TAMARKIN'S PROOF OF KONTSEVICH FORMALITY THEOREM

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Abstract. In 1998 D. Tamarkin announced a proof of Kontsevich formality theorem based on the existence of structure of homotopy Gerstenhaber algebra in the Hochschild cochains of an associative algebra. In this note we give a detailed explanation of Tamarkin's result.

1. Introduction

1.1. This is an extended version of lectures given at Luminy colloquium "Quantification par déformation" held at December, 1999.

In this note we explain Tamarkin's proof [T] of the following affine algebraic version of Kontsevich's formality theorem.

1.2. **Theorem.** *Let A be a polynomial algebra over a field k of characteristic zero and let $\mathcal{C} = C^*(A;A)$ be the cohomological Hochschild complex of A with coefficients at A. The dg Lie algebra $\mathcal{C}[1]$ is formal, that is $\mathcal{C}[1]$ is isomorphic in the homotopy category of dg Lie algebras to its cohomology.*

Our sources are the original Tamarkin's note [T] and the recent paper of Tamarkin-Tsygan [TT] where a simplification (following Etingof's suggestion) of the original proof is sketched.

The aim of the note is to provide all necessary details of the proof of this important theorem. Some of these details are only hinted or have to be guessed in [T].

1.3. Tamarkin's approach to the proof of Theorem 1.2 can be shortly described as follows. It is known since the pioneering work of Gerstenhaber [G] that the Hochschild cohomology $H(\mathcal{C}) = H^*(A;A)$ of any associative algebra A admits a structure of an odd version of Poisson algebra (now called *Gerstenhaber algebra*; this is an algebra over the operad $\mathcal{G}$ defined in 5.2.2).

Tamarkin proves, using Etingof-Kazhdan theory of quantization of Lie bialgebras, that the Gerstenhaber algebra structure on $H(\mathcal{C})$ mentioned above comes from a certain *homotopy Gerstenhaber algebra* structure on the Hochschild complex $\mathcal{C}$. This structure is not unique: it depends (as everything in the Etingof-Kazhdan theory) on the choice of associator. This can explain the role the Grothendieck-Teichmüller group plays in the deformation theory of associative algebras.

Once the Hochschild complex $\mathcal{C} = C^*(A; A)$ is endowed with a structure of homotopy Gerstenhaber algebra, one can use a more or less classical obstruction theory to prove the formality of $\mathcal{C}$. To make sure that $\mathcal{C}$ is formal, one has to calculate the cohomology of $H(\mathcal{C})$ as a Gerstenhaber algebra and to make sure that the obstructions to formality vanish.

1.4. The key words for Tamarkin's proof of Theorem 1.2 are *obstructions to formality* and (an analog of) *Deligne conjecture.*

1.4.1. *Obstructions to formality.* Let X be a commutative dg algebra. Halperin and Stasheff [HS] constructed an infinite sequence of obstructions, depending on the graded commutative algebra $H(X)$, whose vanishing ensures that X is formal. In order to prove formality of the homotopy Gerstanhaber algebra $\mathcal{C} = C^*(A; A)$ where A is a polynomial algebra, Tamarkin uses a version of Halperin-Stasheff theory. Our Theorem 4.1.3 is exactly what Tamarkin needs for his proof. It seems that in this generality the result is new, though we assume it might have been known before to some people. In any case, it is worthwhile to have a written version of it.

1.4.2. *Deligne conjecture.* The question about the algebraic structure of Hochschild complex is usually attributed to Deligne. His original question was whether the Gerstenhaber algebra structure on the Hochschild cohomology $H^*(A; A)$ comes from an action on the Hochschild complex of a chain operad corresponding to the operad of small discs. In order to deduce formality of the Hochschild complex from the intrinsic formality of its cohomology, Tamarkin proves the following analog of Deligne's conjecture.

Theorem (cf. 5.3.3). *There exists a natural homotopy Gerstanhaber algebra structure on the Hochschild cochains $C^*(A; A)$ inducing the standard Gerstenhaber algebra structure on the Hochschild cohomology $H^*(A; A)$.*

This form of Deligne conjecture was suggested by Getzler-Jones in [GJ]. Unfortunately, the proof of this theorem presented in [GJ], contains a gap.

The connection between Theorem 5.3.3 and the original Deligne's question is not obvious. In [T2] Tamarkin proves that the operad of small discs is formal. This implies that that Theorem 5.3.3 and Deligne conjecture are essentially equivalent. By now there exist a number of different proofs of the original Deligne's conjecture, see [MS, KS].

Tamarkin proves the above theorem as follows. There is a dg operad $\mathcal{B}_\infty$ naturally acting on the Hochschild complex, see [GJ] and 5.5 below. Let $\mathcal{G}$ be the operad for Gerstenhaber algebras and $\mathcal{G}_\infty$ be the operad for homotopy Gerstanhaber algebras. In order to endow a Hochschild complex with a canonical structure of $\mathcal{G}_\infty$-algebra, one has to present the obvious morphism of operads $\mathcal{G}_\infty \to \mathcal{G}$ as a composition

$$\mathcal{G}_\infty \to \mathcal{B}_\infty \to \mathcal{G}. \quad (1)$$

It is convenient to present Tamarkin's construction of (1) as follows. First of all, we define another operad $\widetilde{\mathcal{B}}$ and a decomposition

$$\mathcal{G}_\infty \to \widetilde{\mathcal{B}} \to \mathcal{G}. \tag{2}$$

This is done in Section 6. Now comes the most striking observation: an isomorphism between the operads $\widetilde{\mathcal{B}}$ and $\mathcal{B}_\infty$ can be easily obtained using Etingof-Kazhdan theory of quantization (and dequantization) of Lie bialgebras. This isomorphism can be given by universal formulas depeending on the choice of associator.

