HIGHER CATEGORIES

8. Yoneda lemma. Applications

In the presentation of Yoneda lemma we follow [KV].

8.1. **Presheaves and Yoneda lemma.**

8.1.1. *The opposite ∞-category.* The functor op on totally ordered finite sets carries any such set to the same set with the opposite order. Considered as a functor $\mathrm{op}:\Delta\to\Delta$, it carries $[n]$ to itself, but carries d_i to d_{n-i} and s_i to s_{n-i}. Let us decide that, given $\mathcal{C}\in\mathtt{ssSet}$, its opposite $\mathcal{C}^{\mathrm{op}}$ will be the bisimplicial set obtained by precomposing it with $\mathrm{op}\times\mathrm{id}:\Delta\times\Delta\to\Delta\times\Delta$. It is clear that this operation carries CSS to CSS.

This means that the "spaces" $\mathcal{C}^{\mathrm{op}}_n$ and $\mathcal{C}_n$ coincide; only the faces and the degeneracies between them reshuffle.

Remark. This is not an obvious choice. For instance, it does not commute with the construction of classifying CSS of a category $C\mapsto B(C)$.

8.1.2. *Presheaves.* Given a CSS $\mathcal{C}$, we define a CSS $P(\mathcal{C})$ as $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$.

Our aim is to construct a fully faithful functor $Y:\mathcal{C}\to P(\mathcal{C})$ called Yoneda embedding.

8.1.3. *Twisted arrows.* We now combine the functor op with identity to get a new functor defined below.

Recall that for two conventional categories C and D their join $C\star D$ is defined by the formulas

$$\mathrm{Ob}(C\star D) = \mathrm{Ob}(C)\sqcup\mathrm{Ob}(D). \tag{1}$$

$$\mathrm{Hom}_{C\star D}(c,c') = \mathrm{Hom}_C(c,c');\ \mathrm{Hom}_{C\star D}(d,d')=\mathrm{Hom}_D(d,d'). \tag{2}$$

$$\mathrm{Hom}_{C\star D}(c,d) = \{*\};\ \mathrm{Hom}_{C\star D}(d,c)=\emptyset. \tag{3}$$

Let I be a finite totally ordered set. We define $\tau(I)=I^{\mathrm{op}}\star I$. This defines a functor $\tau:\Delta\to\Delta$ and a pair of natural transformations $\mathrm{id}\to\tau,\ \mathrm{op}\to\tau$ defined by the obvious embeddings $I\to\tau(I),\ I^{\mathrm{op}}\to\tau(I)$.

For a simplicial object X we define a new simplicial object $\mathrm{Tw}(X)$ as the composition

$$\Delta^{\mathrm{op}}\xrightarrow{\tau}\Delta^{\mathrm{op}}\xrightarrow{X}\mathtt{Set}.$$

As a result, we have a simplicial object $\mathrm{Tw}(X)$ endowed with a canonical map $p:\mathrm{Tw}(X)\to X\times X^{\mathrm{op}}$.

8.1.4. **Proposition.** *Let $\mathcal{C}$ be a Segal space. Then the map $p : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a left fibration.*

Proof. We skip the verification of the fact that p is a Reedy fibration.

We have to check that for any n the map

$$\mathrm{Tw}(\mathcal{C})_n \to \mathrm{Tw}(\mathcal{C})_0 \times_{\mathcal{C}_0 \times \mathcal{C}_0^{\mathrm{op}}} (\mathcal{C}_n \times \mathcal{C}_n^{\mathrm{op}}) \tag{4}$$

is a trivial fibration. The map (4) can be rewritten as the map

$$\mathrm{Map}(B, \mathcal{C}) \to \mathrm{Map}(A, \mathcal{C})$$

where $A = d\left((\Delta^0 \star \Delta^0) \coprod^{\Delta^0 \sqcup \Delta^0} (\Delta^n \sqcup (\Delta^n)^{\mathrm{op}})\right)$ and $B = d(\Delta^n \star \Delta^n)$. In other words, $A = d(\Delta^n \sqcup^{\Delta^0} \Delta^1 \sqcup^{\Delta^0} \Delta^n)$ and $B = d(\Delta^{2n+1})$. Such map is a trivial fibration for any Segal space $\mathcal{C}$. ☐

Now, given $x \in \mathcal{C}$, we denote $Y(x)$ the left fibration over $\mathcal{C}^{\mathrm{op}}$ obtained from $\mathrm{Tw}(\mathcal{C})$ via the base change with respect to the map

$$\mathcal{C}^{\mathrm{op}} = \{x\} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

8.1.5. Let fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ gives rise to a map

$$\widetilde{Y} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$$

which can be rewritten as a map

$$Y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) = P(\mathcal{C}).$$

This is Yoneda embedding. The image of $x \in \mathcal{C}$ is precisely the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ corresponding to the left fibration $Y(x)$.

