A note on stable sets, groups and theories with NIP

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Let M be an arbitrary structure. We say that an M-formula $\phi(x)$ defines a stable set in M if every formula $\phi(x) \wedge \alpha(x, y)$ is stable. We prove: If G is an M-definable group and every definable stable subset of G has U-rank at most n (the same n for all sets) then G has a maximal connected stable normal subgroup H such that G/H is purely unstable. The assumptions holds for example when the structure M is interpretable in an o-minimal structure.

More generally, an M-definable set X is called weakly stable if the M-induced structure on X is stable. We observe that, by results of Shelah, every weakly stable set in theories with NIP, is stable.

1 Introduction and definitions

In this note we prove that in a definable group G, with a uniform finite bound on the U-rank of definable stable subsets, there is a maximal (up to finite index) normal stable subgroup H, such that G/H is purely unstable (see the definitions below and Theorem 2.1).

We started to work in this article while studying structures which are interpretable in o-minimal theories. All of these satisfy our assumption on stable subsets. We later generalized the results to groups satisfying NIP with a bound on the U-rank of stable sets. It turned out, following a suggestion of the referee, that by modifying slightly our definition of a stable set, the NIP assumptions can be omitted. This is the reason why we examine two variations on the notion of a stable definable set.

In the first section we review some definitions; in Section 2 we prove the main result; in Section 3 we discuss how rosy theories fit with the main result of the paper; in Section 4 we discuss weakly stable sets in theories with NIP and prove that they are stable. We then give some examples and characterize a large family of theories that satisfy the conditions required for our main theorem.

Throughout this paper we work with a model M which is an elementary substructure some "monster" model $\mathcal{C}$.

Recall the following definitions.

Definition 1.1.
Let M be any structure and let $\phi(\bar{x}, \bar{y})$ be a formula in $\mathcal{L}(M)$.

- *A formula $\phi(\bar{x}, \bar{y})$ is said to have* the order property *if there are infinite sequences $\langle \bar{a}_i \rangle_{i \in \omega}$ and $\langle \bar{b}_j \rangle_{j \in \omega}$ of tuples from M such that $M \models \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leq j$. A formula $\phi(\bar{x}, \bar{y})$ is stable if it does not have the order property. M* is stable *if no formula in a monster model of $Th(M)$ has the order property.*

- *A formula $\phi(\bar{x}, \bar{y})$ is said to have* the strict order property *if there is an infinite sequence $\langle \bar{a}_i \rangle_{i \in \omega}$ of tuples such that $N \models \forall \bar{x}(\phi(\bar{x}, \bar{a}_i) \Rightarrow \phi(\bar{x}, \bar{a}_j))$ if and only if $i \leq j$.*

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- *A theory T* satisfies NIP *if there is no model N of T and no $\phi(\bar{x}, \bar{y})$ (may contain parameters in N) and a sequence $\langle \bar{a}_i \rangle$ of indiscernible tuples from N such that for all finite disjoint sets I, J there is some $\bar{b}$ such that $M \models \phi(\bar{a}_i, \bar{b})$ for all $i \in I$ and such that $M \models \neg\phi(\bar{a}_j, \bar{b})$ for all $j \in J$. We will define a structure M to* satisfy NIP *if $Th(M)$ does.*

We will now define some new concepts we will need for this paper.

Definition 1.2.

- *We say that a definable set X is* weakly stable *if there is no formula $\phi(\bar{x}, \bar{y})$ (may contain parameters in M) and sequences of tuples $\langle \bar{a}_i, \bar{b}_i \rangle$ whose elements are in X and such that $M \models \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leq j$.*

- *We will say that a definable X is* stable *if there is no formula $\phi(\bar{x}, \bar{y})$ and a sequence of tuples $\langle \bar{a}_i, \bar{b}_i \rangle$ with $\bar{a}_i \in X^n$ for some n, such that $M \models \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leq j$. A set will be defined to be* unstable *if it is not stable. Equivalently, if X is defined by a formula $\phi(x)$ (possibly over some parameters) then X is stable if and only if for every formula $\delta(x, y)$, the formula $\phi(x) \wedge \delta(x, y)$ is a stable formula.*

- *A structure M is* purely unstable *if no infinite definable subset X of M is stable.*

- *We define $U(X) := \sup\{U(p) | \text{“}x \in X\text{”} \in p\}$.*

The notion of weak stability of a set X is natural if one wants the *structure* $X(M)$, with all its M-induced structure, to be stable. When M is sufficiently saturated then, as is easy to verify, if X is weakly stable in M then it remains so in every elementary equivalent structure. Such $X(M)$ indeed posses some of the good properties of stable structures. However, if $X(M)$ is defined using a formula $\phi(x, a)$ then it does not follow that $\phi(x, y)$ is stable, neither it follows that every formula $\psi(x, y)$ which is implied by $\phi(x, a)$ is a stable formula (see the proposed example below). This turns out to be a serious draw-back for our purposes and that is why we defined the notion of a stable set to be as above. However, as we observe below, either the stable embeddedness of X or NIP imply the equivalence of stability and weak stability.