1.5. The note is organized as follows. In the first part (Sections 2– 3) we review some basic facts on operads and Koszul operads. In Section 4 we study formality of algebras over a Koszul operad. Following Halperin-Stasheff [HS], we call a graded algebra H over a graded Koszul operad $\mathcal{O}$ *intrinsically formal* if any dg $\mathcal{O}$-algebra with cohomology isomorphic to H is formal. We prove Theorem 4.1.3 which gives a sufficient condition of intrinsic formality of a graded algebra over a Koszul operad in terms of its cohomology. We present as well a very easy proof of a version of Homotopy Perturbation Lemma which we need in the proof of Theorem 4.1.3.

In Section 5 we calculate the cohomology of Gerstenhaber algebra $H(\mathcal{C})$, $\mathcal{C} := C^*(A;A)$ being the Hochschild complex of a smooth commutative k-algebra A. The calculation shows that $H(\mathcal{C})$ is intrinsically formal when A is a polynomial algebra. We recall in 5.5 the definition of the operad $\mathcal{B}_\infty$ and its action on the Hochschild complex.

The last two sections are devoted to the construction of homotopy Gerstenhaber algebra structure on the Hochschild complex of an associative algebra A. In Section 6 we define the operad $\widetilde{\mathcal{B}}$ and the morphisms $\mathcal{G}_\infty \to \widetilde{\mathcal{B}}$, $\widetilde{\mathcal{B}} \to \mathcal{G}$ appearing in (2); in Section 7 we recall Etingof-Kazhdan theory and prove that the operads $\widetilde{\mathcal{B}}$ and $\mathcal{B}_\infty$ are isomorphic.

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2. Basic definitions

In this section we recall the basic definitions of operads and operad algebras. Among different approaches to the theory of operads, the one of [GJ] is the most convenient for us. Below we recall all necessary definitions and constructions. For a more detailed exposition of this approach see [GJ], Section 1.

2.1. **Operads.** Let `Vect` be the category of vector spaces over a field k of characteristic zero.

By definition, an $\mathbb{S}$-object in `Vect` is a collection $X = \{X(n)\},\ n \geq 0$, of objects of `Vect` endowed with a right action of the symmetric groups S_n. The category of $\mathbb{S}$-objects in `Vect` admits a (non-symmetric) monoidal structure defined as follows.

Any $\mathbb{S}$-object X defines a functor $\mathcal{S}(X)$ (Schur functor) on `Vect` by the formula

$$\mathcal{S}(X) : V \mapsto \bigoplus X(n) \otimes_{S_n} V^{\otimes n}. \tag{3}$$

2.1.1. **Lemma.** *There is a uniquely defined monoidal operation $\circ$ on the category of $\mathbb{S}$-vector spaces giving rise to a canonical isomorphism*

$$\mathcal{S}(X \circ Y) = \mathcal{S}(X) \circ \mathcal{S}(Y).$$

2.1.2. **Definition.** An operad $\mathcal{O} = \{\mathcal{O}(n)\}$ in `Vect` is a monoid in the category of $\mathbb{S}$-vector spaces. The category of operads in `Vect` is denoted `Op`(`Vect`).

In more conventional terms, an operad is an $\mathbb{S}$-vector space $\{\mathcal{O}(n)\}$ endowed with equivariant operations

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \ldots \otimes \mathcal{O}(m_n) \to \mathcal{O}(\sum m_i) \tag{4}$$

and with a unit element $1 \in \mathcal{O}(1)$ satisfying natural associativity and unit conditions.

2.1.3. For any vector space V one defines an operad $\mathrm{Endop}(V)$ as a $\mathbb{S}$-vector space

$$n \mapsto \mathrm{Hom}(V^{\otimes n}, V)$$

with the obvious composition and action of the symmetric groups.

2.1.4. **Definition.** An algebra A over an operad $\mathcal{O}$ is a map of operads

$$\mathcal{O} \to \mathrm{Endop}(A).$$

In other terms, an $\mathcal{O}$-algebra structure on A is given by a collection of S_n-equivariant maps

$$\mathcal{O}(n) \otimes A^{\otimes n} \to A$$

satisfying natural associativity and unit properties.

2.1.5. *Examples.* There are operads `ASS`, `COM`, `LIE` such that corresponding algebras are associative, commutative and Lie algebras respectively.

2.2. **Other tensor categories.** The definitions of the previous subsection make sense in any tensor (= monoidal symmetric) category $\mathcal{A}$. The following cases will be of a special interest for us.

2.2.1. $\mathcal{A} =$ `Vectgr` — the category of $\mathbb{Z}$-graded vector spaces. The canonical isomorphism (called *commutativity constraint*) $X \otimes Y \xrightarrow{\sim} Y \otimes X$ is defined by the standard formula

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x, \tag{5}$$

where $x \in X^{|x|}$, $y \in Y^{|y|}$.

2.2.2. $\mathcal{A} = C(k)$ — the category of complexes over k. The commutativity constraint in this case is given by the same formula (5.5.3).

2.2.3. Let $\mathcal{O} \in \mathtt{Op}(\mathcal{A})$ for a tensor category $\mathcal{A}$ and let $\alpha : \mathcal{A} \to \mathcal{B}$ be a tensor functor. Then $\alpha(\mathcal{O})$ is an operad over $\mathcal{B}$. This obvious construction allows one, for example, to consider graded or dg Lie algebras as algebras over the operad `LIE` in `Vectgr` or in $C(k)$ respectively.

2.2.4. Let $k[n]$ be a standard one-dimensional vector space concentrated at degree $-n$. The n-shift functor $X \mapsto X[n]$ is defined by the formula

$$X[n] = k[n] \otimes X.$$

This formula makes sense both in `Vectgr` and in $C(k)$.