The left fibration $Y(x) \to \mathcal{C}^{\mathrm{op}}$ has another description — this is the category opposite to the overcategory $\mathcal{C}_{/x}$. Here is the definition of the corresponding ∞-categorical notion.

8.1.6. **Definition.** Given a Reedy fibrant $\mathcal{C}$ and $x \in \mathcal{C}$, the "overcategory" $\mathcal{C}_{/x}$ is defined as the fiber of the map

$$\mathrm{Fun}(d(\Delta^1), \mathcal{C}) \to \mathcal{C}$$

induced by $\{1\} \to [1]$, at x.

8.1.7. **Exercise.** The map $\mathcal{C}_{/x} \to \mathcal{C}$ induced by the restriction along $\{0\} \to [1]$, is a right fibration.

In particular, if $\mathcal{C}$ is a CSS, so is $\mathcal{C}_{/x}$.

It turns out $Y(x)$ is equivalent to the left fibration

$$(\mathcal{C}_{/x})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}.$$

8.1.8. **Lemma.** *There is a natural homotopy equivalence $Y(x) \to (\mathcal{C}_{/x})^{\mathrm{op}}$ of left fibrations over $\mathcal{C}^{\mathrm{op}}$ carrying $\mathrm{id}_x \in Y(x)$ to $\mathrm{id}_x \in (\mathcal{C}_{/x})^{\mathrm{op}}$.*

Proof. Each one of the functors involved can be represented by a cosimplicial object in pointed spaces as follows.

We define $A^n = \Delta^0 \sqcup^{\Delta^n} \tau(\Delta^n)$. This is a pointed space (simplicial set) with the marked point Δ^0. We also define $B^n = \Delta^0 \sqcup^{\Delta^n} \mathrm{Fun}(\Delta^1, \Delta^n)$, also considered as a pointed space.

One has

$$Y(x)_n = \mathrm{Map}_x(d(A^n), \mathcal{C}) \text{ and } (\mathcal{C}_{/x})^{\mathrm{op}}_n = \mathrm{Map}_x(d(B^n), \mathcal{C})$$

functorially in n, where Map_x denotes the space of maps carrying the base point to x. One does not have an obvious map between A^n and B^n; instead one has the functorial maps $A^n \to C^n \leftarrow B^n$ inducing homotopy equivalences

$$\mathrm{Map}_x(d(A^n), \mathcal{C}) \leftarrow \mathrm{Map}_x(d(C^n), \mathcal{C}) \to \mathrm{Map}_x(d(B^n), \mathcal{C}).$$

The pointed spaces C^n are defined here as Δ^{n+1}, with the marked point $\{0\} \in [n+1]$. □

One has

8.1.9. **Proposition.** *Given $F \in P(\mathcal{C})$ and $x \in \mathcal{C}$, the natural map (evaluation)*

$$\mathrm{Map}_{P(\mathcal{C})}(Y(x), F) \to F(x)$$

is an equivalence.

Proof. Lemma 8.1.8 allows one to replace $Y(x)$ in the claim with the left fibration $(\mathcal{C}_{/x})^{\mathrm{op}}$.

Thus, we have to verify that the evaluation map induces an equivalence

$$\mathrm{Map}_{\mathrm{Left}(\mathcal{C}^{\mathrm{op}})}((\mathcal{C}_{/x})^{\mathrm{op}}, F) \to F(x).$$

We know that $\mathrm{Left}(\mathcal{C}^{\mathrm{op}})$ is a full subcategory of the CSS underlying $\mathtt{ssSet}_{/\mathcal{C}^{\mathrm{op}}}$. Thus, the map space on the right can be calculated in $\mathtt{ssSet}_{/\mathcal{C}^{\mathrm{op}}}$.

