CORRECTION TO "DEFINABLE GROUPS AS HOMOMORPHIC IMAGES OF SEMI-LINEAR AND FIELD-DEFINABLE GROUPS".

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The goal of this note is to fix an error in the proof of [2, Proposition 3.6]. The statement of the theorem requires a a small change, and so does the proof.

Recall that for a definable abelian group $\langle H, +\rangle$, a definable subset $X \subseteq H$ is called H-linear if for every $g, h \in X$, there is a neighborhood U of $0 \in H$, such that $(g - X) \cap U = (h - X) \cap U$. If in addition $0 \in X$ then we call X a local subgroup of G.

Given definable abelian groups $\langle G_1, +\rangle$ and $\langle G_2, \oplus\rangle$, G_1-linear subset $X \subseteq G_1$ and G_2-linear subset $Y \subseteq G_2$, a map $\phi : X \to Y$ is called *an isomorphism of X and Y* if ϕ is a bijection and in addition for every $x_1, x_2, x_3 \in X$,

(1) $x_1 - x_2 + x_3 \in X$ if and only if $\phi(x_1) \ominus \phi(x_2) \oplus \phi(x_3) \in Y$, in which case
(2) $\phi(x_1 - x_2 + x_3) = \phi(x_1) \ominus \phi(x_2) \oplus \phi(x_3)$.

The error in the proof of [2, Proposition 3.6] is in the last sentence of the second paragraph: the identity map need not be an isomorphism between $\langle C_b, +\rangle$ and $\langle C_b, \oplus\rangle$. Earlier in this paragraph we remarked that the identity map is locally an isomorphism between those sets. But this does not imply that it is an isomorphism. What we essentially prove below is that the identity map is a bijective homomorphism between these two sets (Definition 0.2), which results in the weaker Proposition 0.1 below. In Section 1.3, we show how this proposition suffices for our purposes.

We suspect that [2, Proposition 3.6] is still true as it is stated, but we do not address this here. An important observation, however, pointed out to us by Eliana Barriga, is that a bijective homomorphism between two G-linear sets need not be an isomorphism. Namely:

Caution *A definable bijection $\phi : X \to Y$ could satisfy one of the implications in (1) without ϕ^{-1} satisfying it, and thus without being an isomorphism. For example, let $G_1 = \langle \mathbb{R}, +\rangle$ and $G_2 = \langle [0, 1), +(mod 1)\rangle$, let $X = (0, 3/4) \subseteq G_1$ and $Y = (0, 3/4) \subseteq G_2$ and let $\phi : X \to Y$ be the identity map.*

It is easy to see that if $x-y+z \in X$ then $x\ominus y\oplus z \in Y$ and then $x-y+z = x\ominus y\oplus z$. However, $2/3 \ominus 1/6 \oplus 2/3 = 1/6 \in Y$ but $2/3 - 1/6 + 2/3 = 7/6 \notin X$.

We now proceed to fix the error. We recall [2, Fact 3.5], and for that we recall some definitions:

By *a definable parallelogram* we mean a set of the form

$$C_0 = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in J_i \right\},$$

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each $J_i = (-a_i, a_i)$ is a long interval (with a_i possibly ∞) and $\lambda_1, \dots, \lambda_k$ are M-independent partial linear maps from $(-a_i, a_i)$ into M^n.

A *k-long cone in M^n* is a set of the form $C = B + C_0$, for a k-long cone C_0, such that for each $x \in C$ there are unique b and t_i's with $x = b + \sum_{i=1}^{k} \lambda_i(t_i)$.

Fact 0.1. [1, Proposition 5.4] *Let $\langle G, \oplus\rangle$ be a definably compact abelian group of long dimension k. Then G contains a definable, generic, bounded k-long cone $CB + C_0$ on which the group topology of G agrees with the o-minimal topology. Furthermore, for every $a \in C$ there exists an open neighborhood $V \subseteq G$ of a such that for all $x, y \in V \cap a + C_0$,*

$$x \ominus a \oplus y = x - a + y. \tag{1}$$

Our goal is to re-formulate and prove Proposition 3.6 from the article. Towards that purpose we make the following definition.