Let $\mathcal{O}$ be an operad in `Vectgr` or $C(k)$. There is a uniquely defined operad $\mathcal{O}\{m\}$ such that a $\mathcal{O}\{m\}$-algebra structure on X is equivalent to a $\mathcal{O}$-algebra structure on $X[m]$. One has

$$\mathcal{O}\{m\}(n) = \Lambda_n^{\otimes m} \otimes \mathcal{O}(n)$$

where Λ_n denotes the graded vector space (or complex) $k[n-1]$ endowed with the sign representation of the symmetric group S_n.

2.3. **Free algebras and free operads.**

2.3.1. Let $\mathcal{O}$ be an operad in a tensor category $\mathcal{A}$. Let V be an $\mathbb{S}$-object in $\mathcal{A}$. The free $\mathcal{O}$-algebra generated by V is defined to be

$$\mathbb{F}_{\mathcal{O}}(V) = \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{S_n} V^{\otimes n} \tag{6}$$

with a canonical $\mathcal{O}$-algebra structure.

2.3.2. Let X be an $\mathbb{S}$-object in $\mathcal{A}$. The forgetful functor from the category of operads to the category of $\mathbb{S}$-objects in $\mathcal{A}$ admits a left adjoint *free operad functor.* Free operad $\mathbb{T}(X)$ generated by X has an explicit description as a direct sum over trees (see [GJ], 1.4).

2.4. **Cooperads and coalgebras.**

2.4.1. The notions of operad and algebra can be dualized. Thus, a cooperad in $\mathcal{A}$ is the same as an operad in the dual category $\mathcal{A}^{\mathrm{opp}}$. Similarly one defines a coalgebra over (under?) a cooperad.

Let $\mathcal{C}$ be a cooperad in `Vect`, `Vectgr` or $C(k)$. A $\mathcal{C}$-coalgebra X is called *nilpotent* if

$$X = \cup_n \operatorname{Ker}(X \to \mathcal{C}(n) \otimes X^{\otimes n}). \tag{7}$$

From now on all coalgebras will be supposed to be nilpotent. We define $\mathtt{Coalg}(\mathcal{C})$ to be the category of nilpotent $\mathcal{C}$-coalgebras. If V is an $\mathbb{S}$-object in $\mathcal{A}$, the cofree (nilpotent) cooalgebra cogenerated by V is defined to be

$$\mathbb{F}^*_{\mathcal{C}}(V) = \bigoplus_{n\geq 0} \left(\mathcal{C}(n)\otimes V^{\otimes n}\right)^{S_n}. \tag{8}$$

Let X be a $\mathcal{C}$-coalgebra and V be an $\mathbb{S}$-object. Any map $X \to V$ of $\mathbb{S}$-objects defines canonically a map of $\mathcal{C}$-coalgebras $X \to \mathbb{F}^*_{\mathcal{C}}(V)$. V is called an $\mathbb{S}$-object of *cogenerators* if the above map is injective.

Cofree cooperad cogenerated by V is denoted $\mathbb{T}^*(V)$. It is isomorphic to $\mathbb{T}(V)$ as an $\mathbb{S}$-object. However, we prefer to have a different notation to stress that this is a cooperad.

2.4.2. Let $\mathcal{O} \in \mathtt{Op}(\mathtt{Vectgr})$ be an operad such that $\mathcal{O}(n)$ are all finite dimensional. Then the collection $\{\mathcal{O}(n)^*\}$ admits an obvious structure of cooperad. This cooperad is denoted by $\mathcal{O}^*$. Coalgebras over $\mathcal{O}^*$ are sometimes called $\mathcal{O}$-coalgebras. In the same style, we will sometimes write $\mathbb{F}^*_{\mathcal{O}}(V)$ instead of $\mathbb{F}^*_{\mathcal{O}^*}(V)$. Thus, COM-coalgebras are just cocommutative coalgebras, LIE-coalgebras are Lie coalgebras, etc.

3. Koszul duality

3.1. **Quadratic operads and quadratic duals.**

3.1.1. **Definition.** An operad $\mathcal{O}$ of graded vector spaces is called *quadratic* if it is generated (as operad) by $\mathcal{O}(2)$ and has only relations of valence 3.

The latter condition means the following. Let V be the $\mathbb{S}$-object in $\mathtt{Vectgr}$ defined by the properties $V(2) = \mathcal{O}(2)$, $V(n) = 0$ for $n \neq 2$. Since $\mathcal{O}$ is generated by its binary operations, the natural map $\mathbb{T}(V) \to \mathcal{O}$ is surjective.

The operad $\mathcal{O}$ is quadratic if the kernel of this map is generated (as an ideal in an operad) by an S_3-invariant subspace $R \subseteq \mathbb{T}(V)(3)$.

Note that $\mathbb{T}(V)(3) = \mathrm{Ind}^{S_3}_{S_2}(V\otimes V)$ where S_2 acts on the tensor product through the trivial action on the first factor.

A quadratic operad $\mathcal{O}$ with generators V and relations R can be described as the pushout in the category of operads

$$\begin{array}{ccc} \mathbb{T}(R) & \longrightarrow & \mathbb{T}(V) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathcal{O} \end{array}$$

where $*$ denotes the trivial operad

$$*(1) = k, \quad *(n) = 0 \text{ for } n \neq 1.$$

Dually, let V be a graded vector space endowed with an action of S_2 and let R be an S_3-invariant subspace of $\mathbb{T}^*(V)(3)$. Denote $Q = \mathbb{T}^*(V)(3)/R$. Then a quadratic cooperad $\mathcal{C}$ cogenerated by V with co-relations R is defined as the pullback

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbb{T}^*(V) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{T}^*(Q) \end{array}$$

3.1.2. **Definition.** 1. Let $\mathcal{O}$ be a quadratic operad with $V = \mathcal{O}(2)$ and the space of relations R. The dual cooperad $\mathcal{O}^\perp$ is cogenerated by the space $V[1]$ with co-relations $\mathcal{O}^\perp(3) = R[2]$.