Since `ssSet` with Reedy model structure is a cartesian model category, and since the map $\{x\} \to (\mathcal{C}_{/x})^{\mathrm{op}}$ is a cofibration, one deduces that the map

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{op}}}((\mathcal{C}_{/x})^{\mathrm{op}}, F) \to F(x)$$

is a fibration. We will now prove it is a trivial fibration. We only have to verify that the fibers of the map are contractible. Let $f \in F(x)$. F is a left fibration, so the map

$$\mathrm{Fun}(d(\Delta^1), F) \to F \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(d(\Delta^1), \mathcal{C}^{\mathrm{op}})$$

is a trivial fibration. Its fiber at $f \in F$ is

$$F_{f/} \to (\mathcal{C}_{/x})^{\mathrm{op}}$$

and it is also a trivial fibration. Our fiber is just the space of sections of this trivial fibration. □

8.2. **Limits and colimits.** We will study colimits only as limits are obtained by passing to the opposite categories.

The most basic notion is that of colimit of an empty diagram.

8.2.1. **Definition.** An object $x\in\mathcal{C}$ is *initial* if $\mathrm{Map}(x,y)$ is contractible for any $y\in\mathcal{C}$.

8.2.2. **Proposition.** *The full subcategory of initial objects in $\mathcal{C}$ is either empty or contractible space.*

Proof. First of all, the projection of this category to its homotopy category is DK equivalence. Furthermore, the respective homotopy category has a unique isomorphism between any two objects. This implies the claim. □

More general colimits are defined using a general notion of undercategory.

8.2.3. *Undercategories.* Let $f:K\to\mathcal{C}$ be a functor. We define $\mathcal{C}_{f/}$ as the fiber product

$$\{f\}\times_{\mathrm{Fun}(K,\mathcal{C})}\mathrm{Fun}(d(\Delta^1)\times K,\mathcal{C})\times_{\mathrm{Fun}(K,\mathcal{C})}\mathcal{C}, \tag{5}$$

where the projections $\mathrm{Fun}(d(\Delta^1)\times K,\mathcal{C})\to\mathrm{Fun}(K,\mathcal{C})$ are given by embeddings $\{0\}\to[1]$ and $\{1\}\to[1]$, and the map $\mathcal{C}\to\mathrm{Fun}(K,\mathcal{C})$ is given by $K\to *$.

The special case $K=*$ played an important role in the theory of left fibrations and in Yoneda lemma. One has in general the following.

8.2.4. **Lemma.** *For $\mathcal{C}$ Segal space the map $\mathcal{C}_{f/}\to\mathcal{C}$ is a left fibration. In particular, if $\mathcal{C}$ is CSS, $\mathcal{C}_{f/}$ is also CSS.*

Proof. The map $\mathcal{C}_{f/}\to\mathcal{C}$ is obtained by base change from $\mathcal{D}_{f/}\to\mathcal{D}$ where $f\in\mathcal{D}=\mathrm{Fun}(K,\mathcal{C})$. Thus, the result follows from Exercise 8.1.7. □

8.2.5. *Colimits.* Given a functor $f:K\to\mathcal{C}$, its colimit is an initial object of the category $\mathcal{C}_{f/}$.

8.2.6. **Exercise.** Prove that f admits a colimit iff the presheaf on $\mathcal{C}^{\mathrm{op}}$ determined by $\mathcal{C}_{f/}$ is representable.

8.3. **Adjoint functors.** The notion of adjoint pair of functors between conventional categories does not automatically fit our picture as it includes morphisms of functors (unit and counit) which are not isomorphisms, so that adjointness is a priori 2-categorical notion. Fortunately, Yoneda lemma allows one to avoid usage of 2-categorical notions.

8.3.1. For conventional categories an adjoint pair $F:C\rightleftarrows D:G$ is uniquely defined by a bifunctor $C^{\mathrm{op}}\times D\to\mathtt{Set}$ carrying a pair (c,d) to the set $\mathrm{Hom}_D(F(c),d)=\mathrm{Hom}_C(c,G(d))$. So, it is natural to expect that an adjunction between two infinity categories can be defined as a left fibration on $C^{\mathrm{op}}\times D$ satifying some representability conditions. Details are explaned below for CSS.

