HIGHER CATEGORIES

Abstract.

1. Introduction. Categories and simplicial sets

1.1. **Introduction.** In topology and homological algebra the language of categories is widely used. However, this language does not completely fulfills the needs. We would like to demonstrate this. This will lead to another type of category theory — the theory of infinity categories. This is the aim of the present course.

1.1.1. Category theory appeared in algebraic topology which studies algebraic invariants of topological spaces. From the very beginning it was well understood that what is important is not just to assign a, say, abelian group to a topogical space, but to make sure that this assignment is functorial, that is carries a continuous map of topological spaces to a homomorphism of groups. Thus, singular homology appears as a functor

$$H:\mathtt{Top}\to\mathtt{Ab}$$

from the category of topological spaces to the category of abelian groups. The next thing to do is to realize that the map $H(f):H(X)\to H(Y)$ does not really depend on $f:X\to Y$, but of equivalence class of f up to homotopy. To make our language as close as possible to the problems we are trying to solve, we may replace the category $\mathtt{Top}$ of topological spaces, factoring the sets $\mathrm{Hom}_{\mathtt{Top}}(X,Y)$ by the homotopy relation. We will get another meaningful category which should better describe the object of study of algebraic topology — but this category has some very unpleasant properties (lack of limits). Another approach, which is closer to the one advocated by infinity-category theory, is to think of the sets $\mathrm{Hom}_{\mathtt{Top}}(X,Y)$ as topological spaces, so that information on homotopies between the maps is encoded in the topology of Hom-sets.

This will lead us, in this course, to define an infinity category of spaces describing, roughly speaking, topological spaces up to homotopy, and this infinity category will be an absolutely canonical, therefore, uniquely defined, object, which can be defined without mentioning a definition of topological space.

1.1.2. We are still with topological spaces, but we will now think about a single topological space X instead of totality of all topological spaces. We want to look at a topological space as a (sort of) infinity groupoid [1].

Let X be a topological space. One defines a groupoid $\Pi_1(X)$ whose objects are the points of X and arrows are the homotopy classes of paths. An interesting property — $\Pi_1(X)$ retains information on π_0 and π_1 of X but forgets all the rest. The reason for this is clear: we took homotopy classes of paths as arrows. Could we have taken instead topological spaces of paths — we would have chance to retain all information about the homotopy type of X. This makes perfect sense in infinity catgeory theory — where spaces and infinity groupoids become just the same thing.

1.1.3. Here is a more algebraic example.

An important notion of homological algebra — the notion of derived functor. Study of derived functors led to the notion of derived category in which the standard ambiguity in the choice of resolutions needed to calculate derived functors, disappears.

Here is, in two paragraphs, the construction of derived category of an abelian category $\mathcal{A}$. As a first step, one constructs the category of complexes $C(\mathcal{A})$ where all projective resolutions, their images after application of functors, live. The second step is similar to the notion of localization of a ring: given a ring R and a collection S of its elements, one defines a ring homomorphism $R \to R[S^{-1}]$ such that the image of each element in S becomes invertible in $R[S^{-1}]$ and universal for this property. Localizing $C(\mathcal{A})$ with respect to the collection of quasiisomorphisms, we get $D(\mathcal{A})$ — the derived category of $\mathcal{A}$.

The construction of derived category $D(\mathcal{A})$ is a close relative to the construction of the category of topological spaces, when we factor the set of continuous maps by an equivalence relation. The notion of derived category is not very convenient, approximately for the same reasons we already mentioned. Here is another inconvenience.

1.1.4. Let $\mathcal{A}$ be the category of abelian sheaves on a topological space X. For an open subset $U \subset X$ let $\mathcal{A}_U$ be the category of sheaves on U. There is a very precise procedure how one can glue, given sheaves $F_U \in \mathcal{A}_U$ and some "gluing data", a sheaf $F \in \mathcal{A}$ whose restrictions to U are F_U. The collection of $\mathcal{A}_U$ is also a sort of a sheaf (of categories). It is still possible to glue $\mathcal{A}$ from the collection of $\mathcal{A}_U$ (even though one needs "2-gluing data" for this [2]). We would be happy to be able to glue the derived categories $D(\mathcal{A}_U)$ into the global $D(\mathcal{A})$. It turns out this is in fact possible, but one needs to replace the derived category with a more refined infinity notion, and of course, use even "higher" gluing data.