2. Dually, for a cooperad $\mathcal{C}$ cogenerated by V with co-relations Q, the quadratic dual operad $\mathcal{C}^\perp$ is generated by $V[-1]$ with relations given by the kernel

$$\mathrm{Ker}(Q[-2] \to (V[-1] \circ V[-1])(3).$$

3.1.3. *Examples*

The operads $\mathtt{COM}$, $\mathtt{ASS}$, $\mathtt{LIE}$ are quadratic (note that $\mathtt{COM}$ and $\mathtt{ASS}$ are operads for *non-unital* algebras). Their quadratic dual cooperads are given by the formulas

- $\mathtt{COM}^\perp = (\mathtt{LIE}\{-1\})^*$,
- $\mathtt{ASS}^\perp = (\mathtt{ASS}\{-1\})^*$,
- $\mathtt{LIE}^\perp = (\mathtt{COM}\{-1\})^*$.

3.1.4. **Definition.** Let $\mathcal{O}$ be a (graded) quadratic operad. A structure of $\mathcal{O}_\infty$-algebra on $X \in \mathtt{Vectgr}$ is given by a differential on the cofree $\mathcal{O}^\perp$-coalgebra cogenerated by X.

3.1.5. The above definition gives rise to an operad $\mathcal{O}_\infty$ *in the category of complexes* $C(k)$.

Let X have a structure of $\mathcal{O}_\infty$-algebra. The differential

$$Q : \mathbb{F}^*_{\mathcal{O}^\perp}(X) \to \mathbb{F}^*_{\mathcal{O}^\perp}(X)[1] \tag{9}$$

is defined uniquely by its composition with the projection onto the degree one component $F^{*1}_{\mathcal{O}^\perp}(X) = X$. Thus, the differential is given by the collection of maps

$$Q_i : \mathbb{F}^{*i}_{\mathcal{O}^\perp}(X) = \left(\mathcal{O}^\perp(i) \otimes X^{\otimes i}\right)^{S_i} \to X[1]. \tag{10}$$

in particular, $d := Q_1$ defines a differential on $X \in \mathtt{Vectgr}$.

Define $\mathcal{O}_\infty = \mathbb{T}(\mathcal{O}^\perp)$ to be the free graded operad generated by $\mathcal{O}^\perp$. The collection of maps Q_i from (10) defines an action of $\mathcal{O}_\infty$ on X.

3.1.6. **Lemma.** *There exists a unique differential on the graded operad* $\mathcal{O}_\infty$ *such that the condition* $Q^2 = 0$ *for a degree one differential* Q *as in (9) is equivalent to the statement that the action of* $\mathcal{O}_\infty$ *on* $(X, d = Q_1)$ *respects the differentials.*

From now on $\mathcal{O}_\infty$ will be considered as an operad in $C(k)$, with the differential described in Lemma 3.1.6.

3.1.7. **Example.** Let X be a complex endowed with a $\mathcal{O}$-algebra structure (dg $\mathcal{O}$-algebra). Define the differential Q on $\mathbb{F}^*_{\mathcal{O}^\perp}(X)$ as follows.

$Q_1 : X \to X[1]$ is the differential of X. $Q_2 : \mathcal{O}^\perp(2) \otimes X^{\otimes 2} \to X[1]$ is defined by the $\mathcal{O}$-algebra structure on X since $\mathcal{O}^\perp(2) = \mathcal{O}(2)[1]$. Q_i are defined to be zero for $i > 2$.

The condition $Q^2 = 0$ can be easily verified. This means that any $\mathcal{O}$-algebra admits a canonical $\mathcal{O}_\infty$-algebra structure.

Example 3.1.7 shows there is a canonical map of operads in $C(k)$

$$\mathcal{O}_\infty \to \mathcal{O} \tag{11}$$

(Here $\mathcal{O}$ is supposed to have zero differential).

3.1.8. **Definition.** A quadratic operad $\mathcal{O}$ is called Koszul if the natural map (11) is a quasi-isomorphism.

Let $\mathcal{O}$ be a quadratic operad and let X be an $\mathcal{O}_\infty$-algebra (for instance, an $\mathcal{O}$-algebra). The homology of X, $H^{\mathcal{O}}(X)$, is defined as the homology of the complex $(\mathbb{F}^*_{\mathcal{O}^\perp}(X), Q)$.

3.1.9. **Examples.** 1. $\mathcal{O} = \mathtt{ASS}$. The complex

$$(\mathbb{F}^*_{\mathcal{O}^\perp}(X), Q) = (\mathbb{F}^*_{\mathtt{ASS}}(X[1]), Q)[-1]$$

is the (homology) Hochschild complex of the associative algebra X.

2. $\mathcal{O} = \mathtt{COM}$. The complex

$$(\mathbb{F}^*_{\mathcal{O}^\perp}(X), Q) = (\mathbb{F}^*_{\mathtt{LIE}}(X[1]), Q)[-1]$$

is the Harrison complex of the commutative algebra X.

3. $\mathcal{O} = \mathtt{LIE}$. The complex

$$(\mathbb{F}^*_{\mathcal{O}^\perp}(X), Q) = (\mathbb{F}^*_{\mathtt{COM}}(X[1]), Q)[-1]$$

is the Chevalley-Eilenberg complex of the Lie algebra X.

Thus, for the operads $\mathcal{O} = \mathtt{LIE}$, $\mathtt{ASS}$, $\mathtt{COM}$ we obtain the homology of the corresponding type of algebras (with trivial coefficients).

If $X = \mathbb{F}_{\mathcal{O}}(V)$ for a graded vector space V, one has a canonical map of complexes

$$(\mathbb{F}^*_{\mathcal{O}^\perp}(X), Q) \to V. \tag{12}$$

The following result can be used to prove koszulity of a quadratic operad.