8.3.2. Given a pair of CSS $\mathcal{C}, \mathcal{D}$, a correspondence from $\mathcal{C}$ to $\mathcal{D}$ is a left fibration $p : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. Such correspondence is called left-representable if for each $x \in \mathcal{C}$ the base change of p with respect to $\mathcal{D} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ determined by x, defines a representable presheaf on $\mathcal{D}^{\mathrm{op}}$.

A correspondence $p : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ is called right-representable if for each $y \in \mathcal{D}$ the base change of p with respect to morphism $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$, determined by y, corresponds to a representable presheaf on $\mathcal{C}$.

8.3.3. **Definition.** A left fibration $\mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ determines an adjoint pair between $\mathcal{C}$ and $\mathcal{D}$ if p is both left and right representable.

A correspondence $p : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ can be interpreted as a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ or as $p_C : \mathcal{C}^{\mathrm{op}} \to P(\mathcal{D}^{\mathrm{op}})$ or even as $p_D : \mathcal{D} \to P(\mathcal{C})$. A left fibration $p : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ is left representable if the functor p_C factors through $\mathcal{D}^{\mathrm{op}}$. It is right representable if p_D factors through $\mathcal{C}$.

8.3.4. *Existence of adjoint.* A functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a composition $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \to P(\mathcal{D}^{\mathrm{op}})$, that is, to a left-representable left fibration $p : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. We say that F admits right adjoint if p is also right-representable. In this case the functor $\mathcal{D} \to P(\mathcal{C})$ corresponding to p, can be factored (uniquely up to contractible space of choices) through a functor $G : \mathcal{D} \to \mathcal{C}$ called *the functor right adjoint to F*.

By definition, in order to verify that F admits right adjoint, it is sufficient to verify that for any $d \in \mathcal{D}$ the composition $\mathcal{C} \to P(\mathcal{D}) \overset{ev_d}{\to} \mathcal{S}$, ev_d being the evaluation at d functor, is (co)representable.

8.3.5. *Colimits and adjoint functors.* One has the following interpretation of colimits in terms of adjoint functors.

Let $\mathcal{C}$ be a CSS and let $K \in \mathtt{ssSet}$.

Proposition. *The following conditions are equivalent.*

1. *Any functor $f : K \to \mathcal{C}$ has a colimit.*
2. *The functor $G : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$, induced by the map $K \to *$, has a left adjoint.*

Proof. Existence of adjoint is verified objectwise. That is, to verify the existence of adjoint, one has to verify that for any $f : K \to \mathcal{C}$ the respective left fibration on $\mathcal{C}^{\mathrm{op}}$ is representable.

The functor $G : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ yields a left fibration over $\mathrm{Fun}(K, \mathcal{C})^{\mathrm{op}} \times \mathcal{C}$ which we will denote as $\mathcal{E}$. The base change $\mathcal{E}_f$ of $\mathcal{E}$ defined by $f \in \mathrm{Fun}(K, \mathcal{C})$ is a left fibration on $\mathcal{C}$. It is homotopy equivalent to $\mathcal{C}_{f/}$ (this is a version of 8.1.8), so its representability is equivalent to existence of initial object in $\mathcal{C}_{f/}$. $\square$

8.3.6. *A source of correspondences.* Let $f : \mathcal{C} \to d(\Delta^1)$ be a Reedy fibration. Denote $\mathcal{C}_0$ and $\mathcal{C}_1$ the fibers of f at 0 and 1 respectively. Yoneda embedding gives rise to a map

$$\mathcal{C}_1 \to \mathcal{C} \xrightarrow{Y} P(\mathcal{C}) \to P(\mathcal{C}_0),$$

where the last arrow is the restriction of a presheaf to a subcategory. In other words, any functor as above gives rise to a correspondence. Later we will see that the opposite is also true.

8.4. **Quillen adjunction.** Given a model category $\mathcal{C}$ with a collection of weak equivalences W, we assigned a CSS as fibrant replacement $B^f(\mathcal{C}, W)$ of Rezk nerve of the relative category $(\mathcal{C}, W)$. In what follows we will denote this CSS model (or any other ∞-categorical model) $L(\mathcal{C})$.

