{\cdot}L^\sharp) = \mathrm{st}(g{\cdot}\Gamma^\sharp{\cdot}L^\sharp)\\ &= \mathrm{st}(g{\cdot}L^\sharp{\cdot}\Gamma^\sharp) = \mathrm{cl}(g{\cdot}L{\cdot}\Gamma) = g{\cdot}L^\Gamma{\cdot}\Gamma.\end{aligned}$$

(2) We first consider the abelian case, namely we assume $G = (\mathbb{R}^n, +)$ and $L \subseteq \mathbb{R}^n$ a linear subspace. We need to show that there is $\beta \in \alpha + L^\sharp$ such that $\beta \notin \mu(\gamma + L_0^\sharp)$, for any $\gamma \in \mathrm{dcl}(\alpha)$ and a proper subspace $L_0 \subseteq L$. It is thus sufficient to show

$$\alpha + L^\sharp \not\subseteq \bigcup\{\overline{B}_1^\sharp + \gamma + L_0^\sharp \colon \gamma \in \mathrm{dcl}(\alpha), L_0 \subsetneq L \text{ a subspace}\}. \tag{5.1}$$

By logical compactnes,s (**??**) follows from the following claim.

Claim. *Let $r \in \mathbb{R}^{\geq 0}$, and $L_1, \ldots, L_k \subseteq L$ be proper subspaces. Then for any $\alpha, \gamma_1, \ldots, \gamma_k \in \mathfrak{R}^n$ we have*

$$\alpha + L^\sharp \not\subseteq \overline{B}_r^\sharp + \bigcup_{i=1}^{k} (\gamma_i + L_i^\sharp).$$

Proof of Claim. Since, for fixed r and $L_1, \ldots, L_k$, the conclusion of the claim can be expressed by a first-order formula, we can work in $\mathbb{R}$ instead of $\mathfrak{R}$;and also, subtracting α from both sides, we only need to consider the case $\alpha = 0$.

We fix proper subspaces $L_1, \ldots, L_k \subseteq L$, and show that for all ≥ 0 and $b_1, \ldots, b_k \in \mathbb{R}^n$, we have $L \not\subseteq \overline{B}_r + \bigcup_{i=1}^k (b_i + L_i)$.

Clearly, by the dimension assumptions, $L \neq L_1 \cup \ldots \cup L_k$.

Next, let us see that $L \not\subseteq \overline{B}_r + \bigcup_{i=1}^k L_i$ for any $r \in \mathbb{R}^{\geq 0}$. Indeed, choose $c \in L \setminus (\bigcup_{i=1}^k L_i)$. Then, for every $i = 1, \ldots, k$, we have $d(c, L_i) > 0$, where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^n$. Since $d(tc, L_i) = td(c, L_i)$ for $t \in \mathbb{R}^{>0}$, we obtain that for given $r \in \mathbb{R}^{\geq}$ and $i = 1, \ldots, k$, for t large enough, $d(tc, L_i) > r$, and hence $tc \in L \setminus (\overline{B}_r + \bigcup_{i=1}^k L_i)$.

Finally, assume that for some $r > 0$ and $b_1, \ldots, b_k \in \mathbb{R}^n$ we would have $L \subseteq \overline{B}_r + \bigcup_{i=1}^k (b_i + L_i)$. Then, choosing $r' > 0$ big enough so that $b_i + \overline{B}_r \subseteq \overline{B}_{r'}$ for $i = 1, \ldots, k$, we would have $L \subseteq \overline{B}_{r'} + \bigcup_{i=1}^k L_i$, a contradiction.

This finishes the proof of Claim, and hence the lemma, in the case that G is abelian.

When G is nilpotent and $L \subseteq G$ is co-commutative we first apply the above to $\pi_{\mathrm{ab}}(\alpha L^\sharp) \subseteq G_{\mathrm{ab}}^\sharp$ and find a type over α with $q \vdash \pi_{\mathrm{ab}}(\alpha L^\sharp)$, such that $\pi_{\mathrm{ab}}(\alpha L)$ is a nearest coset to q. Now, choose a type p over α such that $p \vdash \alpha L$ and $\pi_{\mathrm{ab}}(p) = q$. A nearest co-commutative coset to p is contained in αL, and projects via π_{ab} onto $\pi_{\mathrm{ab}}(\alpha L)$ (see Lemma **??**). Since $L \supseteq [G, G]$, it follows that αL is a nearest co-commutative coset to p. This finishes the proof of the claim. □

End of the proof of the lemma.

□

We are going to need the following result.

Lemma 5.3. *If H is a proper real algebraic subgroup of G then $\pi_{\mathrm{ab}}(H)$ is a proper subgroup of G_{ab}*

Proof. By [?nilpotent-book, Theorem 1.1.13] there is a chain of real algebraic subgroups

$$\{e\} = H_0 \subsetneq \cdots \subsetneq H = H_m \subsetneq H_{m+1} \subsetneq \cdots \subsetneq H_n = G,$$

with $n = \dim(G)$ and $\dim H_{i+1} = \dim H_i + 1$. By [?nilpotent-book, Lemma 1.1.8], $[G,G] \subseteq H_{n-1}$. Hence $H_{n-1}/[G,G]$ is a proper subgroup of G_{ab} and so is $H/[G,G]$. □

Proposition 5.4. *Let $\Gamma \subseteq G$ be a lattice, and $L_1, \ldots, L_k$ proper Γ-rational subgroups of G. If $K \subset G$ is a compact set then there is a subgroup $\Gamma_0 \subseteq \Gamma$ of finite index such that for any $g_1, \ldots, g_k \in G$, we have*

$$K \cdot \bigcup_{i=1}^{k} g_i \cdot L_i \cdot \Gamma_0 \neq G.$$

Proof. We first consider the case when G is abelian. So we assume $G = (\mathbb{R}^n, +)$.

Claim. *Let $\Gamma \subseteq \mathbb{R}^n$ be a lattice, $K \subseteq \mathbb{R}^n$ a compact set, and $L \subseteq \mathbb{R}^n$ be a proper Γ-rational subspace. Then for any $m \in \mathbb{N}$, there is a subgroup $\Gamma' \subseteq \Gamma$ of finite index such that for some $b_1, \ldots, b_m \in \mathbb{R}^n$, the translates $b_i + K + L + \Gamma'$, $i = 1, \ldots, m$, are pairwise disjoint.*

Proof. Replacing $\mathbb{R}^n$ by $\mathbb{R}^n/L$ if needed, we may assume that L is the trivial subspace $\{0\}$.

Since K is compact, it is bounded, hence there are $b_1, \ldots b_m \in \mathbb{R}^n$ such that the translates $b_1 + K, \ldots, b_m + K$ are pair-wise disjoint. Let $B = \bigcup_{i=1}^m (b_i + K)$. Obviously B is compact and hence the set $B' = B - B = \{b - b' \colon b, b' \in B\}$ is compact as well.

Since Γ is discrete, the intersection $\Gamma \cap B'$ is finite. Every finitely generated abelian group is residually finite, i.e. the intersection of all subgroups of finite index is trivial, hence there is a subgroup $\Gamma' \subseteq \Gamma$ of finite index with $\Gamma' \cap B' = \{0\}$. It is not hard to see that the sets $b_i + K + \Gamma'$, $i = 1, \ldots m$, are pairwise disjoint.

This finishes the proof of the claim. □

We return to the proof of the proposition for $G = (\mathbb{R}^n, +)$. We apply the above claim to each L_i with $m = k + 1$, and for each $i = 1, \ldots, k$, obtain a subgroup $\Gamma_i \subseteq G$ of finite index such that $K + L_i + \Gamma_i$ has $k + 1$ disjoint translates.

Since every abelian group is amenable, there is a G-invariant finitely additive probability measure $\lambda\colon \mathscr{P}(G) \to [0,1]$. By our choice of Γ_i, we have $\lambda(K+L_i+\Gamma_i) \leq 1/(k+1)$.

We take $\Gamma_0 = \bigcap_{i=1}^k \Gamma_i$. For any $g_1,\dots,g_k \in G$ we have

$$\lambda\Big(\bigcup_{i=1}^k g_i + K + L_i + \Gamma_0\Big) \leq \sum_{i=1}^k \lambda(g_i + K + L_i + \Gamma_0) \leq \sum_{i=1}^k \lambda(g_i + K + L_i + \Gamma_i) \leq k/(k+1) < 1.$$

Hence $\bigcup_{i=1}^k g_i + K + L_i + \Gamma_0 \neq G$. This finishes the proof of the abelian case.

Assume now that G is an arbitrary unipotent group. Let $K_{\text{ab}} = \pi_{\text{ab}}(K)$, $\Gamma_{\text{ab}} = \pi_{\text{ab}}(\Gamma)$, and, for $i = 1,\dots,k$, let $L_i^{\text{ab}} = \pi_{\text{ab}}(L_i)$. Obviously K_{ab} is a compact subset of G_{ab}, Γ_{ab} is a lattice in G_{ab} by Fact **??**, and it is not hard to see that each L_i^{ab} is $\widetilde{\Gamma}$-rational subgroup of G_{ab}. It also follows from Lemma **??** that each L_i^{ab} is a proper subgroup of G_{ab}.

We now use the abelian case and find a subgroup $\Gamma_0' \subseteq \Gamma_{\text{ab}}$ such that for any $b_1,\dots,b_m \in G_{\text{ab}}$ we have $K_{\text{ab}}\cdot \bigcup_{i=1}^k b_i \cdot L_i^{\text{ab}} \cdot \Gamma_0' \neq G_{\text{ab}}$.