[1] Groupoid is just a category whose arrows are invertible.

[2] The reason is that categories $\mathcal{A}_U$ are the objects of a 2-category, that of categories.

1.1.5. There is a pleasant "side effect" in replacing the derived category $D(\mathcal{A})$ with an infinity category. As it is well-known, $D(\mathcal{A})$ is a triangulated category, a very nontrivial and (but) very non-natural notion. The respective notion for infinity categories is very natural: this is just a property of infinity category (stability) rather than a collection of extra structures (shift functor, collection of exact triangles, etc.)

1.2. **Categories. Simplicial sets.**

1.2.1. *First definitions.* Categories, functors, category of functors. Equivalence of categories.

In short: category $\mathcal{C}$ has a set (sometimes big) $\mathrm{Ob}\,\mathcal{C}$, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(x,y)$ for each pair of objects x,y of $\mathcal{C}$, an associative composition

$$\mathrm{Hom}(y,z)\times\mathrm{Hom}(x,y)\to\mathrm{Hom}(x,z)$$

for each triple of objects, units $\mathrm{id}_x\in\mathrm{Hom}(x,x)$.

For two categories $\mathcal{C}$, $\mathcal{D}$ a functor $f:\mathcal{C}\to\mathcal{D}$ is a map $f:\mathrm{Ob}\,\mathcal{C}\to\mathrm{Ob}\,\mathcal{D}$ and a collection of maps $\mathrm{Hom}_{\mathcal{C}}(x,y)\to\mathrm{Hom}_{\mathcal{D}}(fx,fy)$ compatible with compositions.

The functors from $\mathcal{C}$ to $\mathcal{D}$ form a new category denoted $\mathrm{Fun}(\mathcal{C},\mathcal{D})$: its objects are the functors, and $\mathrm{Hom}(f,g)$ is defined as the collection of morphisms of functors defined as follows.

A morphism $u:f\to g$ assigns to each $x\in\mathrm{Ob}\,\mathcal{C}$ an arrow $u(x)\in\mathrm{Hom}_{\mathcal{D}}(f(x),g(x))$ such that for any arrow $a\in\mathrm{Hom}_{\mathcal{C}}(x,y)$ the diagram in $\mathcal{D}$ presented below is commutative.

$$\begin{array}{ccc} f(x) & \xrightarrow{u(x)} & g(x) \\ {\scriptstyle f(a)}\big\downarrow & & {\scriptstyle g(a)}\big\downarrow \\ f(y) & \xrightarrow{u(y)} & g(y) \end{array}.$$

One can compose functors — so that the categories form a category `Cat`. However, this notion has almost no sense. The reason for this is that most of categorical constructions are defined "up to" canonical isomorphism. For instance,

1.2.2. **Definition.** Let $x,y\in\mathcal{C}$. Their product is an object p together with a pair of arrows $p\to x$ and $p\to y$, satisfying a universal property: for any $q,q\to x,q\to y$ there exists a unique arrow $q\to p$ such that the diagrams are commutative.

1.2.3. **Lemma.** *A product, if exists, is unique up to a unique isomorphism.*

This leads to the following important notion.

1.2.4. **Definition.** A functor $f:\mathcal{C}\to\mathcal{D}$ is an equivalence if there exists a functor $g:\mathcal{D}\to\mathcal{C}$ and a pair of isomorphisms of functors

$$g\circ f\xrightarrow{\sim}\mathrm{id}_{\mathcal{C}},\ f\circ g\xrightarrow{\sim}\mathrm{id}_{\mathcal{C}}.$$

If you believe in Axiom of choice (I do), here is an equivalent definition.