Definition 0.2. Given groups $\langle G_1, +\rangle$ and $\langle G_2 \oplus\rangle$, and given a G_1-linear set $X \subseteq G_1$ and a G_2-linear set $Y \subseteq G_2$, a definable $\phi : X \to Y$ is *a homomorphism from X to Y* if for all $x_1, x_2, x_3 \in G_1$, if $x_1 - x_2 + x_3 \in X$ then $\phi(x_1) \ominus \phi(x_2) \oplus \phi(x_3)$ is in Y, and we have

$$\phi(x_1 - x_2 + x_3) = \phi(x_1) \ominus \phi(x_2) \oplus \phi(x_3).$$

Notice that if, in the above definition, X is an actual subgroup of G_1 and $\phi(0) = 0$ then $\phi(X)$ is a subgroup of G_2 and in particular, if ϕ is injective then it **is** an isomorphism of groups.

Proposition 0.3. *Let $\langle G, \oplus\rangle$ be a definably compact, definably connected abelian group. Then there exists a definably connected, k-dimensional local subgroup $H \subseteq G$ and a definable short set $B \subseteq G$, $\dim(B) = \dim(G) - k$, satisfying:*

1. *(1) There exist $e_1, \dots, e_k > 0$ in M, each tall in M, and there exists a definable bijective homomorphism $\phi : H' \to H$, between the M^n-linear set H' and the G-linear set H. In particular, $\dim H = lgdim H = k$.*
2. *(2) The set $B \oplus H = \{b \oplus h : b \in B\ h \in H\}$ is generic in G.*

Notice that the difference between the above formulation and the original one is that the bijective homomorphism between H and H' is not assumed to be an isomorphism any more.

1. Proving Proposition 0.3

1.1. **A preliminary result.** We work in an o-minimal expansion $\mathcal{M} = \langle M, <, +, \dots\rangle$ of an ordered group. Let $G = \langle G, \oplus, e_G\rangle$ be a definable group. It has a group topology, the G-topology. We fix an open parallelogram $C_0 \subseteq M^n$ that is contained in G. Observe that this does not mean C_0 is an open set. We know that C_0 is affine, connected and definable. Suppose that the subspace topology on C_0 coincides with the G-topology on it. By "open $V \subseteq C_0$" we mean that V is relatively open in C_0 (in either topology). We assume the following *local property* holds: for every $a \in C_0$, there is an open $V \subseteq C_0$ containing a, such that for every $x, y \in V$,

$$x \ominus a \oplus y = x - a + y.$$

Our first goal is to prove in details Proposition 1.6 below.

Claim 1.1. *Let $a \in C_0$ and open $V \subseteq C_0$ witnessing the local property around a. Then there is open $U \subseteq V$ containing a such that*

$$U \ominus U \oplus a \subseteq V.$$

Proof. Because C_0 is affine, there is open $U \subseteq V$ containing a such that

$$U + a - U \subseteq V.$$

We then have, for every $x \in U$,

$$U \subseteq V - a + x = V \ominus a \oplus x,$$

and hence $U \ominus x \oplus a \subseteq V$. Therefore $U \ominus U \oplus a \subseteq V$. □

Lemma 1.2. *For every $a \in C_0$, there is an open $U \subseteq C_0$ containing a, such that for every $x, z \in U$,*

$$x \ominus z \oplus a = x - z + a.$$

Proof. Take V witnessing the local property around a. By Claim 1.1, there is open $U \subseteq V$ containing a such that

$$U \ominus U \oplus a \subseteq V.$$

For every $x, z \in U$, we have

$$x \ominus z \oplus a = k \Leftrightarrow x = k \ominus a \oplus z = k - a + z,$$

since $k, z \in V$, and hence

$$x \ominus z \oplus a = x - z + a,$$

as required. □

Lemma 1.3. *For every $a \in C_0$, there is an open $U \subseteq C_0$ containing a, such that for every $x, y, z \in U$,*

$$x \ominus z \oplus y = x - z + y.$$

Proof. Fix $a \in C_0$ and let $V \subseteq C_0$ open containing a witnessing the local property around a. Let also $U \subseteq C_0$ open containing a provided by Lemma 1.2. We may shrink U if necessary, so that $U \subseteq V$ and

$$U - U + a \subseteq V.$$

Then for every $x, y, z \in U$, we have

$$x \ominus z \oplus y = (x \ominus z \oplus a) \ominus a \oplus y = (x - z + a) \ominus a \oplus y = (x - z + a) - a + y = x - z + y,$$

as needed. □

We can now show a special case of Proposition 1.6 (with $z = a$ below).