We take $\Gamma_0 = \pi_{\text{ab}}^{-1}(\Gamma_0') \cap \Gamma$. □

We are now ready to prove one of the main theorems of this paper.

Theorem 5.5. *Let G be a unipotent group, $A \subseteq \mathfrak{R}$, and let $\mathcal{X} \subseteq G^\sharp$ be a set $\mathcal{L}_{\text{om}}$-definable over A.*

For a lattice $\Gamma \subseteq G$, the following are equivalent:

(a) The set $\mathcal{X}$ is strongly Γ-dense in G.
(b) $L^\Gamma = G$ for some $L \in \mathscr{L}_{\max}(\mathcal{X})$.
(c) Some type $p \in S_{\mathcal{X}}(A)$ is Γ-dense.

Proof. By Theorem **??**, $(b) \Leftrightarrow (c)$, and, by Lemma **??**, $(c) \Rightarrow (a)$.

Let us show that $(a) \Rightarrow (b)$.

Let $\Gamma \subseteq G$ be a lattice. We choose L_i and a_i, $i = 1,\dots,k$, as in Theorem **??** with $r = 1$.

Assume (b) fails, namely, $L^\Gamma \neq G$ for all $L \in \mathscr{L}_{\max}(\mathcal{X})$. Then clearly, $L_i^\Gamma \neq G$, for all $i = 1,\dots,k$. For any subgroup $\Gamma_0 \subseteq G$ of finite index we have

$$\mathcal{X}\cdot\Gamma_0^\sharp \subseteq \overline{B}_1^\sharp \cdot \bigcup_{i=1}^k a_i \cdot L_i^\sharp \cdot \Gamma_0^\sharp,$$

hence

$$\mathrm{st}(\mathcal{X}\cdot\Gamma_0^\sharp) \subseteq \mathrm{st}\Big(\overline{B}_1^\sharp\cdot\bigcup_{i=1}^k a_i\cdot L_i^\sharp\cdot\Gamma_0^\sharp\Big) = \bigcup_{i=1}^k \mathrm{st}(\overline{B}_1^\sharp\cdot a_i\cdot L_i^\sharp\cdot\Gamma_0^\sharp).$$

Using Lemma **??**(1) we choose $g_1,\dots,g_k \in G$ such that $\mathrm{st}(\overline{B}_r^\sharp\cdot a_i\cdot L_i^\sharp\cdot\Gamma_0^\sharp) = \overline{B}_r\cdot g_i\cdot L_i^{\Gamma_0}\cdot\Gamma_0$. By fact **??**, $L_i^{\Gamma_0} = L_i^{\Gamma}$, hence

$$\mathrm{st}(\mathcal{X}\cdot\Gamma_0^\sharp) \subseteq \overline{B}_r\cdot\bigcup_{i=1}^k g\cdot L_i^\Gamma\cdot\Gamma_0.$$

By Proposition **??**, there exists $\Gamma_0 \subseteq \Gamma$ of finite index for which the set on the right is a proper subset of G, hence (a) fails. Thus, $(a) \Rightarrow (b)$. □

Recall that for $\pi_{\mathrm{ab}} : G \to G_{\mathrm{ab}}$, and $\Gamma \subseteq G$ a lattice, we let $\Gamma_{\mathrm{ab}} = \pi_{\mathrm{ab}}(\Gamma)$.

Corollary 5.6. *Let G be a unipotent group, and let $\mathcal{X} \subseteq G^\sharp$ be an $\mathcal{L}_{\mathrm{om}}$-definable set. For a lattice $\Gamma \subseteq G$, the set $\mathcal{X}$ is strongly Γ-dense in G if and only if $\pi_{\mathrm{ab}}(\mathcal{X})$ is strongly Γ_{ab}-dense in G_{ab}.*

Proof. The "only if" part follows from Fact **??**.

For the "if" part, assume that $\pi_{\mathrm{ab}}(\mathcal{X})$ is strongly Γ_{ab}-dense in G_{ab}. Choose a set $A \subseteq \mathfrak{R}$ such that $\mathcal{X}$ is $\mathcal{L}_{\mathrm{om}}$-definable over A. Applying Theorem **??**(2) to $\pi_{\mathrm{ab}}(\mathcal{X})$, we obtain a type $q(x) \in S_{\pi_{\mathrm{ab}}(\mathcal{X})}(A)$ that is Γ_{ab}-dense in G_{ab}. Let $p \in S_{\mathcal{X}}(A)$ be a type on $\mathcal{X}$ with $\pi_{\mathrm{ab}}(p) = q$. By Theorem **??**, the type p is Γ-dense in G, hence $\mathcal{X}$ is Γ-dense in G as well. □

6. Interpreting the results as Hausdorff limits

6.1. **Hausdorff limits.** We first recall some definitions.

Let (M,d) be a compact metric space, and $X_1, X_2 \subseteq X$. The Hausdorff distance $d_H(X_1,X_2)$ between X_1 and X_2 is defined as follows: First, for $x \in M$, we let $d(x,X_i) = \inf_{y\in X_i} d(x,y)$. Next,

$$d_H(X_1,X_2) = \max\{\sup_{x\in X_1} d(x,X_2),\ \sup_{x\in X_2} d(x,X_1)\}.$$

An equivalent definition is given by:

$$d_H(X_1,X_2) = \inf\{r \geq 0 : \forall x_i \in X_i, i=1,2,\ \ d(x_1,X_2), d(x_2,X_1) \leq r\}.$$

Remark 6.1. For $X_1, X_2 \subseteq M$ we have $d_H(X_1,X_2) = 0$ if and only if $\mathrm{cl}(X_1) = \mathrm{cl}(X_2)$.

Denoting by $\mathcal{K}(M)$ the set of all compact subsets of M, it is known that the restriction of d_H to $\mathcal{K}(M)$ makes it into a compact metric space.

The topology induced by d_H on $\mathcal{K}(M)$ does not depend on the metric d but only on the topology of M. It coincides with the Vietoris topology.

Given a family $\mathcal{F} \subseteq \mathcal{K}(M)$, a set $Y \in \mathcal{K}(M)$ is *a Hausdorff limit* of $\mathcal{F}$ if for every $\varepsilon > 0$ there is $F \in \mathcal{F}$ with $d_H(Y, F) < \varepsilon$. Using Remark **??**, we extend this definition to a family $\mathcal{F} \subseteq \mathcal{P}(M)$ of arbitrary subsets of M by saying that $Y \in \mathcal{K}(M)$ is a Hausdorff limit of $\mathcal{F}$ if it is a Hausdorff limit of the family $\{\mathrm{cl}(F) \colon F \in \mathcal{F}\}$.

6.1.1. *Limits at infinity.* We denote by I_∞ the interval $(0, +\infty) \subseteq \mathbb{R}$. We abreviate "for all sufficiently large t" by "$t \gg 0$".

We define:

Definition 6.2. Let $\mathcal{F} = \{F_t \colon t \in I_\infty\}$ be a family of subsets of M.

(1) A set $Y \in \mathcal{K}(M)$ is *a Hausdorff limit at* ∞ *of the family* $\mathcal{F}$ if for all $\epsilon > 0$ and $r > 0$ there is $t > r$ with $d_H(Y, F_t) < \varepsilon$.
(2) We say that *the family* $\mathcal{F}$ *converges to a set* $Y \in \mathcal{K}(M)$ *at* ∞ if Y is the unique Hausdorff limit of $\mathcal{F}$ at ∞. In this case, since $\mathcal{K}(M)$ is compact, Y is **the** limit of $\mathcal{F}$ as t goes to ∞, namely for any $\epsilon > 0$ there is $R \in \mathbb{R}$ such that for all $t > R$, $d_H(Y, F_t) < \varepsilon$.

6.2. **Haudorff Limits via the standard part map.** We fix a compact set $M \subseteq \mathbb{R}^n$ with the metric d induced by the Euclidean metric of $\mathbb{R}^n$, and we view both M and d as definable in $\mathbb{R}_{\text{full}}$.

Since M is compact, $M^\sharp \subseteq \mathcal{O}^n$, and we denote by st_M the restriction of the standrad part map $\mathrm{st}\colon \mathcal{O}^n \to \mathbb{R}^n$ to $M^\sharp$. It is not hard to see that $\mathrm{st}_M \colon M^\sharp \to M$ maps $\alpha \in M^\sharp$ to the unique $a \in M$ such that $\alpha \in U^\sharp$ for every neighborhood U of a.

Let $\mathcal{F} = \{F_t \colon t \in T\}$ be a family of subsets of M indexed by a set $T \subseteq \mathbb{R}^m$. We can view this family also as the family of fibers of the set $F = \{(x, t) \in X \times T \colon x \in F_t\}$, with respect to the second projection, and hence as a family definable in $\mathbb{R}_{\text{full}}$. Thus for $\tau \in T^\sharp$ we also have a "non-standard" fiber $F^\sharp_\tau = \{x \in M^\sharp : (x, \tau) \in F^\sharp\}$. Using [?narens, Theorem 4.4] we obtain:

Fact 6.3. *In the above setting, a set* $Y \in \mathcal{K}(M)$ *is a Hausdorff limit of the family* $\mathcal{F} = \{F_t \colon t \in T\}$ *if and only if there is* $\tau \in T^\sharp$ *such that* $Y = \mathrm{st}_M(F^\sharp_\tau)$.