Example 1.3. *Let $\mathcal{L} := \{G(x), R(x, y)\}$ where G is a unary predicate and R is a binary relation and let T be the theory of the random bipartite graph where $R(x, y) \Rightarrow (G(x) \wedge \neg G(y))$. In any model $M \models T$ the definable set $G(M)$ is weakly stable but not stable.*

Indeed, let M be a model of T. The theory T has quantifier elimination so, if we let M_G be the full structure induced by M on $G(M)$ then M_G is a structure in the language $\mathcal{L}_G := \{P_a(x)\}_{a \in M \setminus G(M)}$ where $M_G \models P_a(b) \Leftrightarrow M \models R(a, b)$ for any $b \in G(M)$.

Because all the atomic relations are unary it is immediate that no forking can take place and hence M_G is superstable of U-rank 1. In particular, G is then a weakly stable set.

On the other hand, for any $N \models T$, any set of elements $\{a_i\}_{i \in \kappa}$ in $\neg G(N)$ and any function $\eta : \kappa \to 2$ the type

$$\bigwedge G(x) \wedge R(x, a_i)^{\eta(i)}$$

is consistent; so by definition and Proposition 1.4 $G(x)$ is not stable.

Proposition 1.4. *Let $X = \phi(M)$ be a definable set (possibly with parameters). Then the following are equivalent.*

1. *The set X is stable.*

2. *Every type extending $\phi(x)$ is definable over the algebraic closure of its parameter set.*

3. *There exists a cardinal λ such that for every $A \subseteq M$ of cardinality λ, there are λ many types over A containing $\phi(x)$.*

Proof. The proof of the equivalence is word by word the same proof as the analogue equivalence for definable sets in stable theories (see for example Lemma 2.2 and Remark 2.3 in []). □

We recall that a 0-definable set X is called *stably embedded in* M if every M-definable subset of X^n is definable using parameters from X. If M is ω-saturated then stable embeddedness of X implies a uniform version: For every formula $\psi(x, y)$ there exists a formula $\delta(x, w)$ such that the following two definable families of sets are the same:

$$\{\psi(X^n, b); b \in M^k\} = \{\delta(X^n, c) : c \in X^r\}.$$

Proposition 1.5. *Let $X = \phi(M)$ be a weakly stable set in an ω-saturated M. Then X is stable if and only if X is stably embedded in M.*

Proof. If X is stable and $\psi(x, a)$ defines a subset of X^n (over a in some small model $M_0 \subseteq M$) then the formula $\alpha(x, y) := \phi(x) \wedge \psi(x, y)$ is stable. It follows (see [] Lemma 2.2) that the α-type of a over $X(M_0)$ is definable over $X(M_0)$, and hence the set $\psi(X(M_0), a)$ is definable using parameters in $X(M_0)$. This implies that X is stably embedded in M.

For the converse, assume that X is stably embedded in M. If X were not stable then for some $\psi(x, y)$ and tuples $\langle a_i, b_i\rangle$, with $a_i \in X$, we have $M \models \psi(a_i, b_j)$ if and only if $i \leq j$. The uniform version of stable embeddedness implies that we can replace the b_i's by tuples in X, contradicting weak stability. □

2 Stable subgroups.

Theorem 2.1. *Let M be an ω-saturated structure, G an M-definable group and assume that there is a uniform bound $n \in \mathbb{N}$ such that the U-rank of every stable subset of G is at most n, and n is minimal such. Then there exists definable stable normal subgroup N of G with $\mathrm{U}(N) = n$, such that N is a maximal (up to finite index) stable subset of G (i.e., any stable set is contained in finitely many cosets of N). In particular, G/N is purely unstable.*

Proof. Let $X \subset G$ be a definable stable set such that $\mathrm{U}(X) = n$, and let $p(x)$ be a global complete type in $\mathcal{C}$ containing $x \in X$ and with U-rank n.