Lemma 1.4. *For every $a, c \in C_0$, there is open $U \subseteq C_0$ containing a, such that for every $x, z \in U$,*

$$(*) \qquad x \ominus z \oplus c = x - z + c.$$

Proof. Fix $a \in C_0$ and let

$$\Gamma = \{c \in C_0 : \text{there is open } U \subseteq C_0 \text{ containing } a \text{ such that for every } x, z \in U, (*) \text{ holds}\}.$$

We have $\Gamma \neq \emptyset$, because, by Lemma 1.2, it contains a. We prove that Γ is open and closed in C_0. Since C_0 is connected, this will imply that $\Gamma = C$.

Γ **open.** Suppose $c \in \Gamma$. Take open $V \subseteq C_0$ containing c witnessing the local property, and open $W \subseteq C_0$ containing a on which (*) holds. Since C_0 is an *open* parallelogram, we can shrink W, if necessary, so that

$$W - W + c \subseteq V.$$

We thus have, for every $x, z \in W$ and $y \in V$,

$$x \ominus z \oplus y = (x \ominus z \oplus c) \ominus c \oplus y = (x - z + c) \ominus c \oplus y = (x - z + c) - c + y = x - z + y,$$

and hence $y \in \Gamma$, as needed.

Γ **closed.** Suppose $y \in cl(\Gamma) \cap C_0$. Let $U \subseteq C_0$ be an open set containing y provided by Lemma 1.2 (for $a = y$). Hence, there is $y' \in U \cap \Gamma$. Let $W \subseteq C_0$ be open containing a witnessing that $y' \in \Gamma$. Shrink W, if necessary, so that also

$$W - W + y' \subseteq U.$$

We have, for every $x, z \in W$,

$$x \ominus z \oplus y = (x \ominus z \oplus y') \ominus y' \oplus y = (x - z + y') \ominus y' \oplus y = (x - z + y') - y' + y = x - z + y,$$

and hence $y \in \Gamma$, as needed. $\square$

Corollary 1.5. *For every $a, c \in C_0$, there is open $U \subseteq C_0^3$ containing (a, a, c), such that for every $(x, z, y) \in U$,*

$$(*) \qquad x \ominus z \oplus y = x - z + y.$$

Proof. Let $a, c \in C_0$ and U as in Lemma 1.4. Take $V \subseteq C_0$ open containing c witnessing the local property around c. By shrinking U if necessary, we may assume that

$$U - U + c \subseteq V.$$

Then $U \times U \times V \subseteq C_0^3$ is open containing (a, a, c), and for every $(x, z, y) \in U \times U \times V$,

$$x \ominus z \oplus y = (x \ominus z \oplus c) \ominus c \oplus y = (x - z + c) \ominus c \oplus y = (x - z + c) - c + y = x - z + y,$$

as needed. $\square$

Proposition 1.6. *For every $a, x, y \in C_0$ with $x - a + y \in C_0$,*

$$x \ominus a \oplus y = x - a + y.$$

Proof. Fix $a, y \in C_0$ and let $K = \{x \in C_0 : x - a + y \in C_0\}$ and

$$\Gamma = \{x \in K : x \ominus a \oplus y = x - a + y\}.$$

We have $\Gamma \neq \emptyset$ because it contains a. We prove Γ is open and closed in K. Observe that since C_0 is a parallelogram, K is easily seen to be connected and open in C_0.

Γ **open.** Let $x \in \Gamma$. Let also $U \subseteq C_0$ containing x provided by Lemma 1.4 for $x, x - a + y$ (the latter is in C_0, since $x \in K$). We have, for every $x' \in U$,

$$x' \ominus a \oplus y = x' \ominus x \oplus (x \ominus a \oplus y) = x' \ominus x \oplus (x - a + y) = x' - x + (x - a + y) = x' - a + y,$$

and hence $x' \in \Gamma$, as needed.