Using the above, we conclude:

Lemma 6.4. *For a family $\mathcal{F} = \{F_t : t \in I_\infty\}$ of subsets of M indexed by I_∞, a set $Y \in \mathcal{K}(M)$ is a Hausdorff limit at ∞ of $\mathcal{F}$ if and only if there is $\tau \in \mathfrak{R}$ with $\tau > \mathbb{R}$, such that $Y = \mathrm{st}_M(F^\sharp_\tau)$.*

Proof. For $r \in \mathbb{R}^{>0}$ let $I_{>r}$ be the interval $(r, +\infty) \subseteq \mathbb{R}$ and $\mathcal{F}_r$ be the family $\mathcal{F}_r = \{F_t : t \in I_{>r}\}$.

It is easy to see that a set $Y \in \mathcal{K}(M)$ is a Hausdorf limit at ∞ of the family $\mathcal{F}$ if and only if for every $r \in \mathbb{R}^{>0}$ the set Y is a Hausdorf limit of the family $\mathcal{F}_r$.

By Fact **??**, the latter condition is equivalent to the following: for every $r \in \mathbb{R}^{>0}$, there is $\tau_r \in I^\sharp_{>r}$ with $Y = \mathrm{st}_M(F^\sharp_{\tau_r})$.

Thus the conclusion of the lemma can be restated as follows:

For a set $Y \in \mathcal{K}(M)$ the following are equivalent:

(a) For every $r \in \mathbb{R}^{>0}$, there is $\tau_r \in I^\sharp_{>r}$ with $Y = \mathrm{st}_M(F^\sharp_{\tau_r})$.
(b) There is $\tau \in \mathfrak{R}$ with $\tau > \mathbb{R}$ such that $Y = \mathrm{st}_M(F^\sharp_\tau)$.

The direction $(b) \Rightarrow (a)$ is obvious, and the opposite direction follows from the $|\mathbb{R}|^+$-saturation of $\mathfrak{R}_{\mathrm{full}}$.

□

6.3. **Hausdorff limits in G/H.** Let G be a connected Lie group and $H \subseteq G$ a closed subgroup such that the space of the left cosets $N = G/H$ is compact, with respect to the quotient topology. We denote by $\pi : G \to N$ the quotient map. Using Whitney embedding theorem we embed G into some $\mathbb{R}^m$ and N into some $\mathbb{R}^n$ as closed subsets, and view G, N and π as definable in $\mathbb{R}_{\mathrm{full}}$.

Given a family $\mathcal{F} = \{F_t : t \in T\}$ of subsets of G, we let

$$\pi(\mathcal{F}) = \{\pi(F_t) : t \in T\}$$

be the corresponding family of subsets of N.

Proposition 6.5. *Let $\mathcal{F} = \{F_t : t \in I_\infty\}$ be a family of subsets of G. A set $Y \in \mathcal{K}(N)$ is a Hausdorf limit at ∞ of the family $\pi(\mathcal{F})$ if and only if there is $\tau \in \mathfrak{R}$ with $\tau > \mathbb{R}$ such that $Y = \pi\big(\mathrm{st}_G(F^\sharp_\tau \cdot H^\sharp)\big)$.*

Proof. For $\tau \in \mathfrak{R}_{\mathrm{full}}$, by Claim **??**, it is sufficient to show that

$$\mathrm{st}_N(\pi^\sharp(F^\sharp_\tau)) = \pi\big(\mathrm{st}_G(F^\sharp_\tau \cdot H^\sharp)\big).$$

For $\alpha \in G^\sharp$, we will show that $\mathrm{st}_N(\pi^\sharp(\alpha)) = \pi\big(\mathrm{st}_G(\alpha \cdot H^\sharp)\big)$. That is clearly enough.

Since G/H is compact, there is $\beta \in \mathcal{O}^m \cap G^\sharp$ with $\beta \in \alpha \cdot H^\sharp$. Let $b = \mathrm{st}_G(\beta)$. Since H is a closed subgroup, the set $b \cdot H$ is closed and, by Lemma **??**(2), we have

$$b \cdot H = \mathrm{st}_G(b \cdot H^\sharp) = \mathrm{st}_G(\beta \cdot H^\sharp).$$

Since $\pi^\sharp$ is invariant under the action of $H^\sharp$ on the right we also have $\pi^\sharp(\alpha) = \pi^\sharp(\beta)$ and we are left to show

$$\mathrm{st}_N(\pi^\sharp(\beta)) = \pi(b).$$

Since π is continuous, the latter follows from Fact **??**. □

6.4. **Hausdorff limits in nilmanifolds.** We go back to our o-minimal structure $\mathbb{R}_{\mathrm{om}}$ and fix a unipotent group G.

For a lattice $\Gamma \subseteq G$, we use π_Γ to denote the projection $\pi_\Gamma \colon G \to G/\Gamma$. When no confusion arises, we omit the subscript Γ. Also, whenever $\Gamma_0 \subseteq \Gamma$ is a subgroup of finite index, we let $\pi_0 \colon G \to G/\Gamma_0$ denote the natural projection.

Given an $\mathbb{R}_{\mathrm{om}}$-definable family $\mathcal{F} = \{F_t : t \in I_\infty\}$, for a lattice $\Gamma \subseteq G$ we consider the possible Hausdorff limits at ∞ of the family $\pi(\mathcal{F}) \subseteq G/\Gamma$. Notice that if $\mathcal{F}$ is a constant family $F_t = F$ then the only Hausdorff limit at ∞ is the closure of $\pi(F)$ and this case was handled in [?nilpotent].

Example 6.6. (1) Consider first $G = (\mathbb{R}^2, +)$ and $\Gamma = \mathbb{Z}^2$.

Let L_0 be the line $L_0 = \{(x, 0) \in \mathbb{R}^2 \colon x \in \mathbb{R}\}$, and $\mathcal{F}_1$ be the family of L_0-translates: $\mathcal{F}_1 = \{L_0 + (0, t) \colon t \in I_\infty\}$. It is not hard to see that the Hausdorff limits at ∞ of $\pi(\mathcal{F}_1)$ are exactly the sets $\pi(L_0 + g)$ for $g \in G$.

Let $\mathcal{F}_2 = \{L_t \colon t \in I_\infty\}$ be the family of lines in G where L_t is the line $L_t = \{(x, y) \in G \colon y = tx\}$. It is not hard to see that the only Hausdorff limit at ∞ is the whole G/Γ.

(2) Assume now that $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$, and let $\mathcal{F} = \{t + [0, 2] : t \in I_\infty\}$. The family $\pi(\mathcal{F})$ is the constant family $\pi(F_t) = \mathbb{R}/\mathbb{Z}$, and hence this is the only Hausdorff limit at ∞. However, for any lattice $\Gamma_0 \subseteq \mathbb{Z}$ with $|\mathbb{Z} : \Gamma_0| \geq 3$, the Hausdorff limits at ∞ of $\pi_0(\mathcal{F})$ are the sets of the form $g + \pi_0([0, 2])$, for $g \in \mathbb{R}/\Gamma_0$, and none of these equals $\mathbb{R}/\Gamma_0$.

Definition 6.7. Let $\mathcal{F} = \{F_t : t \in I_\infty\}$ be an $\mathbb{R}_{\mathrm{om}}$-definable family of subsets of a unipotent group G, and $\Gamma \subseteq G$ be a lattice.

We say that the family $\pi(\mathcal{F})$ *converges strongly* to G/Γ at ∞ if $\pi_0(\mathcal{F})$ converges to G/Γ_0 at ∞, for any subgroup $\Gamma_0 \subseteq \Gamma$ of finite index, as in Definition **??**.

The next observation immediately follows from Proposition **??**:

Proposition 6.8. *Let $\mathcal{F} = \{F_t : t \in I_\infty\}$ be an $\mathbb{R}_{\mathrm{om}}$-definable family of subsets of a unipotent group G, and $\Gamma \subseteq G$ be a lattice. Then:*

(1) G/Γ is a Hausdorff limit of $\pi(\mathcal{F})$ $\Leftrightarrow$ there exists $\tau \in \mathfrak{R}$, $\tau > \mathbb{R}$, such that $F^\sharp_\tau$ is Γ-dense in G.

(2) $\pi(\mathcal{F})$ *converges to* $G/\Gamma \Leftrightarrow$ *for all* $\tau \in \mathfrak{R}$ *with* $\tau > \mathbb{R}$, $F^\sharp_\tau$ *is* Γ*-dense in* G.

(3) $\pi(\mathcal{F})$ *converges strongly to* $G/\Gamma \Leftrightarrow$ *for all* $\tau \in \mathfrak{R}$, *with* $\tau > \mathbb{R}$, $F^\sharp_\tau$ *is strongly* Γ*-dense in* G.

For the next result, recall the notation at the beginning of Section **??**, regarding the abelianization of G.

Corollary 6.9. *Let* G *be a unipotent group and let* $\mathcal{F} = \{F_t : t \in I_\infty\}$ *be an* $\mathcal{L}_{\mathrm{om}}$*-definable family of subsets of* G*. We denote by* $\mathcal{F}_{\mathrm{ab}}$ *the family* $\pi_{\mathrm{ab}}(\mathcal{F})$ *of subsets of* G_{ab}*. For a lattice* $\Gamma \subseteq G$*, we let* $\pi : G \to G/\Gamma$*, and* $\pi^* : G_{\mathrm{ab}} \to G_{\mathrm{ab}}/\Gamma_{\mathrm{ab}}$ *be the quotient maps.*

Then, the family $\pi(\mathcal{F})$ *converges strongly to* G/Γ *at* ∞ *if and only if* $\pi^*(\mathcal{F})$ *converges strongly to* $G_{\mathrm{ab}}/\Gamma_{\mathrm{ab}}$ *at* ∞.