3.1.10. **Theorem.** *(cf.* [GK]*, Thm. 4.2.5) A quadratic operad $\mathcal{O}$ is Koszul iff for any graded vector space V the canonical map (12) is quasi-isomorphism.*

Theorem 3.1.10 implies that the operads COM, ASS, LIE are Koszul.

4. Deformations and formality

4.1. **Intrinsic formality.** In this section $\mathcal{O}$ is a fixed graded Koszul operad.

Let X be an $\mathcal{O}_\infty$-algebra. The cohomology $H(X)$ has a natural structure of algebra over $H(\mathcal{O}_\infty) = \mathcal{O}$. This gives, via (11), an $\mathcal{O}_\infty$-algebra structure in $H(X)$.

4.1.1. **Definition.** A $\mathcal{O}_\infty$-algebra X is called to be *formal* if there exists a pair of quasi-isomorphisms of $\mathcal{O}_\infty$-algebras $X \leftarrow F \rightarrow H(X)$.

4.1.2. **Definition.** A graded $\mathcal{O}$-algebra H is *intrinsically formal* if any $\mathcal{O}_\infty$-algebra X with $H(X) = H$ is formal.

Note. The results of [H] inmply that the above notion of intrinsic formality is equivalent to the one mentioned in 1.5: H is intrinsically formal iff any $\mathcal{O}$-algebra X with $H(X) = H$ is formal. This follows from the fact that the homotopy categories of algebras over quasi-isomorphic operads, are equivalent.

The aim of this section is to prove a criterion of intrinsic formality.

Let H be a graded $\mathcal{O}$-algebra and let $\mathfrak{g}$ be the dg Lie algebra of coderivations of the corresponding dg $\mathcal{O}^\perp$-coalgebra $(\mathbb{F}^*_{\mathcal{O}^\perp}(H), Q)$. Since $\mathbb{F}^*_{\mathcal{O}^\perp}(H)$ is cofree, any coderivation is uniquely defined by its composition with the projection onto H. Therefore, $\mathfrak{g}$ considered as a graded vector space, is isomorphic to $\mathrm{Hom}(\mathbb{F}^*_{\mathcal{O}^\perp}(H), H)$. We denote

$$\mathfrak{g}_{\geq 1} = \mathrm{Hom}(\oplus_{i\geq 2}\mathbb{F}^{*i}_{\mathcal{O}^\perp}(H), H).$$

This is a dg Lie subalgebra of $\mathfrak{g}$.

4.1.3. **Theorem.** *Suppose that the map $H^1(\mathfrak{g}_{\geq 1}) \rightarrow H^1(\mathfrak{g})$ is zero. Then H is intrinsically formal.*

4.2. **Proof of Theorem 4.1.3.** The following standard lemma results from the fact that $\mathcal{O}_\infty$ is *cofibrant* (see [H], Sect. 6). This result is traditionally called *Homology Perturbation Theory*; it has been widely used in different special cases since 70-ies by Gugenheim, Stasheff and others.

4.2.1. **Lemma.** *Let X be a $\mathcal{O}_\infty$-algebra. There exists a $\mathcal{O}_\infty$-algebra structure on $H(X)$ so that X and $H(X)$ are quasi-isomorphic $\mathcal{O}_\infty$-algebras (i.e., there exists a pair of quasi-isomorphisms of $\mathcal{O}_\infty$-algebras $X \leftarrow F \rightarrow H(X)$).*

For an easy proof of this (and a more general) fact see 4.3.

4.2.2. Let H be a graded $\mathcal{O}$-algebra. Let X be a $\mathcal{O}_\infty$-algebra so that $H = H(X)$ as $\mathcal{O}$-algebras. Choose a $\mathcal{O}_\infty$-algebra structure on H guaranteed by Lemma 4.2.1. One has $\mathcal{O}_\infty(2) = \mathcal{O}(2)$ and the $\mathcal{O}$-algebra structure on H is the restriction of the $\mathcal{O}_\infty$-algebra structure. To fix a notation, let the collection of maps

$$Q_n : \mathbb{F}^{*n}_{\mathcal{O}^\perp}(H) \to H[1], \tag{13}$$

$n \geq 2$, define the said $\mathcal{O}_\infty$-algebra structure on H. The $\mathcal{O}$-algebra structure on H is given by the collection $\{Q^0_n\}$ with $Q^0_2 = Q_2$; $\quad Q^0_i = 0$ for $i > 2$.

4.2.3. **Lemma.** *Let $\lambda \in k$. Put $Q^\lambda_n = \lambda^{n-2} Q_n$. The collection $\{Q^\lambda_n\}_{n\geq 1}$ defines a collection of $\mathcal{O}_\infty$-algebra structures on H parametrized by $\lambda \in k$. This gives the structure $\{Q_n\}$ for $\lambda = 1$ and $\{Q^0_n\}$ for $\lambda = 0$.*

Proof. The only property we have to check to make sure that the collection $\{Q^\lambda_n\}$ defines a $\mathcal{O}_\infty$-algebra structure, is the identity looking like

$$d(Q^\lambda_n) = P_n(Q^\lambda_2, \ldots, Q^\lambda_{n-1})$$

where P_n is a quadratic (non-commutative) polynomial. Since H has zero differential (this means $Q_1 = 0$ in our notation) the left-hand side vanishes. The right hand side vanishes for $\lambda = 1$ since the collection of Q_i does define a $\mathcal{O}_\infty$-action. Since the polynomials P_n are homogeneous, one has

$$P_n(Q^\lambda_2, \ldots, Q^\lambda_{n-1}) = \lambda^{n-1} P_n(Q_2, \ldots, Q_{n-1}).$$

This proves the claim. ∎

4.2.4. Put $C = (\mathbb{F}^*_{\mathcal{O}^\perp}(H), Q^0)$. This is a differential graded $\mathcal{O}^\perp$-coalgebra. The collection $\{Q^\lambda_n\}$ defines a $k[\lambda]$-linear differential Q^λ on the $\mathcal{O}^\perp$-coalgebra $C[\lambda]$. We wish to construct an isomorphism

$$\theta : (C[\lambda], Q^0) \to (C[\lambda], Q^\lambda)$$

which is identity modulo λ.