Γ **closed.** Let $x \in cl(\Gamma) \cap K$. Let $U \subseteq C_0^3$ be open containing $(x, x, x - a + y)$ provided by Corollary 1.5 for $x, x - a + y$. We may shrink $\pi_1(U)$ if necessary so that for all $x' \in \pi_1(U)$, $x' - a + y \in \pi_3(U)$. Now, since $x \in cl(\Gamma)$, there is $x' \in \pi_1(U) \cap \Gamma$. We have:

$$x \ominus a \oplus y = x \ominus x' \oplus (x' \ominus a \oplus y) = x \ominus x' \oplus (x' - a + y) = x - x' + (x' - a + y) = x - a + y,$$

and hence $x \in \Gamma$, as needed. $\square$

1.2. **The proof of Proposition 0.1.** We have a strongly long parallelogram $C_0 \subseteq M^n$ and a short set $B \subseteq M^n$, such that for every $b \in B$, and $x, y, z \in C_0$, with $x - y + z \in C_0$, we have

$$(b + x) - (b + y) + (b + z) = (b + x) \ominus (b + y) \oplus (b + z),$$

or written differently,

$$b + (x - y + z) = (b + x) \ominus (b + y) \oplus (b + z).$$

(In this notation the cone C from the article is $B + C_0$).

For $b \in B$, we let $f_b(x) = b + x$ a map from C_0 into C. By the above we have for all $x, y, x + y \in C_0$,

$$f_b(x + y) = f_b(x) \oplus f_b(y) \ominus b. \tag{2}$$

We now define the binary relation on B: $b_1 \sim b_2$ iff there exists $g \in G$ such that for all $x \in C_0$ we have $f_{b_1}(x) = f_{b_2}(x) \oplus g$. It is easy to see that this defines an equivalence relation on B (this is true for any independent of the linearity property above).

We need:

Claim 1.7. *Assume that for $b_1, b_1 \in B$ there exists an open set $W \subseteq C_0$ and $g \in G$, such that for every $x \in W$ we have $f_{b_1}(x) = f_{b_2}(x) \oplus g$. Then $b_1 \sim b_2$.*

Proof. We first claim that there exists a neighborhood $W_1 \ni 0$ and a constant element $g_1 \in G$ such that for all $x_1 \in W_1$, $f_{b_1}(x_1) = f_{b_2}(x_1) \oplus g_1$.

Indeed, fix $x_0 \in W$ and choose $W_1 \ni 0$ such that $x_0 + W_1 \subseteq W$. On one hand we have, for all $x_1 \in W_1$,

$$f_{b_1}(x_0 + x_1) = f_{b_1}(x_0) \oplus f_{b_1}(x_1) \ominus b_1 = (f_{b_2}(x_0) \oplus g) \oplus f_{b_1}(x_1) \ominus b_1.$$

On the other hand,

$$f_{b_1}(x_0 + x_1) = f_{b_2}(x_0 + x_1) \oplus g = f_{b_2}(x_0) \oplus f_{b_2}(x_1) \ominus b_2 \oplus g.$$

It follows that for all $x_1 \in W_1$ we have

$$f_{b_1}(x_1) = f_{b_2}(x_1) \oplus (b_1 \ominus b_2).$$

We now fix $g_1 = b_1 \ominus b_2$ and define

$$C_{b_1,b_2} = \{x \in C_0 : f_{b_1}(x) = f_{b_2}(x) \oplus g_1\}.$$

In order to show that $b_1 \sim b_2$ we need to prove that $C_{b_1,b_2} = C_0$. Because C_0 is definably connected, we need to verify that it is closed and open in C_0:

Each of the maps $f_{b_i} : C \to G$ is continuous with respect to the M^n-topology in the domain and the G-topology in the range (because $B + C_0$ is open in both topologies). Thus also the map $f_{b_2}(x) \oplus g$ is continuous from M^n into G. It follows that C_{b_1,b_2} is closed in C_0. Let us see that it is also open in C_0, so let $x_0 \in C_{b_1,b_2}$. We already saw that 0 is an interior point so fix $W_1 \subseteq C_{b_1,b_2}$ an open neighborhood of 0 such that $x_0 + W_1 \subseteq C_0$.