Proof. By Proposition **??**, $\pi(\mathcal{F})$ converges strongly to G/Γ at ∞ if and only if for all $\tau > \mathbb{R}$ in $\mathfrak{R}$, $F^\sharp_\tau$ is strongly Γ-dense in G. By Corollary **??**, this is equivalent to $\pi_{\mathrm{ab}}(F^\sharp_\tau)$ being strongly Γ_{ab}-dense in G_{ab}, for all $\tau > \mathbb{R}$, which again, by Proposition **??**, is equivalent to $\pi_{\mathrm{ab}}(\mathcal{F})$ strongly converging to $G_{\mathrm{ab}}/\Gamma_{\mathrm{ab}}$ at ∞.

□

Before the next theorem we observe:

Lemma 6.10. *Let* G *be a unipotent group and let* $\mathcal{F} = \{F_t : t \in I_\infty\}$ *be an* $\mathbb{R}_{\mathrm{om}}$*-definable family of subsets of* G*. Then, for all* $\tau, \tau' \in \mathfrak{R}$*, with* $\tau, \tau' > \mathbb{R}$, $\mathscr{L}_{\max}(F^\sharp_\tau) = \mathscr{L}_{\max}(F^\sharp_{\tau'})$.

Proof. We use the fact that τ and τ' have the same $\mathcal{L}_{\mathrm{om}}$-type over $\mathbb{R}$.

Clearly, it is enough to show $\mathscr{L}(F^\sharp_\tau) = \mathscr{L}(F^\sharp_{\tau'})$, and, by symmetry, it is sufficient to show $\mathscr{L}(F^\sharp_\tau) \subseteq \mathscr{L}(F^\sharp_{\tau'})$.

Let $L \in \mathscr{L}(F^\sharp_\tau)$. We choose $\alpha \in \mathrm{dcl}(\tau)$ such that the coset αL is a nearest co-commutative coset to some type on $F^\sharp_\tau$. Let $a(t)$ be an $\mathbb{R}_{\mathrm{om}}$-definable function with $a(\tau) = \alpha$,

By saturation of $\mathfrak{R}_{\mathrm{om}}$, the coset $a(\tau')L$ is a nearest co-commutative coset to some type on $F^\sharp_{\tau'}$. □

The above lemma justifies the following definition

Definition 6.11. For an $\mathbb{R}_{\mathrm{om}}$-definable family $\mathcal{F} = \{F_t : t \in I_\infty\}$ of a unipotent group G, we denote by $\mathscr{L}_{\max}(\mathcal{F})$ the finite set of co-commutative subgroups $\mathscr{L}_{\max}(F^\sharp_\tau)$, for some (any) $\tau > \mathbb{R}$.

The next theorem is one of our main results.

Theorem 6.12. *Let* G *be a unipotent group,* $\mathcal{F} = \{F_t : t \in I_\infty\}$ *an* $\mathbb{R}_{\mathrm{om}}$*-definable family of subsets of* G.

For every lattice $\Gamma \subseteq G$ *we have:*

(1) $L^\Gamma = G$ *for some* $L \in \mathscr{L}_{\max}(\mathcal{F})$ *if and only if* $\pi(\mathcal{F})$ *converges strongly to* G/Γ *at* ∞*.*

(2) $L^\Gamma \neq G$ *for all* $L \in \mathscr{L}_{\max}(\mathcal{F})$ *if and only if there exists a subgroup* $\Gamma_0 \subseteq \Gamma$ *of finite index such that all Hausdorff limits at* ∞ *of* $\pi_0(\mathcal{F})$ *are proper subsets of* G/Γ_0*.*

Note that the condition given in (2) is formally stronger than the negation of the condition in (1), thus both need to be proved separately.

Proof. (1) By Proposition **??**, $\pi(\mathcal{F})$ converges strongly to G/Γ if and only if for all $\tau > \mathbb{R}$, $F^\sharp_\tau$ is strongly Γ-dense in G, which by Theorem **??**, is equivalent to $L^\Gamma = G$ for some $L \in \mathscr{L}_{\max}(\mathcal{F})$.

(2) Assume that for all $L \in \mathscr{L}_{\max}(\mathcal{F})$ we have $L^\Gamma \neq G$, and for contradiction assume that for every subgroup $\Gamma_0 \subseteq \Gamma$ of finite index, G/Γ_0 is one of the Hausdorff limits at ∞, of the family $\pi_0(\mathcal{F})$. Equivalently, it follows from Proposition **??** and Lemma **??**, that for every subgroup $\Gamma_0 \subseteq \Gamma$ of finite index, there is a $\tau' \in \mathfrak{R}$ with $\tau' > \mathbb{R}$, such that $G = \mathrm{st}(F^\sharp_{\tau'} \cdot \Gamma^\sharp_0)$.

Claim 6.13. *There exists* $\tau^* \in \mathfrak{R}$ *with* $\tau^* > \mathbb{R}$ *such that for every* $\Gamma_0 \subseteq \Gamma$ *of finite index,* $\mathrm{st}(F^\sharp_{\tau*} \cdot \Gamma^\sharp_0) = G$*.*

Proof of the claim. For every $g \in G = G(\mathbb{R})$, $r \in \mathbb{R}^{>0}$ and $\Gamma_0 \subseteq \Gamma$ of finite index, we consider the following formula $\phi_{g,r,\Gamma_0}(t)$:

$$t > r \,\&\, g \in B_{1/r}(e) \cdot F_t \cdot \Gamma_0.$$

We let $p(t)$ be the type consisting of all ϕ_{g,r,Γ_0}, as g, r, Γ_0 vary over all $g \in G$, $r \in \mathbb{R}^{>0}$ and $\Gamma_0 \subseteq \Gamma$ of finite index, respectively. We claim that p is finitely consistent. Indeed, given finitely many subgroups of Γ of finite index, let Γ_1 be their intersection. Clearly Γ_1 has finite index in Γ. By our assumption, there is $\tau > \mathbb{R}$ such that $G(\mathbb{R}) \subseteq \mu \cdot F^\sharp_\tau \cdot \Gamma^\sharp_1$, which implies that for any $g_1, \ldots, g_k \in G$ and $r_1, \ldots, r_k \in \mathbb{R}$, $\phi_{g_i,r_i,\Gamma_1}(\tau)$ holds. It follows that $p(t)$ is finitely consistent, so by the saturation of $\mathfrak{R}_{\mathrm{full}}$, there exists $\tau^* \in \mathfrak{R}$ realizing $p(t)$.

Now, given $\Gamma_0 \subseteq \Gamma$ of finite index, and $g \in G$, we have $g \in B_\epsilon(e)^\sharp \cdot F^\sharp_{\tau*} \cdot \Gamma^\sharp_0$, for all $\epsilon \in \mathbb{R}^{>0}$. Using saturation again, it follows that $g \in \mu \cdot F^\sharp_{\tau*} \cdot \Gamma^\sharp_0$, and hence $G = \mathrm{st}(F^\sharp_{\tau*} \cdot \Gamma^\sharp_0)$, proving the claim. □

For τ^* as in the above claim, $F^\sharp_{\tau*}$ is strongly Γ-dense in G and therefore, by Theorem **??**, there is $L \in \mathscr{L}_{\max}(\mathcal{F})$ with $L^\Gamma = G$, contradiction.

The opposite implication of (2) follows from (1).

□

6.5. **The abelian case.** In the unipotent case, Theorem **??** tells us when the family $\pi(\mathcal{F})$ converges (strongly) to G/Γ. In the abelian case we can say more about the possible Hausdorff limits of $\pi(\mathcal{F})$, due to the following theorem.

Theorem 6.14. *Let $G = (\mathbb{R}^m, +)$ and let $\mathcal{F} = \{F_t : t \in I_\infty\}$ be an $\mathbb{R}_{\text{om}}$-definable family of subsets of G.*

For every $r \in \mathbb{R}^{>0}$, there are subspaces $L_1, \ldots, L_k \subseteq G$ (possibly with repetitions and with k depending on r), with $\mathscr{L}_{\max}(\mathcal{F}) \subseteq \{L_1, \ldots, L_k\}$, and there are $\mathbb{R}_{\text{om}}$-definable functions $a_1(t), \ldots, a_k(t) \colon I_\infty \to G$, such that

(1) For some $\tau \in \mathfrak{R}, \tau > \mathbb{R}$, each $a_i(\tau) + L_i$ is a nearest coset to some type $p \in S_{F^\sharp_\tau}(\tau)$.

(2) For $t \gg 0$,

$$F_t \subseteq \overline{B}_r + \bigcup_{i=1}^{k} a_i(t) + L_i.$$

(3) Let $\Gamma \subseteq G$ be a lattice and $\pi : G \to G/\Gamma$ the projection. For every $s \in \mathbb{R}^{>0}$, there exists $t_s > 0$ such that for all $t > t_s$, and all $i = 1, \ldots, k$.

$$\pi(a_i(t) + L_i) \subseteq \pi(\overline{B}_s) + \pi(F_t).$$

Proof. Let $\tau \in I^\sharp_\infty$ with $\tau > \mathbb{R}$, and let $A = \text{dcl}(\tau)$.

