CORRECTION TO "EXPANSIONS OF ALGEBRAICALLY CLOSED FIELDS IN O-MINIMAL STRUCTURES", *SELECTA MATH. NS 7 (2001) 409-445*

Y. PETERZIL AND S. STARCHENKO

This note is written in order to fix an error in [2], noted by Alex Wilkie and his student Javier Utreras. We thank them for pointing it out.

The error comes at the very end of the proof of Lemma 2.25, where we (implicitly) apply Fact 2.23 to conclude that the winding number of f along the boundary of D_1 is 0. However, while Fact 2.23 assumes that the curve is "star-shaped" the curve to which the result is applied is not star-shaped.

We resolve the matter by strengthening the statement of Fact 2.23. Before doing that we need a general topological statement. We thank Alessandro Berarducci for helping us with this proof.

We work in an o-minimal expansion $\mathcal{M}$ of a real closed field. We say that $\mathcal{C} \subseteq M^2$ is *a definable simple closed curve* if $\mathcal{C}$ is the image under definable continuous injective map $\sigma : [0, 1) \to M^2$, with $\lim_{t\to 1} \sigma(t) = \sigma(0)$. By the o-minimal version of Jordan's Lemma (see [3]), the curve divides M^2 into two components. We call the bounded component the interior of $\mathcal{C}$ and denote it by D.

Lemma 0.1. *If $\mathcal{C} \subseteq M^2$ is a definable simple closed curve then $B = D \cup \mathcal{C}$ is definably contractible to any point in B. Namely, for every $p \in B$ there exists a definable $H : B \times [0, 1] \to B$ such that for all $x \in B$, $H(x, 0) = x$ and $H(x, 1) = p$.*

Proof. We first use the triangulation theorem to triangulate M^2, compatibly with $\mathcal{C}$ and D, inside M^2 (see Theorem 8.2.9 in [1]). Namely, there is a simplicial complex $K \subseteq M^2$ and a definable homeomorphism $h : M^2 \to K$ such that $h(\mathcal{C})$ and $h(D)$ are sub-complexes of K.

It is not hard to see (or, by the theorem on Invariance of domain, see [3]), that the underlying set of K is an open subset of M^2, and therefore $h(\mathcal{C})$ is a semilinear simple closed curve in $K \subseteq M^2$. Clearly, $h(D)$ is the interior of $h(\mathcal{C})$, so we may assume from now one that $\mathcal{C}$ is a polygonal curve.

We assume it to be known that under these assumptions there is a semialgebraic contraction of B but for the sake of completeness we include a proof of this fact. The proof is by induction on the number of linear pieces, n, of $\mathcal{C}$. Clearly, $n \geq 3$, and if $n = 3$ the set B is a triangle which is definably contractible to any of its points. Assume then that $n > 3$. The following is known.

Fact There exist vertices $p_1 \neq p_2 \in \mathcal{C}$ such that the line segment connecting them, call it e, lies entirely in D (the segment is called "a diagonal").

Now, the curve $\mathcal{C}$ gives rise to two simple closed curves $\mathcal{C}_1, \mathcal{C}_2$ which share a common edge e. Each curve is obtained by one of the two pieces of $\mathcal{C}$ which connect p_1 and p_2, together with e. The interiors of $\mathcal{C}_1$ and $\mathcal{C}_2$ are contained in D. We fix a point p on e. Each $\mathcal{C}_i$, has less than n edges and therefore, by induction, $\mathcal{C}_i \cup Int(\mathcal{C}_i)$ can be contracted to p. These contractions can be patched to a contraction of B. □

We can now prove the following strong version of Lemma 2.23 from [2].

Lemma 0.2. *Let $\mathcal{C} \subseteq M^2$ be a definable simple closed curve with interior D. Let $f : D \cup \mathcal{C} \to M^2$ be a definable continuous map. If $w \notin f(D \cup \mathcal{C})$ then $W_{\mathcal{C}}(f, w) = 0$.*

Proof. Let $B = D \cup \mathcal{C}$, and fix a definable contraction $H : B \times [0, 1] \to B$ to a point $p \in D$.

Because $w \neq f(p)$, we can find U_0 small enough around p and a point s_0 on the unit circle, such that for every $w' \in f(U_0)$, $(w' - w)/|w' - w| \neq s_0$. In particular, the map $(f(u) - w)/|f(u) - w|$ is not surjective from U_0 to the unit circle S^1. Because H is continuous and $B \times [0, 1]$ definably compact, there is r close enough to 1 such that the image of $B \times [r, 1]$ under H is contained in U_0.

For every $t \in [0, 1]$, and $z \in \mathcal{C}$, we let

$$f_t^*(z) = \frac{f(H(z,t)) - w}{|f(H(z,t)) - w|}$$

be a map from $\mathcal{C}$ into S^1 (the map is well defined since $H(z, t) \in B$ and $w \notin f(B)$).

By [2, Lemma 2.13(4)], we have $W(f_0^*) = W(f_r^*)$. But, by definitions, $W(f_0^*) = W_{\mathcal{C}}(f, w)$, and by [2, Lemma 2.13(2)], $W(f_r^*) = 0$. □

References

[1] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.

[2] Y. Peterzil and S. Starchenko, *Expansions of algebraically closed fields in o-minimal structures*, Selecta Math. (N.S.) **7** (2001), no. 3, 409–445.

[3] A Woerheide, *O-minimal homology, PhD thesis (1996)*, U. of Illinois at Urbana-Champaign.