1.2.5. **Definition.** A functor $f : \mathcal{C} \to \mathcal{D}$ is an equivalence if

- It is essentially surjective, that is for any $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ and an isomorphism $f(x) \xrightarrow{\sim} y$.
- It is fully faithful, that is for all $x, x' \in \mathcal{C}$ the map
$$\mathrm{Hom}_{\mathcal{C}}(x, x') \to \mathrm{Hom}_{\mathcal{D}}(fx, fx')$$
is an isomorphism.

Remark. For an equivalence f the functor g, "quasi-inverse" to f, is not defined uniquely — but uniquely up to unique isomorphism. We left this as an exercise.

1.2.6. *Yoneda lemma. Representable functors.* Probably the most important example of category is the category of sets, denoted `Set`.

Sometimes functors do not preserve arrows, but invert them. This justifies the following definition.

Definition. Opposite category $\mathcal{C}^{\mathrm{op}}$. It has the same objects and inverted morphisms:
$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x).$$

Functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ are sometimes called the contravariant functors.

Definition. $P(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathtt{Set})$ — the category of presheaves.

Here is the origin of the name. Let X be a topological space. A sheaf on X (of, say, abelian groups) F assigns to each open set $U \subset X$ an abelian group $F(U)$, and for $V \subset U$ a homomorphism $F(U) \to F(V)$ (restriction of a section to an open subset) such that certain additional gluing properties are satisfied. Presheaf is a collection of abelian groups $F(U)$ with restruction maps without extra gluing properties. In our terms, this is just a contravariant functor from the category of open subsets of X to the abelian groups.

We define Yoneda embedding as the functor
$$Y : \mathcal{C} \to P(\mathcal{C})$$
carrying $x \in \mathcal{C}$ to the functor $Y(x)$, $Y(x)(y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$.

Lemma. *Yoneda embedding is fully faithful.*

Meaning: in order to describe an object $x \in \mathcal{C}$ up to unique isomorphism, it suffices to describe the functor $Y(x)$ (called: the functor represented by x).

A presheaf isomorphic to $Y(x)$ for some x is called representable presheaf.

Yoneda lemma is a direct consequence of yet stronger result which is also called Yoneda lemma.

Lemma. *Let $\mathcal{C}$ be a category and $F \in P(\mathcal{C})$. Then for any $x \in \mathcal{C}$ the map*

$$\mathrm{Hom}_{P(\mathcal{C})}(Y(x), F) \to F(x)$$

carrying any morphism of functors $a : Y(x) \to F$ to $a(\mathrm{id}_x) \in F(x)$, is a bijection.

We suggest to prove the lemma as an exercise. An important step in our course will be an infinity version of Yoneda lemma.

1.2.7. *Adjoint functors.* Here is a standard definition.

Definition. A pair of functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$, together with morphisms $\alpha : L \circ R \to \mathrm{id}_D$, $\beta : \mathrm{id}_{\mathcal{C}} \to R \circ L$, is called *an adjoint pair* if the compositions below give identity of L and of R respectively.

$$L \stackrel{1\circ\beta}{\to} LRL \stackrel{\alpha\circ 1}{\to} L \tag{1}$$

$$R \stackrel{\beta\circ 1}{\to} RLR \stackrel{1\circ\alpha}{\to} R \tag{2}$$

Here is a more digestible definition: this is a pair of functors L, R, together with a natural isomorphism (=isomorphism of bi-functors)

$$\mathrm{Hom}_{\mathcal{D}}(Lx, y) = \mathrm{Hom}_{\mathcal{C}}(x, Ry).$$

Thus, a primary datum is a bifunctor

$$F : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathtt{Set}.$$

This bifunctor can be equivalently rewritten as a functor $\mathcal{C} \to P(\mathcal{D})$ or as a functor $\mathcal{D} \to P(\mathcal{C})$. By Yoneda lemma, if the first functor has its essential image in $\mathcal{D} \subset P(\mathcal{D})$, there is a unique functor L up to unique isomorphism presenting F as

$$F(x, y) = \mathrm{Hom}_{\mathcal{D}}(L(x), y).$$

Similarly, if the second functor has essential image in $\mathcal{C} \subset P(\mathcal{C})$, there exists $R : \mathcal{D} \to \mathcal{C}$ so that $F(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, R(y))$.