The isomorphism θ is uniquely defined by a collection of maps

$$\theta_n : \mathbb{F}^{*n}_{\mathcal{O}^\perp}(H) \to H[\lambda]$$

with $\theta_1 = \mathrm{id}_H$. We will be looking for θ satisfying the following property.

$$\theta_n = \phi_n \cdot \lambda^{n-1} \text{ for some } \phi_n : \mathbb{F}^{*n}_{\mathcal{O}^\perp}(H) \to H. \tag{14}$$

An automorphism θ satisfying (14) is constructed in 4.2.6 below. Then, tensoring θ by $k[\lambda]/(\lambda - 1)$, we get an isomorphism of dg $\mathcal{O}^\perp$-coalgebras

$$\overline{\theta} : (C, Q^0) \xrightarrow{\sim} (C, Q).$$

This will prove Theorem 4.1.3.

4.2.5. Define an action of the multiplicative group k^* on $C[\lambda]$ by the formulas

$$\mu * x = \mu^n \cdot x \text{ for } x \in \mathbb{F}^{*n}_{\mathcal{O}^\perp}(H); \quad \mu * \lambda = \mu \cdot \lambda.$$

The differentials Q and Q^λ have both degree -1 with respect to this action:

$$\mu * Q(\mu^{-1} * x) = \mu^{-1} \cdot Q(x); \quad \mu * Q^\lambda(\mu^{-1} * x) = \mu^{-1} \cdot Q^\lambda(x).$$

The condition (14) means that θ has degree zero with respect to the defined action of k^*.

4.2.6. The map θ will be constructed by induction.

Suppose we have constructed an isomorphism

$$\theta : (C[\lambda]/(\lambda^n), Q^0) \xrightarrow{\sim} (C[\lambda]/(\lambda^n), Q^\lambda)$$

satisfying the property $\theta_k = \phi_k \cdot \lambda^{k-1}$ for some $\phi_k : \mathbb{F}^{*k}_{\mathcal{O}^\perp}(H) \to H$ for all k. This means in particular that $\theta_k = 0$ for $k > n$.

Our aim is to lift θ to a map

$$\widetilde{\theta} : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q^\lambda)$$

such that its components $\widetilde{\theta}_k$ satisfy the same property.

First of all, we lift θ to the isomorphism

$$\theta' : (C[\lambda]/(\lambda^{n+1}), Q') \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q^\lambda)$$

taking $\theta'_k = \theta_k$ for all k, where Q' is some differential uniquely defined by the above formula. The differential Q' has also degree -1. Since Q' coincides with Q_0 modulo λ^n, one has actually an equality $Q'_k = Q^0_k$ for $k \leq n+1$ and $Q'_{n+2} = \lambda^n \cdot z$ for some $z : \mathbb{F}^{*n+2}_{\mathcal{O}^\perp}(H) \to H$. One easily observes that the element z considered as a derivation, is a cycle. Therefore, there is a derivation $u \in \mathfrak{g}^0$, such that $z = du$. This gives an isomorphism

$$\eta = \exp(\lambda^n \cdot u) : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q')$$

which is identity modulo λ^n. The inductive step will be acomplished if we are able to find an isomorphism between $(C[\lambda]/(\lambda^{n+1}), Q)$ and $(C[\lambda]/(\lambda^{n+1}), Q')$ having degree zero.

The components η_k of η are divisible by λ^n for $k > 1$. An easy calculation shows that the collection $\kappa_k : \mathbb{F}^{*k}_{\mathcal{O}^\perp}(H) \to H$ given by the formulas

$$\kappa_1 = \mathrm{id}_H, \ \kappa_{n+1} = \eta_{n+1}, \ \kappa_i = 0 \text{ for } i \neq 1, n+1,$$

defines an isomorphism

$$\kappa : (C[\lambda]/(\lambda^{n+1}), Q) \xrightarrow{\sim} (C[\lambda]/(\lambda^{n+1}), Q').$$

The composition of κ with θ' is the isomorphism $\widetilde{\theta}$ we were looking for.

The construction of isomorphism θ satisfying (14), and, therefore, the proof of Theorem 4.1.3, is acomplished.

4.3. **Proof of Lemma 4.2.1.** We will prove a more general statement.

4.3.1. **Proposition.** *Let k be a commutative ring and let $\mathcal{O}$ be a cofibrant operad* [H] *in $C(k)$. Let A be endowed with an $\mathcal{O}$-algebra structure. Suppose, finally, $B \in C(k)$, $\pi : A \to B$ and $\sigma : B \to A$ be quasi-inverse homotopy equivalences with $\pi\sigma = \mathrm{id}_B$.*

Then there exists a $\mathcal{O}$-algebra structure on B such that A and B become weakly equivalent $\mathcal{O}$-algebras.