Let $X \cdot X^{-1} := \{x \cdot y \mid x, y^{-1} \in X\}$. Because $X \cdot X^{-1}$ is in definable bijection with X^2, it is clearly stable. Let

$$\mathit{Stab}(p) = \{g \in G : gp(\mathcal{C}) = p(\mathcal{C})\} = \{g \in X \cdot X^{-1} : gp = p\}.$$

The type p is definable, by Proposition 1.4, hence the group $\mathit{Stab}(p)$ is a type-definable subgroup of G, contained in $X \cdot X^{-1}$. Since the induced structure on $X \cdot X^{-1}$ is stable, the group $\mathit{Stab}(p)$ must be the intersection of definable groups and, by compactness, one of such groups, call it H, must be contained in $X \cdot X^{-1}$. We claim that $\mathrm{U}(H) = n$:

Fix a, b two realizations of p such that $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$. We now have

$$\mathrm{U}(tp(a/b)) = \mathrm{U}(tp(ab^{-1}/b)) = n,$$

and $ab^{-1} \in \mathit{Stab}(p)$. It follows that $\mathrm{U}(\mathit{Stab}(p)) = n$, and hence $\mathrm{U}(H) = \mathrm{U}(H) \geq n$. By maximality, $\mathrm{U}(H) = n$.

We will now obtain a normal group. For $a \in G$, let $H^a := aHa^{-1}$. We claim that $H^a \cap H$ has finite index in H.

Indeed, H^a is definably bijective with H therefore it is stable and $\mathrm{U}(H^a) = n$. The set $H \cdot H^a$ is in definable bijection with H^2 therefore, by maximality, we also have $\mathrm{U}(H \cdot H^a) = n$. Now, the set $H \cdot H^a$ can be written as a definable union of cosets of H^a so there are only finitely many cosets in this union. The map $h \cdot (H^a \cap H) \mapsto h \cdot H^a$ is injective from H/H^a into $(H \cdot H^a)/H^a$, therefore $H^a \cap H$ has finite index in H for all a.

By the stable embeddedness of H in M, the family of subgroups $\{H^a \cap H : a \in G\}$ is definable using parameters in H. By Baldwin-Saxl (applied to the induced stable structure on H), there are finitely many $a_1, \ldots, a_k$ such that

$$N := \bigcap_{a \in G} H^a = H^{a_1} \cap \cdots \cap H^{a_k}.$$

It follows that the normal subgroup N has finite index in H and therefore $\mathrm{U}(N) = n$.

The maximality of N follows: Let $\phi : G \to G/N$ be map of G into the quotient group. If X is any stable subset of G which is not covered by finitely many cosets of N then, by counting the number of types (and using Proposition 1.4), we see that the set $X \cdot N$ is stable as well. It follows that $\mathrm{U}(X \cdot N) > \mathrm{U}(N) = n$, contradiction. Similarly, one shows that G/N is purely unstable. □

3 Stable sets in theories of finite $\mathrm{U}^{\text{þ}}$-rank.

In this section we will recall a couple of easy results on þ-forking to prove that in any group interpretable in a theory of finite $\mathrm{U}^{\text{þ}}$-rank the assumptions of Theorem 2.1 hold.

We will start by recalling the definition of þ-forking. All the definitions and results in this section can be found in [] and [].

Definition 3.1. *A formula $\delta(x, a)$* strongly divides over *A if $tp(a/A)$ is non-algebraic and $\{\delta(x, a')\}_{a' \models tp(a/A)}$ is k-inconsistent for some $k \in \mathbb{N}$.*

We say that $\delta(x, a)$ þ-divides over *A if we can find some tuple c such that $\delta(x, a)$ strongly divides over Ac.*

A formula þ-forks over *A if it implies a (finite) disjunction of formulas which þ-divide over A.*

We say that the type $p(x)$ þ-divides over *A if there is a formula in $p(x)$ which þ-divides over A; þ-forking is similarly defined. We say that a is* þ-independent from b over A, *denoted $a \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_A b$, if $tp\,(a/Ab)$ does not þ-fork over A.*

Definition 3.2. *We define the $U^{þ}$-rank on types inductively as follows.*

Let p be a type over some set A, let α be any ordinal and let λ be a limit ordinal.