Corollary.
1. *If $L : \mathcal{C} \to \mathcal{D}$ admits a right adjoint functor, it is unique up to unique isomorphism.*
2. *L admits a right adjoint iff for any $y \in \mathcal{D}$ the functor $x \mapsto \mathrm{Hom}_{\mathcal{D}}(L(x), y)$ is representable.*

Proof. Exercise. □

1.3. **Exercises.** Prove everything formulated above without proof.

1.4. **Category Δ. Category sSet.**

1.4.1. *Category* Δ. Δ is a very important category, "the category of combinatorial simplices".

Its objects are $[n] = \{0, \ldots, n\}$, considered as ordered sets. Morphisms are maps of ordered sets (preserving the order). In particular, $[0]$ consists of one element and so is the terminal object in Δ.

By the way,

Definition. An object $x \in \mathcal{C}$ is terminal if $\mathrm{Hom}_{\mathcal{C}}(y, x)$ is a singleton for all y. An object x is initial if $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a singleton for all y.

The category Δ has no nontrivial isomorphisms. This is sometimes convenient; otherwise I would prefer to define Δ as the category of totally ordered finite nonempty sets. It would be equivalent to the one we defined, but would look more natural.

Here are special arrows in Δ.

Faces $\delta^i : [n-1] \to [n]$, the injective map missing the value $i \in [0, n]$.

Degeneracies $\sigma^i : [n] \to [n-1]$ the surjective map for which the value $i \in [0, n-1]$ is repeated twice.

Any map $[m] \to [n]$ can be uniquely presented as a surjective map followed by an injective map. Any injective map is a composition of faces, and any surjective map is a composition of degeneracies.

The latter presentations are not unique. For instance, $\delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}$ for $i < j$.

Exercise. Prove this. Try to find and prove all the identities.

Definition. A simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. A simpicial object in sets is called a simplicial set. The category of simplicial sets will be denoted `sSet`. In other words, $\mathtt{sSet} = P(\Delta)$.

The category of simplicial sets is the one where most of the homotopy theory lives. Let us describe in more detail what is a simplicial set.

To each $[n]$ it assigns a set X_n called "the set of n-simplices of X". Any map $\alpha : [m] \to [n]$ defines $\alpha^* : X_n \to X_m$. In particular, we will usually denote $d_i = (\delta^i)^*$ and $s_i = (\sigma^i)^*$. Here how a simplicial set looks like (this is only a small part of it):

Draw X_0, X_1, X_2 and the maps between them.

The only examples of simplicial sets that we can produce immediately are representable by the objects of Δ: any object $[n]$ defines a simplicial set Δ^n whose m-simplices are maps $[m] \to [n]$.

1.5. **Singular simplices. Nerve of a category.** Simplicial sets and, more generally, simplicial objects, are everywhere, but at the moment we can hardly give an example of such.

1.5.1. *Singular simplices.* Let X be a topological space. We assign to it a simplicial set $\mathrm{Sing}(X)$ as follows. The set of n-simplices $\mathrm{Sing}_n(X)$ is the set of continuous maps from the standard n-simplex

$$\Delta[n] = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0,\ \sum x_i = 1\}$$

to X.

The faces and the degeneracies are defined via maps

$$\delta^i : \Delta[n-1] \to \Delta[n]$$

and

$$\sigma^i : \Delta[n] \to \Delta[n-1]$$

where δ^i inserts 0 at the place i and σ^i puts $x_i + x_{i+1}$ at the place i.

1.5.2. *Nerve of a category.* There is a very similar construction in the world of categories — this is something that allows one to guess that categories and topological spaces are somehow connected.

Given a category $\mathcal{C}$, we define a simplicial set $N(\mathcal{C})$, the nerve of $\mathcal{C}$, as follows. Its n-simplices are functors $[n] \to \mathcal{C}$ where $[n]$ is now considered as the category defined by the corresponding ordered set (the objects are numbers $0, \ldots, n$, and there is a unique arrow $i \to j$ for $i \leq j$.)