4.3.2. **Corollary.** *Structure of algebra over a cofibrant operad can be transferred along homotopy equivalences.*

4.3.3. *Proof of 4.3.1.* The maps π, σ define a morphism of endomorphism operads $\tau : \mathrm{Endop}(B) \to \mathrm{Endop}(A)$ as the composition

$$\tau(n) : \mathrm{Endop}(B)(n) = \mathrm{Hom}(B^{\otimes n}, B) \xrightarrow{\pi^{\otimes n}} \mathrm{Hom}(A^{\otimes n}, B) \xrightarrow{\sigma} \mathrm{Hom}(A^{\otimes n}, A). \tag{15}$$

The collection of maps $\tau(n)$ forms an operad morphism since σ splits π. Morphism τ is a quasi-isomorphism of operads since π and σ are homotopy equivalences. Since the category of operads in $C(k)$ admits a model category structure (see [H]), τ can be presented as a composition

$$\mathrm{Endop}(B) \xrightarrow{i} E \xrightarrow{p} \mathrm{Endop}(A),$$

where p is an acyclic fibration (=surjective quasi-isomorphism) and i is an acyclic cofibration. Moreover, acyclic cofibrations split, so there exists $q : E \to \mathrm{Endop}(B)$ such that $qi = \mathrm{id}$.

Now, an $\mathcal{O}$-algebra structure on A is given by a morphism $a : \mathcal{O} \to \mathrm{Endop}(A)$. This morphism lifts to a morphism $e : \mathcal{O} \to E$ since $\mathcal{O}$ is cofibrant. This defines a composition $qe : \mathcal{O} \to E \xrightarrow{q} \mathrm{Endop}(B)$ which gives an $\mathcal{O}$-algebra structure on B satisfying the required property.

5. Hochschild complex

5.1. **Hochschild complex.** Let A be an associative k-algebra. Its Hochschild complex $\mathcal{C} := C^*(A; A)$ has components defined by the formula

$$\mathcal{C}^n = C^n(A; A) = \mathrm{Hom}(A^{\otimes n}, A), \quad n = 0, 1, \ldots$$

The graded vector space $\mathcal{C}$ admits a $\mathtt{LIE}\{1\}$-algebra structure which comes from the identification of $\mathcal{C}[1]$ with the collection of coderivations of the cofree coalgebra (with counit) cogenerated by $A[1]$.

An explicit formula for the Lie bracket is given in 5.5.4 below.

The multiplication $\mu : A^{\otimes 2} \to A$ belongs to $\mathcal{C}^2$; therefore the operator $\mathrm{ad}\,\mu$ has degree 1. An easy calculation shows that $(\mathrm{ad}\,\mu)^2 = 0$; $\mathcal{C}$ endowed with the differential $\mathrm{ad}\,\mu$ becomes a dg $\mathtt{LIE}\{1\}$-algebra.

5.2. $H(\mathcal{C})$ **is a $\mathcal{G}$-algebra.** In order to prove Theorem 1.2 it would be enough to check that $H = H(C^*(A;A))$ is intrinsically formal as a Lie algebra. This, however, is not true. Tamarkin's idea is to prove that H becomes intrinsically formal when it is considered as an algebra over an operad $\mathcal{G}$ described below. Since $\mathcal{G}$ contains $\mathtt{LIE}\{1\}$ as a suboperad, this implies Theorem 1.2.

Define $m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ by the formula

$$m(x \otimes y) = \mu \circ (x \boxtimes y) \tag{16}$$

where $x \boxtimes y : A^{\otimes m+n} \to A^{\otimes 2}$ is defined to be the tensor product of the maps $x : A^{\otimes m} \to A$ and $y : A^{\otimes n} \to A$.

The following lemma is due to M. Gerstenhaber [G].

5.2.1. **Lemma.** *The map m induces a commutative associative multiplication on $H(\mathcal{C})$. The bracket on $H(\mathcal{C})$ is a derivation with respect to m.*

5.2.2. **Definition.** Operad $\mathcal{G}$ is the operad generated by the operations $m \in \mathcal{G}(2)^0, \quad \ell \in \mathcal{G}(2)^{-1}$ satisfying the following identities:

- m is commutative associative
- ℓ is Lie
- ℓ is a derivation with respect to m.

Lemma 5.2.1 above means that the cohomology $H(C^*(A;A))$ admits a natural $\mathcal{G}$-algebra structure.

5.2.3. The following construction assigns a $\mathcal{G}$-algebra to any Lie algebra $\mathfrak{g}$. Put $X = \mathbb{F}_{\mathtt{COM}}(\mathfrak{g}[-1]) = \oplus_{i>0} S^i(\mathfrak{g}[-1])$. There is a unique $\mathtt{LIE}\{1\}$-algebra structure on X extending that on $\mathfrak{g}[-1]$ such that X becomes a $\mathcal{G}$-algebra. This is *the $\mathcal{G}$-algebra generated by a Lie algebra* $\mathfrak{g}$.

5.2.4. There is a twisted (=sheaf) version of the above construction. Let $\mathfrak{g}$ be a Lie algebroid over a commutative algebra A. This mean that $\mathfrak{g}$ is a Lie algebra, an A-module, and a map of Lie algebras and A-modules $\pi : \mathfrak{g} \to \mathrm{Der}(A, A)$ is given so that

$$[f, ag] = a[f, g] + \pi(f)(a)g$$

for $a \in A, \quad f, g \in \mathfrak{g}$.

Then a $\mathcal{G}$-algebra structure on the A-symmetric algebra without unit $S_A^{\geq 1}(\mathfrak{g}[-1])$ is naturally defined. If one defines $A = S_A^0(\mathfrak{g}[-1])$ to commute with $S_A^{\geq 1}(\mathfrak{g}[-1])$, one obtains a $\mathcal{G}$-algebra structure on the A-symmetric algebra $S_A(\mathfrak{g}[-1])$.

5.3. **Koszulity.** The operad $\mathcal{G}$ is obviously quadratic. The quadratic dual cooperad $\mathcal{G}^\perp$ has as cogenerators elements $\widetilde{m}, \quad \widetilde{\ell}$ of degrees -1 and -2 respectively. A simple calculation gives

5.3.1. **Lemma.**

$$\mathcal{G}^{\perp} = \mathcal{G}^*[2].$$

One has the following important

5.3.2. **Proposition.** *([GJ]) $\mathcal{G}$ is Koszul.*

For an easy proof of this fact see 5.4.6.