Fix $r \in \mathbb{R}^{>0}$. Let $L_1, \ldots, L_k \in \mathscr{L}_A(F^\sharp_\tau)$ be subgroups and $\alpha_1, \ldots, \alpha_k \in \text{dcl}(A)$ be as in Theorem **??**. Thus we have

$$F^\sharp_\tau \subseteq \overline{B}^\sharp_r + \bigcup_{i=1}^{k} \alpha_i + L^\sharp_i, \quad \text{with } \mathscr{L}_{\max}(F^\sharp_\tau) \subseteq \{L_1, \ldots, L_k\},$$

and, by Theorem **??**, for every $i = 1, \ldots, k$, and $s > 0$ we also have

$$\alpha_i + L^\sharp_i \subseteq \overline{B}^\sharp_s + F^\sharp_\tau + \Gamma^\sharp.$$

Since each $\alpha_i \in \text{dcl}(\tau)$, for $i = 1, \ldots, k$, we choose $\mathbb{R}_{\text{om}}$-definable functions $a_i(t) \colon \mathbb{R} \to G$, such that $\alpha_i = a_i(\tau)$. We have

$$F^\sharp_\tau \subseteq \bigcup_{i=1}^{k} \overline{B}^\sharp_r + a_i(\tau) + L^\sharp_i.$$

Since the $\mathcal{L}_{\text{om}}$-type of τ over $\mathbb{R}$ is implied by $\{x > r \colon r \in \mathbb{R}\}$ and the above inclusion can be expressed by an $\mathcal{L}_{\text{om}}$-formula over τ, we obtain that for $t \gg 0$,

$$F_t \subseteq \bigcup_{i=1}^{k} \overline{B}_r + a_i(t) + L_i.$$

This proves (1) and (2).

Assume (3) fails. Then, there is a lattice $\Gamma \subseteq G$ and $s \in \mathbb{R}^{>0}$, such that for some $i_0 = 1, \dots, k$, the set

$$\{t \in I_\infty \colon \pi(a_i(t) + L_i) \nsubseteq \pi(\overline{B}_s) + \pi(F_t)\}$$

is unbounded in $\mathbb{R}$. Without loss of generality, we assume $i_0 = 1$.

Using saturation of $\mathfrak{R}_{\text{full}}$, we can find $\tau' \in \mathfrak{R}$ with $\tau' > \mathbb{R}$, such that $\pi(a_1(\tau') + L_1^\sharp) \nsubseteq \pi(\overline{B}_s^\sharp) + \pi(F_{\tau'}^\sharp)$, so in particular,

$$a_1(\tau') + L_1^\sharp \nsubseteq \overline{B}_s^\sharp + F_{\tau'}^\sharp + \Gamma^\sharp.$$

Since τ and τ' realize the same $\mathcal{L}_{\text{om}}$-type over $\mathbb{R}$, and $a_1(\tau) + L_1^\sharp$ is the nearest coset to some type on $F_\tau^\sharp$, the coset $a_1(\tau') + L_1^\sharp$ is the nearest coset to a type, call it p, on $F_{\tau'}^\sharp$. However, using Theorem **??**, we have

$$a_1(\tau') + L_1^\sharp \subseteq \mu + a_1(\tau') + L_1^\sharp = \mu + p(\mathfrak{R}) + \Gamma^\sharp \subseteq \overline{B}_s^\sharp + F_{\tau'}^\sharp + \Gamma^\sharp,$$

contradiction.

□

We can now show that, in the abelian case, every Hausdorff limit of $\pi(\mathcal{F})$ at ∞ is trapped between a finite union of cosets and a "thickening" of it.

Corollary 6.15. *Let $G = (\mathbb{R}^m, +)$ and let $\mathcal{F} = \{F_t : t \in I_\infty\}$ be an $\mathbb{R}_{\text{om}}$-definable family of subsets of G.*

For every $r \in \mathbb{R}^{>0}$, there are subspaces $L_1, \dots, L_k \subseteq G$ (possibly with repetitions and with k depending on r), with $\mathscr{L}_{\max}(\mathcal{F}) \subseteq \{L_1, \dots, L_k\}$, such that for any lattice $\Gamma \subseteq G$, and any Hausdorff limit X of the family $\pi(\mathcal{F})$ at ∞, there are $g_1, \dots, g_k \in G$ with

$$\pi\Big(\bigcup_{i=1}^k (g_i + L_i)\Big) \subseteq X \subseteq \pi(\overline{B}_r) + \pi\Big(\bigcup_{i=1}^k (g_i + L_i)\Big).$$

Proof. By Proposition **??**, $X = \pi(\operatorname{st}(F_\tau^\sharp + \Gamma^\sharp))$, for some $\tau > \mathbb{R}$. We now apply Theorem **??** and obtain linear subspaces $L_1, \dots, L_k \subseteq \mathbb{R}^m$ and definable functions $a_1(t), \dots, a_k(t) \colon I_\infty \to G$ as in the theorem. Given a lattice $\Gamma \subseteq G$, clause (2) of that theorem implies that

$$F_\tau^\sharp + \Gamma^\sharp \subseteq \overline{B}_r^\sharp + \bigcup_{i=1}^k a_i(\tau) + L_i^\sharp + \Gamma^\sharp,$$

while (3) (applied for all $r \in \mathbb{R}^{>0}$) implies

$$\bigcup_{i=1}^k a_i(\tau) + L_i^\sharp + \Gamma^\sharp \subseteq \mu + F_\tau^\sharp + \Gamma^\sharp.$$

Using Lemma **??**, for $g_i \in \mathrm{st}(a_i(\tau) + \Gamma^\sharp)$, we obtain

$$\bigcup_{i=1}^{k} g_i + L_i^\Gamma + \Gamma = \bigcup_{i=1}^{k} \mathrm{st}(a_i(\tau) + L_i^\sharp + \Gamma^\sharp) \subseteq \mathrm{st}(F_\tau^\sharp + \Gamma^\sharp) \subseteq \overline{B}_r + \bigcup_{i=1}^{k} g_i + L_i^\Gamma + \Gamma.$$

Applying π the result follows. □

7. Polynomial dilations

Let $G \subseteq \mathrm{GL}(n,\mathbb{R})$ be a unipotent group. In several articles ([?KSS], [?fish]) a particular type of families of subsets of G, given by dilations of an initial curve, was considered in the unipotent setting. We first make some definitions.

7.1. **The setting.**

Definition 7.1. *A polynomial $m \times k$ matrix M_t (over $\mathbb{R}$)* is a matrix $M_t \in M_{m\times k}(\mathbb{R}[t])$, namely, a matrix all of whose entries are polynomials in $\mathbb{R}[t]$. It can be written as $\sum_{j=0}^{d} t^j A_j$, with each A_j is an $m \times k$ matrix over $\mathbb{R}$.

Let G be a unipotent group of dimension m. We identify the underlying vector space of its Lie algebra $\mathfrak{g}$ with $\mathbb{R}^m$, and we let $\exp : \mathfrak{g} \to G$ be the exponential map (which is a polynomial bijection with a polynomial inverse, see [?nilpotent-book]). For a polynomial $m \times k$ matrix M_t we consider the family of "dilations" $\rho_t \colon \mathbb{R}^k \to G$ given by $x \mapsto \exp(M_t x)$ and for a set $X \subseteq \mathbb{R}^k$, the family $\{\rho_t(X) : t \in I_\infty\}$ of subsets of G.

In [?KSS], the authors start with a measure ν on $\mathfrak{g}$ given as the pushforward of the Lebesgue measure on $(0,1)$ via a real analytic map $\phi : (0,1) \to \mathfrak{g}$, then "dilate" the measure ν using multiplication by a polynomial $m \times m$ matrix M_t, and consider the limit of the measures as $t \to \infty$. In [?KSS, Theorem 1.1] the authors prove that under some assumptions on $\Phi = \mathrm{Image}(\phi)$, given in terms of kernels of particular characters of G, the associated family of measures μ_t, on G/Γ is "equidistributed", roughly saying that for any Borel $D \subseteq G/\Gamma$, the family $\mu_t(D)$ converges to the canonical Haar measure of D. Translated to the topological language this would imply (but a-priori might not be equivalent) that the family $\{\pi_\Gamma(\rho_t(\Phi)) \colon t \in I_\infty\}$ converges to G/Γ at ∞.

In the current section we extend the topological corollary by replacing the one-dimensional set Φ with an $\mathbb{R}_{\mathrm{om}}$-definable set $X \subseteq \mathbb{R}^k$ of arbitrary dimension. Using Theorem **??**, one can formulate conditions, similar to those in [?KSS], as to when the family $\{\pi(\exp(M_t{\cdot}X)) \colon t \in I_\infty\}$ converges strongly to G/Γ at ∞. Instead, we consider the special case, when the polynomial matrix M_t does not have a constant term, and describe first in full, in the abelian case, all possible Hausdroff limits at ∞ of the family.

7.2. **Polynomial dilations in vector spaces.** We fix some notations.

Notation 7.2. (1) We call a map $a(t): \mathbb{R} \to \mathbb{R}^m$ *a polynomial map* if $a(t) = \sum_{i=0}^{d} t^i a_i$ for some $a_0, \dots, a_d \in \mathbb{R}^m$. We say that the polynomial map is *proper* if a_0=0.

(2) By *a polynomial family of dilations* we mean a family of maps $\{\rho_t \colon \mathbb{R}^k \to \mathbb{R}^m \colon t \in I\}$ with $I \subseteq \mathbb{R}$, such that for some polynomial $m \times k$-matrix M_t, for all $t \in I$ and $x \in \mathbb{R}^k$, we have $\rho_t(x) = M_t x$. We say that the family is *proper* if the constant term of M_t is the zero $m \times k$-matrix.