1.5.3. *Example: BG.* Let G be a discrete group. Denote BG the category having one object and G as its group of automorphisms. A functor $BG \to \mathtt{Vect}$ is the same as representation of G.

Nerve of BG is the simplicial set whose n-simplices are sequences of n elements of G; degeneracies insert $1 \in G$ and faces d_i, $i = 1, \ldots, n-1$ multiply two neghboring elements of G.

What do d_0 and d_n do?

1.5.4. *Geometric realization.* The functor $\mathrm{Sing} : \mathtt{Top} \to \mathtt{sSet}$ admits a left adjoint (called geometric realization and denoted $|X|$ for $X \in \mathtt{sSet}$.)

We already know that it is sufficient to check that for each $X \in \mathtt{sSet}$ the functor $\mathtt{Top} \to \mathtt{Set}$ carrying T to $\mathrm{Hom}(X, \mathrm{Sing}(T))$, is (co)representable. We definitely know this for $X = \Delta^n$ — then by definition the functor is corepresented by $\Delta[n]$.

We will prove existence of left adjoint after a discussion of colimits (=inductive limits). Meanwhile, we calculated $|\Delta^n| = \Delta[n]$ which is very nice.

1.6. **Topological spaces versus simplicial sets.**

1.6.1. *Generalities: limits and colimits.* Examples: intersection or union of decreasing or incresing sequence of sets, fiber product et cetera.

General setup: given a functor $F : I \to \mathcal{C}$ we are looking for a universal object $x \in \mathcal{C}$ endowed with compatible collection of maps $F(i) \to x$ (explain what is compatible) and universal with respect to this property (explain universality). This object x with all extra information (the maps $F(i) \to x$) is called the colimit of F.

One can describe the notion of (co)limit using the language of adjoint functors.

Functors from I to $\mathcal{C}$ live in $\mathrm{Fun}(I, \mathcal{C})$; one has an obvious functor

$$\mathrm{Fun}(I, \mathcal{C}) \leftarrow \mathcal{C} : \mathrm{const} \tag{3}$$

assigning to any object $x \in \mathcal{C}$ the constant functor with value x. Then colimit and limit appear as left and right adjoint functors to const.

Of course, limits and colimits do not always exist.

Note that for $F \in \mathrm{Fun}(I, \mathcal{C})$, a canonical adjunction yields a map $F \to \mathrm{const}(\mathrm{colim}(F))$. This is to stress that the compatible collection of maps $F(i) \to \mathrm{colim}\, F$ is a part of the data for $\mathrm{colim}\, F$.

1.6.2. *Colimits versus adjoint functors: cont.* Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair of functors. Let $a : I \to \mathcal{C}$ be a functor and let $A = \mathrm{colim}\ a$. Then $F(a)$ is a colimit of $F \circ a$. We explained that a left adjoint functor preserves all colimits (that exist in $\mathcal{C}$). Dually, a right adjoint functor preserves limits.

1.6.3. *Adjoint pair of functors between topological spaces and simplicial sets.* The functor $\mathrm{Sing} : \mathtt{Top} \to \mathtt{sSet}$ admits left adjoint called geometric realization. By definition, geometric realization $|X|$ of a simplicial set X satisfies the following universal property:

$$\mathrm{Hom}_{\mathtt{Top}}(|X|, Y) = \mathrm{Hom}_{\mathtt{sSet}}(X, \mathrm{Sing}(Y)). \tag{4}$$

The right-hand side is a compatible collection of continuous maps $\tilde{x} : \Delta[n] \to Y$ given for each $x \in X_n$, the compatibility meaning that for any $a : [m] \to [n]$ and $x \in X_n,\ y = a^*(x) \in X_m$ one has $\tilde{y} = \tilde{x} \circ a$. This means that $|X|$ is a colimit of the functor we will now define.