We have

$$f_{b_1}(x_0+x_1) = f_{b_1}(x_0) \oplus f_{b_1}(x_1) \ominus b_1 = (f_{b_2}(x_0) \oplus (b_1 \ominus b_2) \oplus f_{b_2}(x_1) \oplus (b_1 \ominus b_2)) \ominus b_1.$$

$$= [f_{b_2}(x_0+x_1) \oplus b_2] \oplus (b_1 \ominus b_2) \ominus b_2 = f_{b_2}(x_0+x_1) \oplus (b_1 \ominus b_2),$$

so $x_0 + W_1$ is contained in C_{b_1,b_2}. □

Lemma 1.8. *There are only finitely many $\sim$-classes in B.*

Proof. This is very similar to the proof in the article. We assume towards contradiction that there are infinitely many classes and by replacing B with a definable set of representatives, we assume that each $\sim$-class contains a single element (and B is infinite).

We now consider the map $F : B \times C_0 \to G$ given by $F(b,x) = f_b(x)$. We replace C_0 by a definably compact, still strongly long $X \subseteq C_0$ of the same dimension. The map F is continuous from $B \times X$ endowed with the M^n-topology into G, endowed with the group topology. As before, for any $b_1 \neq b_2$ in B, we obtain an open set $V'' \subseteq C_0$, such that the map $f_{b_1}(x) \ominus f_{b_2}(x)$ is constant on V''. Namely, there exists $g \in G$ such that for all $x \in V''$, $f_{b_1}(x) = f_{b_2}(x) \oplus g$. By Claim 1.7, we have $b_1 \sim b_2$, contradicting our assumption. □

As in our article, we may replace B by one of the equivalence classes B_i such that $B_i + C_0$ is still generic in G, thus we may assume that for all $b_1, b_2 \in B$ there exists $g = g(b_1, b_2)$ such that $f_{b_1}(x) = f_{b_2}(x) \oplus g$ for all $x \in C_0$.

Fix a cone $C \subseteq M^n$ of the form $B + C_0$ (for B as above), fix $b_0 \in B$ and for every $b \in B$ choose $g(b) \in G$ such that $f_b(x) = f_{b_0}(x) \oplus g(b)$. We now define

$$B' = \{g(b) \oplus b_0 : b \in B\}$$

and

$$H = \{f_{b_0}(x) \ominus b_0 : x \in C_0\}.$$

For every $b \in B$ and $x \in C_0$, we have

$$b + x = f_b(x) = f_{b_0}(x) \oplus g(b) = (f_{b_0}(x) \ominus b_0) \oplus (g(b) \oplus b_0),$$

hence $C = B + C_0 = B' \oplus H$. Furthermore, $0_G \in H$ and the map $\sigma(x) = f_{b_0}(x) \ominus b_0$ from C_0 onto H is injective and satisfies for all $x_1, x_2, x_1 + x_2 \in C_0$, $\sigma(x_1 + x_2) = \sigma(x_1) \oplus \sigma(x_2)$, and in particular, $\sigma(x_1) \oplus \sigma(x_2) \in H$. Note however, that we do not claim that if $\sigma(x_1) \oplus \sigma(x_2) \in H$ then $x_1 + x_2$ is in C_0. We can finish as in the article (but with the disclaimer that σ^{-1} is not a local homomorphism because of our remark).

1.3. **On Section 4.1.** We now need to clarify the first few paragraphs of 4.1 . Actually, the argument there does not really use the existence of an inverse homomorphism from H into C_0.

Consider then the bijection $\sigma : C_0 \to H$, which is a homomorphism when defined. Since C_0 is convex, for every $n \in \mathbb{N}$ and $x \in C_0$, we have $\sigma(x) = \sigma(x/n) \oplus \cdots \oplus \sigma(x/n)$, where the sum on the right is taken n-times (since C is convex in M^n, each x/n belongs to C_0). Using the fact that $\langle M^n, + \rangle$ is torsion-free it now immediately follows:

Corollary 1.9. *The map σ can be extended to a locally definable homomorphism from the group generated by C_0 in $\langle M^n, + \rangle$ onto $\langle H \rangle$ the subgroup of G generated by H.*

The rest of the argument remains the same. Because the set of short elements in C_0 is a subgroup of $\langle M^n, +\rangle$ the restriction of σ to it is now an isomorphism of groups onto its image and we can proceed as before.

References

[1] Pantelis E. Eleftheriou, *Local analysis for semi-bounded groups*, Fundamentae Mathematica **216** (2012), 223-258.

[2] Pantelis E. Eleftheriou and Ya'acov Peterzil, *Definable groups as homomorphic images of semilinear and field-definable groups*, Selecta **18** (2012), no. 4, 905-940.