(3) By a *a polynomial family of cosets* we mean a family $\{a(t)+L \colon t \in I\}$, where $I \subseteq \mathbb{R}$, $L \subseteq \mathbb{R}^m$ is a subspace, and $a(t)\colon \mathbb{R} \to \mathbb{R}^m$ a polynomial map, We say that the family is *proper* if $a(t)$ is a proper polynomial map.

(4) By *a polynomial family of multi-cosets* we mean a family of the form $\{\bigcup_{j=1}^{n}(a_j(t) + L_j) \colon t \in I\}$, where $I \subseteq \mathbb{R}$, each $a_j(t)\colon \mathbb{R} \to \mathbb{R}^m$ is a polynomial map and each $L_j \subseteq \mathbb{R}^m$ is a subspace. We say that the polynomial family of multi-cosets is *proper* if each $a_j(t)$ is proper.

We refer the readers to [?KSS, Remark 2.5] for an example explaining the reason to work with proper polynomial dilations instead of general ones.

Remark 7.3. Let $\mathcal{M} = \{\bigcup_{j=1}^{n}(a_j(t) + L_j) \colon t \in I\}$ be a polynomial family of multi-cosets. It is not hard to see, using Lemma **??**(2), that $\mathscr{L}_{\max}(\mathcal{M})$ is exactly the set of maximal (by inclusion) subspaces in $\{L_1, \dots, L_n\}$.

Our main result is:

Theorem 7.4. *Let $\{\rho_t \colon \mathbb{R}^k \to \mathbb{R}^m \colon t \in I_\infty\}$ be a proper family of polynomial dilations, $X \subseteq \mathbb{R}^k$ an $\mathbb{R}_{\text{om}}$-definable set, and $\mathcal{F} = \{\rho_t(X) \colon t \in I_\infty\}$.*

Then, there is a proper polynomial family of multi-cosets

$$\mathcal{M} = \left\{\bigcup_{j=1}^{n}(p_j(t) + L_j) \colon t \in I_\infty\right\}$$

such that

(1) $\rho_t(X) \subseteq \bigcup_{j=1}^{n}(p_j(t) + L_j)$ *for all* $t \in \mathbb{R}$*;*
(2) $\mathscr{L}_{\max}(\mathcal{F}) = \mathscr{L}_{\max}(\mathcal{M})$*;*

(3) for any lattice $\Gamma \subseteq \mathbb{R}^m$, the families $\pi(\mathcal{F})$ and $\pi(\mathcal{M})$ have the same Hausdorff limits at ∞.

Proof. Let $A_1, \ldots, A_d$ be $m \times k$-matrices such that $\rho_t(x) = \sum_{i=1}^d t^i A_i x$.

We fix $\tau \in \mathfrak{R}$ with $\tau > \mathbb{R}$, and let $\mathcal{Y} = \rho_\tau(X^\sharp)$.

Applying Theorem **??** with $r = 1$, we obtain subspaces $L_1, \ldots L_n \subseteq \mathbb{R}^m$ and $\mathbb{R}_{\mathrm{om}}$-definable maps $a_1(t), \ldots, a_n(t) \colon I_\infty \to \mathbb{R}^m$ such that

$$\text{each } a_j(\tau) + L_j \text{ is a nearest coset to some type in } S_{\mathcal{Y}}(\tau), \text{ and}$$

$$\rho_t(X) \subseteq \overline{B}_1 + \bigcup_{j=1}^n \big(a_j(t) + L_j\big) \text{ for } t \gg 0. \tag{7.1}$$

We pick $a_i(t)$ and L_i, $j = 1, \ldots, n$, as above, with the minimal possible n.

We claim that $\mathscr{L}_{\max}(\mathcal{F})$ is exactly the set of maximal elements of $\{L_1, \ldots, L_n\}$. Indeed, assume $L \in \mathscr{L}_{\max}(\mathcal{F})$ and $p \in S_{\mathcal{Y}}(\tau)$ is such that $L_p = L$. Then, for some $j = 1, \ldots, n$, $p \vdash a_j(\tau) + L_j + \overline{B}_1$. It follows from Lemma **??** that $L \subseteq L_j$, so by maximality $L = L_j$. Since every L_i is contained in some $L \in \mathscr{L}_{\max}(\mathcal{F})$, the claim follows.

For each $j = 1, \ldots, n$, let $L_j^\perp \subseteq \mathbb{R}^m$ be the orthogonal complement to L_j with respect to the standard inner product on $\mathbb{R}^m$, and $\pi_j^\perp \colon \mathbb{R}^m \to L_j^\perp$ be the corresponding projection, whose kernel is L_j.

Replacing each $a_j(t)$ with $\pi_j^\perp(a_j(t))$, if needed, we assume $a_j(t) \in L_j^\perp$ for all $t \in I_\infty$.

For $j = 1, \ldots, n$, let

$$X_j = \{x \in X \colon \text{ for } t \gg 0,\ \rho_t(x) \in \overline{B}_1 + a_j(t) + L_j\}.$$

Notice that each X_j is $\mathbb{R}_{\mathrm{om}}$-definable,

$$X = \bigcup_{j=1}^n X_j,$$

and, by the minimality of n, each X_j is non-empty.

Claim A. *For each $j = 1, \ldots, n$, and every $i = 1, \ldots, d$, the set $A_i X_j$ is contained in a single coset of L_j.*

Proof of Claim A. We fix $j \in \{1, \ldots, n\}$.

Clearly it is sufficient to show that for any $i = 1, \ldots, d$, every $\mathbb{R}$-linear function $F \colon \mathbb{R}^m \to \mathbb{R}$ that vanishes on L_j is constant on $A_i X_j$.

Assume not, and for some $\mathbb{R}$-linear function $F \colon \mathbb{R}^m \to \mathbb{R}$ vanishing on L_j there are $r \in \{1, \ldots, d\}$ and $b_1, b_2 \in X_j$ with $F(A_r b_1) \neq F(A_r b_2)$.

Consider the map $q(t) = F(\rho_t(b_1) - \rho_t(b_2))$.

Since $\rho_t(b_1) - \rho_t(b_2) = \sum_{i=1}^d t^i A_i(b_1 - b_2)$, by linearity of F, we have

$$q(t) = \sum_{i=1}^{d} t^i F(A_i(b_1 - b_2)),$$

so $q(t)$ is a polynomial map from $\mathbb{R}^k$ to $\mathbb{R}$.

By our assumptions, the coefficient of t^r in $q(t)$ is not zero, hence $q(t)$ is a non-zero polynomial and $\lim_{t\to\infty} |q(t)| = \infty$.

On the other hand, by the definition of X_j, we have $\rho_t(b_1) - \rho_t(b_2) \in L_j + \overline{B}_1 - \overline{B}_1$ for $t \gg 0$. Since F vanishes on L_j, we obtain $q(t) \in F(\overline{B}_1 - \overline{B}_1)$, which is compact set. A contradiction with $\lim_{t\to\infty} |q(t)| = \infty$.

This finishes the proof of the claim. $\square$

Let $j = 1, \dots, n$. By Claim A, for $i = 1, \dots, d$, we may choose $a_{ij} \in L_j^\perp$ with $A_i X_j \subseteq a_{ij} + L_j$. For $x \in X_j$ and $t \in I_\infty$ we have

$$\rho_t(x) = \sum_{i=1}^{d} t^i A_i x \in \sum_{i=1}^{d} t^i (a_{ij} + L_j) = \sum_{i=1}^{d} t^i a_{ij} + \sum_{i=1}^{d} t^i L_j.$$

Since L_j is closed under multiplication by scalars, setting $p_j(t) = \sum_{i=1}^d t^i a_{ij}$ we obtain

$$\rho_t(X_j) \subseteq p_j(t) + L_j \text{ and } \rho_t(X) \subseteq \bigcup_{j=1}^{n} (p_j(t) + L_j) \text{ for all } t \in \mathbb{R}. \tag{7.2}$$

Notice that each $p_j(t)$ is a proper polynomial map, and takes values in $L_j^\perp$.

Let $\mathcal{M}$ be the proper polynomial family of multi-cosets

$$\mathcal{M} = \Big\{ \bigcup_{j=1}^{n} (p_j(t) + L_j) : t \in I_\infty \Big\}.$$

By (**??**), $\mathcal{M}$ satisfies clause (1) of the theorem, and by the explanation right after (**??**), $\mathscr{L}_{\max}(\mathcal{F}) = \mathscr{L}_{\max}(\mathcal{M})$, implying clause (2). Let us see that clause (3) also holds.

Claim B. *For every $j = 1, \dots, n$, there exists $c_j \in L_j^\perp$, such that $c_j = \lim_{t\to\infty}(a_j(t) - p_j(t))$.*

Proof of Claim B. We fix $j = 1, \dots, n$, and choose $b \in X_j$. By the definition of X_j, we have

$$\rho_t(b) \in \overline{B}_1 + a_j(t) + L_j \text{ for } t \gg 0,$$

and by (**??**),

$$\rho_t(b) \in p_j(t) + L_j.$$

Since both $a_j(t)$ and $p_j(t)$ take values in $L_j^{\perp}$, we have

$$p_j(t) \in a_j(t) + \pi_j^{\perp}(\overline{B}_1).$$

Thus $a_j(t) - p_j(t) \in \pi_j^{\perp}(\overline{B}_1)$. The set $\pi_j^{\perp}(\overline{B}_1)$ is compact, hence, by o-minimality, $\lim_{t\to+\infty}(a_j(t) - p_j(t))$ exists and it belongs to $L_j^{\perp}$, call it c_j. This finishes the proof of Claim B. □

Recall that each $a_j(\tau) + L_j$ is a nearest coset to some $q_j \in S_{\mathcal{Y}}(\tau)$. It follows that $p_j(\tau) + c_j + L_j$ is also a nearest coset to q_j.