We will show now that a Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

gives rise to an adjunction of the respective CSS. For understandable reasons, we will denote the respective functors

$$\mathbf{L}F : L(\mathcal{C}) \rightleftarrows L(\mathcal{D}) : \mathbf{R}G.$$

We proceed as follows. We realize $L(\mathcal{C})$ as a simplicial localization $L^H(\mathcal{C}^c, W)$ and $L(\mathcal{D})$ as $L^H(\mathcal{D}^f)$ [1]. Then we construct a simplicial category $\mathcal{M}$ with a functor $\mathcal{M} \to [1]$ as follows.

The fiber at 0 is $L^H(\mathcal{C}^c, W)^f$ (the functorial fibrant replacement in `sCat`) and the fiber at 1 is $L^H(\mathcal{D}^f, W)^f$. Finally, for $c \in \mathcal{C}$ and $d \in \mathcal{D}$ we define $\mathrm{Map}_{\mathcal{M}}(c, d) = \mathrm{Map}_{L^H(\mathcal{D},W)^f}(F(c), d)$. Compositions are defined and are strictly associative by functoriality of construction of hammock localization. Passing to CSS, we get a CSS over $d(\Delta^1)$ and we claim that it defines a required adjoint pair. To do so, we have to verify left and right representability of the respective correspondence.

Details can be found in [H.L].

8.5. **DK localization as ∞-localization.** Using the ∞-categorical notion of adjoint functor, we will be able to define localization of infinity categories. Then we will find out that DK localization calculates this ∞-version of localization.

8.5.1. *Spaces and categories.* Recall that $\mathtt{Cat}_\infty$ is the infinity category of (small) infinity categories (e.g., of CSS), and $\mathcal{S}$ is the full subcategory of spaces (realized, for instance, as essentially constant CSS). We will see that the embedding $\mathcal{S} \to \mathtt{Cat}_\infty$ admits both left and right adjoint.

The easiest way to present an adjoint is via Quillen adjunction. One has the Quillen adjunctions

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[1] Recall that $\mathcal{C}^c$(resp., $\mathcal{C}^f$) is the full subcategory of cofibrant (resp., fibrant) objects in $\mathcal{C}$.

$$\texttt{sSet}\rightleftarrows(\texttt{ssSet},R)\rightleftarrows(\texttt{ssSet},CSS), \tag{6}$$

with the first right adjoint functor carrying $X \in \texttt{ssSet}$ to X_0 and the second Quillen adjunction being Bousfield localization.

Thus, the embedding of $\mathcal{S}$ into $\texttt{Cat}_\infty$ has a right adjoint assigning to an infinity category $\mathcal{C}$ its maximal subspace $\mathcal{C}^{eq}$, the space of its objects (or, equivalently, the infinity category obtained by discarding non-equivalences).

We will now describe the left adjoint functor. Here is the easiest way.

The standard model structure on the simplicial sets can be considered as a Bousfield localization of the Joyal model structure. This yields a Quillen pair

$$\mathrm{id} : (\texttt{sSet}, J)\rightleftarrows(\texttt{sSet}, Q) : \mathrm{id}$$

with the right adjoint functor representing the embedding $\mathcal{S} \to \texttt{Cat}_\infty$. Thus, the left adjoint functor is the respective left derived functor of id. In this model it consists of replacing a quasicategory with its Kan fibrant replacement.

The above construction is equivalent to Dwyer-Kan "total localization". In fact, let $\mathcal{C}$ be a fibrant simplicial category. Its total localization is a Kan fibrant replacement of the nerve $\mathcal{N}(\mathcal{C})$. Total DK localization can be described as the total localization of a cofibrant replacement $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. According to Dwyer-Kan, their localization does not alter the homotopy type of the nerve; since total localization of $L(\widetilde{\mathcal{C}},\widetilde{\mathcal{C}})$ is a simplicial groupoid, it becomes Kan after the application of $\mathcal{N}$.

Thus, total ∞-localization is represented by total DK localization.

8.5.2. *∞-localization.* The functor of maximal subspace defined above yields a functor $K : \texttt{Cat}_\infty \to \mathrm{Fun}(\Delta^1, \texttt{Cat}_\infty)$ carrying $\mathcal{C}$ to the embedding of the maximal subspace $\mathcal{C}^{eq}$ of $\mathcal{C}$ into $\mathcal{C}$.