The functor is defined on the category whose objects are pairs $(n, x \in X_n)$ and whose arrows are pairs (a, x) where $a : [m] \to [n]$ is an arrow in Δ and $x \in X_n$. We denote this category $N_*(X)$ — the category of simplices in X. The functor $F_X : N_*(X) \to \mathtt{Top}$ assigns to (a, x) the standard (topological) n-simplex $\Delta[n]$. It is easy to see that one has

$$|X| = \mathrm{colim}\, F_X.$$

1.7. **First notions in homotopy theory.**

Weak equivalences, Kan fibrations, Kan simplicial sets. Singular simplices of a topological space are Kan.

The categories of topological spaces and of simplicial sets are not equivalent; but they both are good to study homotopical properties of topological spaces. Thus, they should be equivalent in another, "homotopic" sense. There are different ways to express this. We will present one of them in some detail later — they are Quillen equivalent.

Before formally presenting necessary machinery, we will try to describe most of features of this equivalence.

In a few words, there is a notion of weak equivalence in both categories so that the adjoint functors induce an equivalence of respective localizations. We will discuss localization later. We will now mention some standard notions in topology.

1.7.1. *Homotopy equivalence.* Two maps $f, g : X \to Y$ in `Top` are homotopic if there is a (continuous) map $F : X \times I \to Y$ which restricts to f and g at $0, 1$. A map $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that the two compositions are homotopic to the respective identity.

1.7.2. *Homotopy groups.* Fix $n > 0$. The n-th homotopy group $\pi_n(X, x)$ (of a pointed space $(X, x \in X)$) is the set of equivalence classes of maps $(D^n, \partial D^n) \to (X, x)$.

One defines $\pi_0(X)$ as the set of (path) connected components of X. One has

- π_1 has a group structure.
- π_n has a commutative group structure for $n > 1$.

1.7.3. *Weak homotopy equivalence.* A map $f : X \to Y$ of topological spaces is called a weak homotopy equivalence if it induces a bijection of connected components, and for each $x \in X$ it also induces an isomorphism of all homotopy groups $\pi_n(X, x) \to \pi_n(Y, f(x))$. It is easy to verify that homotopy equivalences satisfy the above properties. It turns out however that weak homotopy equivalence is to some extent more preferable in homotopy theory. This can be explained as follows. Yes, we are mostly interested in genuine homotopy equivalence; but we are also mostly interested in good topological spaces (CW complexes). Fortunately, the two notions of equivalences coincide on good topological spaces, see below.

Theorem.

1.7.4. *Homotopy groups of simplicial sets.* Let $X \in$ `sSet`. We define $\pi_n(X, x)$ as $\pi_n(|X|, x)$. In general, there is no easy combinatorial way to define $\pi_n(X, x)$ (see homotopy groups of spheres).

It is easy for Kan simplicial sets (see definition below). One has

Theorem. *Let X be a Kan simplicial set. Then $\pi_n(X,x)$ is the set of equivalence classes of maps $(\Delta^n, \partial\Delta^n) \to (X,x)$, with equivalence given by homotopies.*

1.7.5. *Degenerate and nondegenerate simplices.* .

Even the smallest (nonempty) simplicial set has infinite number of simplices. This is because degeneracy maps $s_i : X_n \to X_{n+1}$ are injective. Therefore, it is interesting to look at the simplices $x \in X_n$ which are not degenerations of any simplex. Such simplices are called nondegenerate. The collection of all simplices of dimension $\leq n$ and of all their degenerations is a simplicial subset of X called n-th skeleton of X, $\mathrm{sk}_n(X)$.

One has $\mathrm{sk}_{n-1}(\Delta^n)$ contains all non-degenerate simplices of Δ^n except for the one of dimension n (corresponding to $\mathrm{id}_{[n]}$). We will denote this simplicial set $\partial\Delta^n$; this is, by definition, the boundary of Δ^n. The following simplicial subsets will be also very important. These are Λ^n_i -simplicial subset of Δ^n spanned by all nondegenerate simplices of $\partial\Delta^n$ but one — $d_i : [n-1] \to [n]$.