Claim C. *For every $j = 1\dots,n$, we have $c_j = 0$, namely the coset $p_j(\tau) + L_j$ is a nearest coset to q_j.*

Proof of Claim C. We proceed by reverse induction on $\dim(L_j)$, so we consider $j_0 \in \{1,\dots,n\}$ and assume that for all j with $\dim(L_j) > \dim(L_{j_0})$, we already know that $c_j = 0$, so $p_j(\tau) + L_j$ is a nearest coset to q_j.

Since $X = \bigcup_{j=1}^n X_j$, there exists $j_1 \in \{1,\dots,n\}$ such that q_{j_0} lies on $\rho_\tau(X_{j_1}^\sharp)$. From (**??**) we conclude

$$q_{j_0} \vdash p_{j_1}(\tau) + L_{j_1}^\sharp, \tag{7.3}$$

hence, by the definition of a nearest coset $L_{j_0} \subseteq L_{j_1}$.

If $L_{j_0} \neq L_{j_1}$, then $\dim(L_{j_1}) > \dim(L_{j_0})$, so by our induction assumption, the coset $p_{j_1}(\tau) + L_{j_1}^\sharp$ is a nearest coset to q_{j_1}. Thus $\mu + p_{j_1}(\tau) + L_{j_1}^\sharp = \mu + a_{j_1}(\tau) + L_{j_1}$, hence

$$a_{j_0}(t) + L_{j_0} \subseteq \overline{B}_1 + a_{j_1}(t) + L_{j_1},\ \text{for } t \gg 0.$$

It follows that

$$\rho_t(X) \subseteq \bigcup_{j=1}^{n}(a_j(t) + L_j + \overline{B}_1) \subseteq \bigcup_{\substack{1\le j\le n \\ j\neq j_0}}(a_j(t) + L_j + \overline{B}_1),$$

contradicting minimality of n.

Thus, we must have $L_{j_0} = L_{j_1}$. From the definition of a nearest coset and (**??**), we get

$$\mu + p_{j_0}(\tau)\ + c_{j_0} + L_{j_0}^\sharp = \mu + p_{j_1}(\tau) + L_{j_1}^\sharp.$$

Since $c_{j_0} \in L_{j_1}^{\perp}$, and both $p_{j_1}(t)$, $p_{j_0}(t)$ take values in $L_{j_1}^{\perp}$, it follows that $\lim_{t\to\infty}(p_{j_1}(t) - p_{j_0}(t)) = c_{j_0}$. Recall that both polynomial $p_{j_1}(t)$ and $p_{j_0}(t)$ have zero constant terms, so $c_{j_0} = 0$.

This finishes the proof of Claim C. □

In order to prove that $\mathcal{M}$ satisfies clause (3), we need to show that for every lattice $\Gamma \subseteq \mathbb{R}^n$, and $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$, the families $\pi(\mathcal{F})$ and $\pi(\mathcal{M})$ have the same Hausdorff limits at ∞. Equivalently, by Proposition **??**, it is sufficient to show, for any $\tau > \mathbb{R}$,

$$\operatorname{st}(\rho_\tau(X^\sharp) + \Gamma^\sharp) = \operatorname{st}\Big(\bigcup_{j=1}^{n} p_j(\tau) + L_j^\sharp + \Gamma^\sharp\Big).$$

We fix $\tau > \mathbb{R}$.

By clause (1), we clearly have the left-to-right inclusion. For the opposite inclusion, for each $j = 1, \dots, n$, by Claim C, there exists a type $q_j \in S_{\mathcal{Y}}(\tau)$, whose nearest coset is $p_j(\tau) + L_j$. By Theorem **??**, $\operatorname{st}(q_j(\mathfrak{R}) + \Gamma^\sharp) = \operatorname{st}(p_j(\tau) + L_j + \Gamma^\sharp)$, thus $\operatorname{st}(\mathcal{Y} + \Gamma^\sharp) \supseteq \operatorname{st}(\bigcup_{j=1}^{n} p_j(\tau) + L_j^\sharp + \Gamma^\sharp)$.

This ends the proof of Theorem **??**.

□

7.2.1. *Hausdorff limits of families of multi-cosets.* In this section we describe Hausdorff limits of families of multi-cosets in real tori. Together with Theorem **??** it provides a complete description of the Hausdorff limits at ∞ of proper families of polynomial dilations.

Let $a_1(t), \dots, a_n(t) \colon I_\infty \to \mathbb{R}^m$ be $\mathbb{R}_{\mathrm{om}}$-definable functions, and let $L_1, \dots, L_n \subseteq \mathbb{R}^m$ be linear subspaces.

For $t \in I_\infty$, let $\mathcal{M}_t$ be the multi-coset $\mathcal{M}_t = \bigcup_{i=1}^{n} \big(a_i(t) + L_i\big)$, and let $\mathcal{M} = \{\mathcal{M}_t : t \in I_\infty\}$ be the corresponding family of multi-cosets. Notice, we do not assume that $a_i(t)$ are polynomials.

Let $\Gamma \subseteq \mathbb{R}^m$ be a lattice. We denote by $\Gamma^n \subseteq (\mathbb{R}^m)^n$ the lattice obtained by the n-fold cartesian power of Γ, and as before, for a subspace $V \subseteq (\mathbb{R}^m)^n$, we denote by V^{Γ^n} the smallest linear Γ^n-rational subspace of $(\mathbb{R}^m)^n$ containing V. For the quotient map $\pi \colon \mathbb{R}^m \to \mathbb{R}^m/\Gamma$, we denote by $\pi^{(n)}$ the quotient map $\pi^{(n)} \colon (\mathbb{R}^m)^n \to (\mathbb{R}^m)^n/\Gamma^n$.

Theorem 7.5. *Let $a_1(t), \dots, a_n(t)$, $L_1, \dots, L_n$ and $\mathcal{M}$ be as above. Then, there exists a coset $\bar{c} + V$ of a linear subspace $V \subseteq (\mathbb{R}^m)^n$, such that for every lattice $\Gamma \subseteq \mathbb{R}^m$, the family of Hausdorff limits of $\pi_\Gamma(\mathcal{M})$ at ∞ is exactly the family*

$$\left\{ \pi_\Gamma\Big(\bigcup_{i=1}^{n}(d_i + L_i^\Gamma)\Big) : (d_1, \dots, d_n) \in \bar{c} + V^{\Gamma^n} \right\}.$$

Proof. Consider the $\mathbb{R}_{\mathrm{om}}$-definable curve $\sigma \colon I_\infty \to (\mathbb{R}^m)^n$, given by $\sigma(t) = (a_1(t), \dots, a_n(t))$. Let $p(x) \in S(\mathbb{R})$ be the unique o-minimal type on $\sigma(t)$ at ∞, whose realization is the set

$$p(\mathfrak{R}) = \{\sigma(\tau)\colon \tau \in \mathfrak{R}, \tau > \mathbb{R}\} \subseteq (\mathfrak{R}^m)^n.$$

Let $\bar{c} + V$ be a nearest coset to $p(x)$. Since p is a type over $\mathbb{R}$, we have $\bar{c} = (c_1, \dots, c_n)$, with each c_i in $\mathbb{R}^m$, and $V \subseteq (\mathbb{R}^m)^n$ is a subspace. We claim that this coset satisfies the conclusion of the theorem.

Let $\Gamma \subseteq \mathbb{R}^m$ be a lattice, and $X \subseteq \mathbb{R}^m/\Gamma$. We let $\pi = \pi_\Gamma$. We denote by $\mathcal{H}_\Gamma$ the set of Hausdorff limits at ∞ of $\pi(\mathcal{M})$

Since π is Γ-invariant, it is sufficient to show that $X \in \mathcal{H}_\Gamma$ if and only if

$$X = \pi\Big(\bigcup_{i=1}^n (d_i + L_i^\Gamma)\Big) \text{ for some } (d_1, \dots, d_n) \in \bar{c} + V^{\Gamma^n} + \Gamma^n.$$

Using Proposition **??**, we have that $X \in \mathcal{H}_\Gamma$ if and only if

$$X = \pi\Big(\mathrm{st} \bigcup_{i=1}^n (a_i(\tau) + L_i^\sharp + \Gamma^\sharp)\Big) \text{ for some } \tau \in \mathfrak{R} \text{ with } \tau > \mathbb{R}.$$

Thus $X \in \mathcal{H}_\Gamma$ if and only if

$$X = \pi\Big(\bigcup_{i=1}^n \mathrm{st}(\alpha_i + L_i^\sharp + \Gamma^\sharp)\Big) \text{ for some } (\alpha_1, \dots, \alpha_n) \in p(\mathfrak{R}).$$

Let $\alpha_1, \dots, \alpha_n \in \mathfrak{R}^n$. By Lemma **??**, for every $i = 1, \dots, n$, the set $\mathrm{st}(\alpha_i + \Gamma^\sharp)$ is non-empty, and for any $d_i \in \mathrm{st}(\alpha_i + \Gamma^\sharp)$ we have $\mathrm{st}(\alpha_i + L_i^\sharp + \Gamma^\sharp) = d_i + L_i^\Gamma + \Gamma$.