We define the general localization as the functor

$$L : \mathrm{Fun}(\Delta^1, \texttt{Cat}_\infty) \to \texttt{Cat}_\infty$$

left adjoint to K. One can easily see that L carries a morphism $\mathcal{W} \to \mathcal{C}$ to the pushout $L(\mathcal{W}) \sqcup^{\mathcal{W}} \mathcal{C}$.

This implies that DK localization represents also this more general version of localization.

8.6. **Functor categories.** Let $\mathcal{C}$ be a combinatorial model category. Let I be a conventional category. By the universal property, we get a canonical map

$$L(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, L(\mathcal{C})). \tag{7}$$

8.6.1. **Theorem.** *This map is an equivalence.*

Proof. The proof is based on the following very important result of D. Dugger [D]. Let J be a small category. Dugger endows the category of simplicial presheaves $U(J) := \mathrm{Fun}(J^{\mathrm{op}}, \mathtt{sSet})$ with the projective model structure; then he proves that for any combinatorial model category $\mathcal{C}$ there exists a Quillen pair $F : U(J) \rightleftarrows \mathcal{C} : G$ and an isomorphism of functors $\mathbf{L}F \circ \mathbf{R}G \to \mathrm{id}_{\mathcal{C}}$. This allows him to construct a Quillen equivalence between a certain Bousfield localization of $U(J)$ and $\mathcal{C}$.

Now the proof goes as follows. First of all, a Quillen equivalence $\mathcal{C} \rightleftarrows \mathcal{D}$ gives rise to equivalence of $L(\mathcal{C})$ and $L(\mathcal{D})$. Thus, we can assume that $\mathcal{C}$ is a Bousfield localization of $U(J)$. Next, a Bousfield localization $M \rightleftarrows M_{\mathrm{loc}}$ identifies the underlying infinity category $L(M_{\mathrm{loc}})$ with the full subcategory of $L(M)$ spanned by S-local objects. Thus, both $\mathrm{Fun}(I, L(\mathcal{C}))$ and $L(\mathrm{Fun}(I, \mathcal{C}))$ are full subcategories of $\mathrm{Fun}(I, L(U(J)))$ and $L(\mathrm{Fun}(I, U(J)))$ respectively. This reduces the claim to the case $\mathcal{C} = \mathtt{sSet}$.

It remains to verify the claim for $\mathcal{C} = \mathtt{sSet}$.

In this case $L(\mathcal{C})$ can be represented by the simplicial category Kan_* of Kan simplicial sets and so $\mathrm{Fun}(I, L(\mathcal{C}))$ is the category of simplicial functors $\mathfrak{C}(I) \to \mathrm{Kan}_*$. The left-hand side is the simplicial category of functors $I \to \mathrm{Kan}_*$. The fact that any simplicial functor $\mathfrak{C}(I) \to \mathrm{Kan}_*$ is equivalent to a genuine functor, is classical. We need a more precise statement which is proven in [L.T], A.3.4, for any simplicial combinatorial model category $\mathcal{C}$. The proof is based on an explicit construction of a path object for the fibrant simplicial category $\mathcal{C}_*^{cf} \in \mathtt{sCat}$.

□

8.7. **(Co)limits.** Let I be a category and let $\mathcal{C}$ be a combinatorial model category. We want to explain that I-indexed limits and colimits in $L(\mathcal{C})$ can be expressed in terms of derived limits and colimits in $\mathcal{C}$.

The functor $c : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ induced by the projection $I \to *$, gives rise to two Quillen pairs.

The first one,

$$\mathrm{colim} : \mathrm{Fun}(I, \mathcal{C}) \rightleftarrows \mathcal{C} : c, \tag{8}$$

identifies $\mathbf{L}\,\mathrm{colim}$ with the functor left adjoint to the constant functor $L(\mathcal{C}) \to L(\mathrm{Fun}(I, \mathcal{C})) = \mathrm{Fun}(I, L(\mathcal{C}))$, that is with the colimit functor between the respective infinity categories.

The second,

$$c : \mathcal{C} \rightleftarrows \mathrm{Fun}(I, \mathcal{C}) : \lim, \tag{9}$$

identifies $\mathbf{R}\lim$ with the limit functor between the respective infinity categories.

8.7.1. **Exercise.**
1. Prove that an object $x \in \mathcal{C}$ is initial iff the left fibration $\mathcal{C}_{x/} \to \mathcal{C}$ is an equivalence.
2. See 8.2.6.

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