- *$U^{þ}(p) \geq \emptyset$ if p is consistent.*
- *$U^{þ}(p) \geq \alpha + 1$ if and only if there is some $B \supset A$ and some type q over B extending p such that q þ-forks over A and $U^{þ}(q) \geq \alpha$.*
- *$U^{þ}(p) \geq \lambda$ if and only if $U^{þ}(p) \geq \sigma$ for all $\sigma < \lambda$.*

If for all type $U^{þ}(p)$ is finite for all types p with finite number of variables, we define

$$U^{þ}(\phi(x, a)) := max \left\{ U^{þ}(p) \mid \phi(x, a) \in p \right\}.$$

It follows from the definitions that any instance of þ-forking is an instance of forking. The converse is not always true. However, we have the following:

Fact 3.3.

1. *Let T be an arbitrary first order theory. If $tp(a/Ab)$ forks over A and this is witnessed by a stable formula $\phi(x, y)$ then $tp(a/Ab)$ þ-forks over A.*
2. *If T is super-rosy of finite $U^{þ}$-rank then for every stable set X, we have $U(X) = U^{þ}(X)$.*

Proof.
(1) Follows from work in [].
(2) We assume that $\theta(x)$ defines a stable set X, let $p(x, Ab)$ be a type containing $\theta(x)$ and let $\delta(x, y)$ be a formula such that $\delta(x, b)$ witnesses forking of $p(x)$ over A.

By Definition 1.2 the formula $\delta(x,y) \wedge \theta(x)$ is a stable formula and $\delta(x,a) \wedge \theta(x)$ is in $p(x)$ so by (1) we have that $tp(a/Ab)$ þ-forks over A. Hence, every forking extension of $p(x)$ is also a þ-forking extension (and vice-versa). It follows now by induction that the U-rank of any type extneding θ equals its $\mathrm{U}^{\text{þ}}$-rank. In particular, $\mathrm{U}^{\text{þ}}(X) = \mathrm{U}(X)$. □

Corollary 3.4. *Let T be super-rosy, of finite $U^{þ}$-rank. If G is an interpretable group in a model of T then it has a stable normal subgroup N which is a maximal (by finite) stable subset of G.*

Proof. By fact3.3 (2), the U-rank of any stable definable subset of G is bounded by the $\mathrm{U}^{\text{þ}}$-rank of G, which by assumption is finite. So G satisfies the assumptions of Theorem 2.1(2) and the corollary follows. □

4 Stable sets in theories with NIP.

In this section we examine the two notions of stable sets and show that they are equivalent under the assumption of NIP. Since our motivating example was to study groups definable in structures which are interpretable in o-minimal ones, the NIP assumption is natural. Such groups also have finite $\mathrm{U}^{\text{þ}}$-rank so all the results for the previous section will hold.

In order to proceed we need the following Lemma. This Lemma is proved within the proof of Lemma II, 4.7 in [], although it is not stated as such, so we include a proof[5].

Lemma 4.1. *Let $\phi(x,y)$ be an unstable formula (x,y are tuples of variables) in a theory satisfying NIP. Then there is a formula $\theta(x,\bar{b})$ such that $\theta(x,\bar{b}) \wedge \phi(x,y)$ has the strict order property.*

Proof. We recall the proof of Lemma II,4.7 in [].

Let $\phi(x,y)$ be an unstable formula and let $\langle a_i \rangle_{i\in\omega}$ and $\langle b_j \rangle_{j\in\omega}$ be indiscernible sequences witnessing the instability of $\phi(x,y)$. By compactness we can assume that the indiscernible sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ are both indexed by $\mathbb{Q}$. Note first that for I and J disjoint subsets of $\mathbb{Q}$ such that $i<j$ for any $i \in I$ and $j \in J$, if $k \in \mathbb{Q}$ is such that $I<k<J$ then, by construction, $a_k \models \neg\phi(x,b_i) \wedge \phi(x,b_j)$ for any $i \in I$ and $j \in J$.

On the other hand, using NIP, there is a finite subset $I \subset \mathbb{Q}$ and a function $\eta_0 : I \to 2$ such that

$$\exists x \bigwedge_{i\in I} \phi(x,b_i)^{\eta_0(i)}$$

is inconsistent. Let $I := \{i_1,\ldots,i_n\}$ with $i_l < i_m$ for $l<m$. We can start with this formula and substitute, one by one, instances of $\phi(x,b_i) \wedge \neg\phi(x,b_{i+1})$ by $\neg\phi(x,b_i) \wedge \phi(x,b_{i+1})$. After a finite number of steps we will clearly arrive at some formula of the form

$$\bigwedge_{i\in I_0} \neg\phi(x,b_i) \wedge \bigwedge_{i\in I_1} \phi(x,b_j)$$

with $I = I_0 \cup I_1$ and $I_0 < I_1$ which is consistent by our previous observation.