Also, clearly, for any $d_1, \dots, d_n \in \mathbb{R}^m$ and $\alpha_1, \dots, \alpha_n \in \mathfrak{R}^m$ we have

$$\bigwedge_{i=1}^n d_i \in \mathrm{st}(\alpha_i + \Gamma^\sharp) \iff (d_1, \dots, d_n) \in \mathrm{st}\big((\alpha_1, \dots, \alpha_n) + (\Gamma^n)^\sharp\big).$$

It follows that $X \in \mathcal{H}_\Gamma$ if and only if

$$X = \pi\Big(\bigcup_{i=1}^n (d_i + L_i^\Gamma)\Big) \text{ for some } (d_1, \dots, d_n) \in \mathrm{st}\big(p(\mathfrak{R}) + (\Gamma^n)^\sharp\big).$$

By Theorem **??** (over the parameter set $\mathbb{R}$),

$$\mathrm{st}\big(p(\mathfrak{R}) + (\Gamma^n)^\sharp\big) = \mathrm{st}\big(\bar{c} + V^\sharp + (\Gamma^n)^\sharp\big),$$

and by Lemma **??**(2), the set on the right equals $\mathrm{cl}(\bar{c} + V + \Gamma)$. By Fact **??**, we have

$$\mathrm{st}\big(p(\mathfrak{R}) + (\Gamma^n)^\sharp\big) = \bar{c} + V^{\Gamma^n} + \Gamma^n.$$

This finishes the proof of the theorem.

□

Theorem **??** and Theorem **??** immediately yield the definability of the family of Hausodrff limits at ∞ of proper families of polynomial dilations, in the following sense.

Corollary 7.6. *Let $\{\rho_t \colon \mathbb{R}^k \to \mathbb{R}^m \colon t \in I_\infty\}$ be a proper family of polynomial dilations, and $X \subseteq \mathbb{R}^k$ an $\mathbb{R}_{\mathrm{om}}$-definable set.*

Then there are linear subspaces $L_1, \dots, L_n \subseteq \mathbb{R}^n$, and there is a coset of a linear space $\bar{c} + V \subseteq (\mathbb{R}^m)^n$ such that for any lattice $\Gamma \subseteq \mathbb{R}^m$, the set of Hausdorff limits at ∞ of the family $\{\pi \circ \rho_t(X) \colon t \in I_\infty\}$ is exactly the family

$$\left\{ \pi\Big(\bigcup_{i=1}^{n}(d_i + L_i^\Gamma)\Big) : (d_1, \dots, d_n) \in \bar{c} + V^{\Gamma^n} \right\}.$$

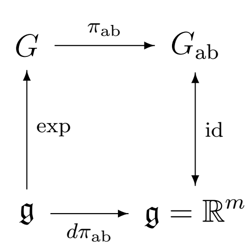
In particular, it is the projection under π of a definable family of subsets of $\mathbb{R}^m$.

7.3. **Polynomial dilations in unipotent groups.** We now consider the case of general unipotent groups. For the setting we refer to Section **??**

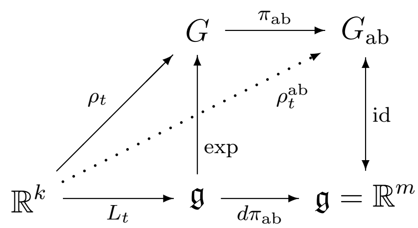
7.3.1. *Abelinization of a family of dilations.* Following [?KSS, Section 1.5] we introduce *the abelinization* of a family of dilations.

Let G be a unipotent group of dimension d, $G_{\mathrm{ab}} = G/[G,G]$ its abelinization and $\pi_{\mathrm{ab}} \colon G \to G_{\mathrm{ab}}$ the projection map. Since G_{ab} is an abelian unipotent group, we identify it with $(\mathbb{R}^m, +)$ for $m = \dim(G_{\mathrm{ab}})$. We also identify the Lie algebra $\mathfrak{g}_{\mathrm{ab}}$ with $(\mathbb{R}^m, +)$, and assume that the exponential map $\exp \colon \mathfrak{g}_{\mathrm{ab}} \to G_{\mathrm{ab}}$ is the identity map.

Let $d\pi_{\mathrm{ab}}$ be the differential of π_{ab} at the identity $e \in G$. We have the following commutative diagram with polynomial maps.



Let M_t be a polynomial $d \times k$ matrix M_t, and $\{\rho_t \colon \mathbb{R}_k \to G : \ t \in I_\infty\}$ the corresponding polynomial family of dilations. For $t \in I_\infty$ we denote by $L_t \colon \mathbb{R}^k \to \mathbb{R}^d$ the linear map $x \mapsto M_t x$. Thus $\rho_t = \exp \circ L_t$, and the following diagram is commutative



with $\rho_t^{\mathrm{ab}} = \pi_{\mathrm{ab}} \circ \rho_t = d\pi_{\mathrm{ab}} \circ L_t$.

Since $d\pi_{\mathrm{ab}}$ is a linear map, there is a $d \times m$ matrix D such that $d\pi_{\mathrm{ab}} \colon x \mapsto Dx$. Thus

$$\rho_t^{\mathrm{ab}} \colon x \to D(M_t x) = (DM_t)x.$$

It is not hard to see that DM_t is a polynomial matrix, hence we obtain that the family $\{\rho_t^{\mathrm{ab}} \colon \mathbb{R}^k \to G_{\mathrm{ab}} \colon t \in I_\infty\}$ is a polynomial family of dilations, and it is proper if the original family $\{\rho_t \colon t \in I_\infty\}$ was.

We call the family $\{\rho_t^{\mathrm{ab}} \colon t \in I_\infty\}$ *the abelinization of the family* $\{\rho_t \colon t \in I_\infty\}$.

We are now ready to prove a strong version of Theorem **??** for proper polynomial dilations.

Theorem 7.7. *Let G be a unipotent group, $\{\rho_t \colon \mathbb{R}^k \to G \colon t \in I_\infty\}$ a proper family of polynomial dilations, $X \subseteq \mathbb{R}^k$ an $\mathbb{R}_{\mathrm{om}}$-definable set, and let $\mathcal{F}$ be the family*

$$\mathcal{F} = \{\rho_t(X) \subseteq G \colon t \in I_\infty\}.$$

Then, for every lattice $\Gamma \subseteq G$ the following conditions are equivalent:

(a) $L^G = G$ for some $L \in \mathscr{L}_{\max}(\mathcal{F})$.
(b) $\pi(\mathcal{F})$ converges strongly to G/Γ at ∞.
(c) $\pi(\mathcal{F})$ converges to G/Γ at ∞.
(d) G/Γ is a Hausdorff limit at ∞ of the family $\pi(\mathcal{F})$.

Proof. By Theorem **??**, (a) snd (b) are equivalent.

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ are obvious.

We are left to show that $(d) \Rightarrow (a)$.

Assume (d) holds. i.e. G/Γ is one of the Hausdorff limits at ∞ of the family $\pi(\mathcal{F})$.

Let π^* be the quotient map $\pi^* \colon G_{\mathrm{ab}} \to \Gamma_{\mathrm{ab}}$, and $\mathcal{F}^{\mathrm{ab}}$ be the family

$$\mathcal{F}^{\mathrm{ab}} = \{\rho_t^{\mathrm{ab}}(X) \subseteq G_{\mathrm{ab}} \colon t \in I_\infty\}.$$

Applying abelianization, we obtain that $G_{\mathrm{ab}}/\Gamma_{\mathrm{ab}}$ is a Hausdorff limit at ∞ of the family $\pi^*(\mathcal{F}^{\mathrm{ab}})$.

Let $\mathcal{M} = \{\bigcup_{j=1}^n (p_j(t) + L_j) \colon t \in I_\infty\}$ be a proper polynomial family of multi-cosets of G_{ab}, as in Theorem **??** applied to the family $\mathcal{F}^{\text{ab}}$. Then $G_{\text{ab}}/\Gamma_{\text{ab}}$ is a Hausdorff limit at ∞ of $\mathcal{M}$, and hence (e,g. by Lemma **??**(1) and Lemma **??**)

$$G_{\text{ab}} = \Gamma_{\text{ab}} + \bigcup_{j=1}^n (a_j + L_j^{\Gamma_{\text{ab}}}),$$

for some $a_1, \dots, a_n \in G_{\text{ab}}$.

Since Γ_{ab} is a discrete subgroup, it follows that $G_{\text{ab}} = L_k^{\Gamma_{\text{ab}}}$ for some $k \in \{1, \dots, n\}$. By Fact **??**, it follows then that $G_{\text{ab}} = L_k^{\Gamma_0}$ for any subgroup $\Gamma_0 \subseteq \Gamma_{\text{ab}}$ of finite index. Thus $\pi^*(\mathcal{M})$ converges strongly to $G_{\text{ab}}/\Gamma_{\text{ab}}$ at ∞, and hence, by the choice of $\mathcal{M}$, the family $\pi^*(\mathcal{F}^{\text{ab}})$ converges strongly at ∞ to $G_{\text{ab}}/\Gamma_{\text{ab}}$ as well.

By Corollary **??**, $\pi(\mathcal{F})$ converges strongly at ∞ to G/Γ. Thus (b) holds.

This finishes the proof of the theorem. □

Remark 7.8. Theorem **??** can be compared to [?KSS, Theorem 1.3]. The latter is an equidistribution result on measures which are associated to polynomial dilations of real analytic curves in nilmanifolds. The set $\mathscr{L}_{\max}$ in our analysis is replaced there by (a-priori infinitely many) kernels of characters. The equidistribution of the measures implies the convergence of the family to G/Γ, under the appropriate assumptions. Our additional input, under the assumption of $\mathbb{R}_{\text{om}}$-definability, is the treatment of higher dimensional sets, as well as the fact that the sets in $\mathscr{L}_{\max}$ work for all lattices.

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