1.7.6. *Kan simplicial sets.* A simplicial set X is Kan (Kan fibrant) if any map $\Lambda^n_i \to X$ extends to a map $\Delta^n \to X$.

Exercise. Prove that $\mathrm{Sing}(X)$ is Kan for any topological space X. Prove that the nerve $N(\mathcal{C})$ is Kan if and only if the category $\mathcal{C}$ is a groupoid.

1.7.7. Here is an expression of the fact that simplicial sets and topological spaces are "practically equivalent".

Theorem. *Let S be a simplicial set and let X be a topological space. A map $f : S \to \mathrm{Sing}(X)$ is a weak equivalence of simplicial sets iff the corresponding map $|S| \to X$ is a weak homotopy equivalence.*

1.7.8. There is an important property of geometric realizations which does not follow from the adjunction.

Theorem. *The functor of geometric realization preserves products.*

Direct product of simplicial sets is given pointwise: $(X \times Y)_n = X_n \times Y_n$.

First of all, one has a canonical map $|X \times Y| \to |X| \times |Y|$. So it remains to verify the given map is a homeomorphism.

Step 1. Prove the claim when $X = \Delta^n$ and $Y = \Delta^m$.

It is worthwhile to explicitly describe $\Delta^n \times \Delta^n$. One can use the following trick: the nerve functor from categories to simplicial sets obviously preserves limits. Since Δ^n is a nerve of the category $[n]$ corresponding to the totally ordered set $\{0, \ldots, n\}$, $\Delta^n \times \Delta^m$ is the nerve of the poset $[n] \times [m]$. In particular, it is glued of $\binom{n+m}{n}$ $n+m$-simplices glued along the boundary. See the case $n = m = 1$ — the square is glued of two triangles.

Step 2. Everything commutes with the colimits.

1.7.9. **Remark.** (Adjoint pairs and simplicial objects)

Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjoint pair of functors. Given an object $d \in \mathcal{D}$, one defines a simplicial object $B_\bullet(d)$ together with a map of simplicial objects

$$B_\bullet(d) \to d$$

(where d is considered as a constant simplicial object) as follows. One defines $B_0(d) = LR(d)$, and, more generally, $B_n(d) = (LR)^{n+1}(d)$. We define $s_0 : B_0(d) \to B_1(d)$ as induced by the map $\mathrm{id} \to RL$ applied to $R(d)$. We define $d_i : B_1 \to B_0$, $i = 0, 1$ as induced by the map $LR \to \mathrm{id}$ applied to the first (resp., the second) pair of L, R. This easily generalizes to all face and degeneracy maps. A lot of resolutions in homological algebra (Bar-resolutions) come from this construction. The same origin has a cosimplicial object known as Cech complex connected to a covering of a topological space.

1.8. **Exercises.**

1.8.1. See 1.3.

1.8.2. See Exercise in 1.4.1.

1.8.3. Forgetful functor $\mathtt{Top} \to \mathtt{Set}$ has both left and right adjoints. Describe them.

1.8.4. Construct a functor left adjoint to the forgetful functor from commutative algebras over a field k to the category of vector spaces over k.

1.8.5. Let $\mathcal{C}$ be a category having finite products. For $x, y \in \mathcal{C}$ we define $\mathcal{H}om(x, y) \in \mathcal{C}$ by the property

$$\mathrm{Hom}_{\mathcal{C}}(z, \mathcal{H}om(x, y)) = \mathrm{Hom}_{\mathcal{C}}(z \times x, y).$$

Prove existence of $\mathcal{H}om$ for $\mathcal{C} = \mathtt{sSet}$.

1.8.6. The same for $\mathcal{C} = \mathtt{Cat}$. Compare to the above.

1.8.7. Let $\mathcal{C}$ has colimits. A colimit preserving functor $F : \mathtt{sSet} \to \mathcal{C}$ is uniquely given by a functor $\Delta \to \mathcal{C}$ (a cosimplicial object in C).

References

[1] S. Gelfand, Y. Manin, Methods of homological algebra, Springer.
[2] S. MacLane, Categories for the working mathematician.

(Everything can be found at the Andrew Ranicky homepage).