So for some $\eta : I \to 2$ and some k we have

$$\exists x \bigwedge_{1\le l\le k-1} \phi(x,b_{i_l})^{\eta(i_l)} \wedge \neg\phi(x,b_{i_k}) \wedge \phi(x,b_{i_{k+1}}) \wedge \bigwedge_{k+2\le l\le n} \phi(x,b_{i_l})^{\eta(i_l)}$$

is consistent but

$$\exists x \bigwedge_{1\le l\le k-1} \phi(x,b_{i_l})^{\eta(i_l)} \wedge \phi(x,b_{i_k}) \wedge \neg\phi(x,b_{i_{k+1}}) \wedge \bigwedge_{k+2\le l\le n} \phi(x,b_{i_l})^{\eta(i_l)}$$

is inconsistent.

[5] Shelah actually quotes Lemma 4.1 in the proof of Observation 1.37 in [] which means that he is aware of having proved the stronger result.

This implies that for any $q_1, q_2 \in \mathbb{Q}$ such that $i_{k-1} < q_1, q_2 < i_{k+2}$ we know that

$$\exists x \bigwedge_{1\le l\le k-1} \phi(x, b_{i_l})^{\eta(i_l)} \wedge \bigwedge_{k+2\le l\le n} \phi(x, b_{i_l})^{\eta(i_l)} \wedge \neg\phi(x, b_{q_1}) \wedge \phi(x, b_{q_2})$$

if and only if $q_1 < q_2$.

Let

$$\psi\left(x; b_{1_1}, \dots, b_{i_{k-1}}, b_{i_{k+2}}, \dots b_{i_n}\right) := \bigwedge_{1\le l\le k-1} \phi(x, b_{i_l})^{\eta(i_l)} \wedge \bigwedge_{k+2\le l\le n} \phi(x, b_{i_l})^{\eta(i_l)}.$$

So

$$\psi(x; \bar{c}) \wedge \neg\phi(x, b_{q_1}) \wedge \phi(x, b_{q_2})$$

is consistent if and only if $q_1 < q_2$ which implies that $\langle b_i \rangle_{i\in\mathbb{Q}\cap(b_{k-1}, b_{k+2})}$ witnesses that $\psi(x, \bar{c}) \wedge \phi(x, y)$ has the strict order property. □

Proposition 4.2. *Assume that M satisfies NIP. Then every weakly stable set in stable.*

Proof. Assume that $X = \phi(M, a)$. Consider the formula

$$\xi(x, z) := \delta(x, z) \wedge \phi(x, a).$$

If $\xi(x, z)$ is unstable then, by symmetry, so is the formula $\rho(z, x) := \xi(x, z)$. By Lemma 4.1, there is a formula $\theta(z, x)$ (with parameters) with the strict order property such that $\theta(z, x) \to \rho(z, x)$. We can now define a quasi ordering with infinite chains on the x's by:

$$\forall z(\theta(z, x_1) \to \theta(z, x_2)).$$

By our definition of ρ and ξ this quasi-ordering is a subset of X, contradicting the stability of the set X. □

Remark 4.3. *Notice that in the proof of Lemma 4.1 all the parameters in $\bar{c}$ were b_i's. This implies that we can prove something stronger than Proposition 4.2. Namely, given any model M of a theory T with NIP and a definable subset X of M, then X is stable if and only if the structure on X whose atomic relations are all the 0-definable (in M) subsets of X^n, $n \in \mathbb{N}$, is stable.*

X (this is, the substructure of M with universe X) is stable.

5 Examples

1. Consider the structure $\langle \mathbb{R}^2, +, P, Q\rangle$, where P is a predicate for the y-axis and Q is a predicate for the interval $(-1, 1)$ in the x-axis. Here, P is a maximal stable subgroup. The quotient $\mathbb{R}^2/P$ is purely unstable because it is a group isomorphic to $\langle \mathbb{R}, +\rangle$ together with a predicate for the interval $(-1, 1)$. It is easy to see that every interval of finite length can be definably linearly ordered in this quotient structure.

As this example shows, there is no purely unstable analogue to the existence of a "largest" normal stable subgroup. Actually, in this example there are no definable subgroups of G which are purely unstable.

2. If $\langle G, \cdot\rangle$ is a definably simple, definably compact, semi-algebraic group then it is purely unstable (with respect to the group structure).

Indeed, as was shown in [] that every semi-algebraic subset of G^n is definable in the pure group language. Thus, it is easy to see then that every definable infinite subset of G^k is unstable.

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