Recall that koszulity of $\mathcal{G}$ means that the natural map (11)

$$\mathcal{G}_\infty \to \mathcal{G}$$

is a quasi-isomorphism of operads. The operad $\mathcal{G}_\infty$ is the operad for *homotopy Gerstenhaber algebras.*

Deformation theory approach to the Formality Theorem is based on the following

5.3.3. **Theorem.** *There is a structure of $\mathcal{G}_\infty$-algebra on $C^*(A;A)$ inducing the described above $\mathcal{G}$-algebra structure on $H(C^*(A;A))$.*

Theorem 5.3.3 will be proven in Sections 6 and 7. In this section we will deduce Formality Theorem 1.2 from Theorem 5.3.3.

5.4. **Calculation.** From now on A is a smooth commutative k-algebra. Our aim is to calculate the cohomology of $H := H(C^*(A;A))$ and to make sure it vanishes when A is a polynomial algebra. This, together with Theorem 5.3.3, gives Formality Theorem.

The following classical result of Hochschild-Kostant-Rosenberg describes the cohomology of $C^*(A;A)$.

5.4.1. **Lemma.** *$H = S_A(T_A[-1])$ where $T_A = \mathrm{Der}(A,A)$. The $\mathcal{G}$-algebra structure on H is defined as in 5.2.4.*

Following 4.1.3, we have to calculate the dg Lie algebra of coderivations of the dg $\mathcal{G}^{\perp}$-coalgebra $(\mathbb{F}^*_{\mathcal{G}^{\perp}}(H), Q)$ corresponding to H.

5.4.2. Note the following formula

$$\mathbb{F}^*_{\mathcal{G}^{\perp}}(X) = \mathbb{F}^*_{\mathtt{COM}}\left(\mathbb{F}^*_{\mathtt{LIE}}(X[1])[1]\right)[-2] \tag{17}$$

which can be obtained using 5.3.1 from the formula dual to the following

$$\mathbb{F}_{\mathcal{G}}(X) = \mathbb{F}_{\mathtt{COM}}(\mathbb{F}_{\mathtt{LIE}\{1\}}(X)). \tag{18}$$

5.4.3. According to 4.1.3, we have to calculate the map $H^1(\mathfrak{g}_{\geq 1}) \to H^1(\mathfrak{g})$ where

$$\mathfrak{g} = \mathrm{Coder}(\mathbb{F}^*_{\mathcal{G}^\perp}(H)) = \mathrm{Hom}(\mathbb{F}^*_{\mathcal{G}^\perp}(H), H) = \mathrm{Hom}(\mathbb{F}^*_{\mathtt{COM}}(\mathbb{F}^*_{\mathtt{LIE}}(H[1])[1]), H[2])$$

with the differential induced by the differential Q of $\mathbb{F}^*_{\mathcal{G}^\perp}(H)$.

The differential Q of $\mathbb{F}^*_{\mathcal{G}^\perp}(H)$ comes from the map $\mathcal{G}(2) \otimes H^{\otimes 2} \to H$ describing the $\mathcal{G}$-algebra structure on H. Therefore, $Q = Q_m + Q_\ell$ where Q_m is induced by the commutative multiplication $m : H \otimes H \to H$, and Q_ℓ is induced by the bracket $\ell : H \otimes H \to H[-1]$. Since the defining relations on operations m and ℓ in $\mathcal{G}$ are homogeneous, one necessarily has

$$Q_m^2 = Q_\ell^2 = Q_m Q_\ell + Q_\ell Q_m = 0.$$

The total diferential Q on $\mathfrak{g}$ is also a sum of two differentials which will be denoted by Q_m and Q_ℓ.

Any cofree coalgebra is naturally graded — see (8). Formula (17) gives rise to a bigrading on the cofree $\mathcal{G}^\perp$-coalgebra $\mathbb{F}^*_{\mathcal{G}^\perp}(H)$ in which the (p,q)-component consists of the elements of COM-degree $-p$ and total LIE-degree $-q$.

This defines a bigrading on $\mathfrak{g}$ so that

$$\mathfrak{g}^{p\bullet} = \mathrm{Hom}(\mathbb{F}^{*1+p}_{\mathtt{COM}}(\mathbb{F}^{*\bullet}_{\mathtt{LIE}}(H[1])[1]), H[2]).$$

Note that

$$\mathfrak{g}_{\geq 1} = \bigoplus_{(p,q)\neq(0,0)} \mathfrak{g}^{pq}. \tag{19}$$

The differentials Q_m and Q_ℓ have degrees $(0,1)$ and $(1,0)$ with respect to this bigrading and $\mathfrak{g}$ lives in the first quadrant. Therefore, one can use the spectral sequence argument to calculate the cohomology of $\mathfrak{g}$.

Let us calculate the first term $E_1^{pq} = H^{pq}(\mathfrak{g}, Q_m)$. To keep track of the differential Q_m in $\mathfrak{g}$ it is convenient to present

$$\mathfrak{g}^{0q} = \mathrm{Hom}(\mathbb{F}^{*1+q}_{\mathtt{LIE}}(H[1])[1], H[2]) = \mathrm{Hom}_H(\mathbb{F}^{*1+q}_{\mathtt{LIE}}(H[1]) \otimes H[1], H[2])$$

and to identify $\mathbb{F}^*_{\mathtt{LIE}}(H[1]) \otimes H$ with the homological Harrison complex $Z := \mathrm{Harr}_*(H,H)$.

Then one can see that $(\mathfrak{g}^{0\bullet}, Q_m)$ coincides with $\mathrm{Hom}_H(Z[1], H[2])$ as a complex; moreover, for each p one has

$$(\mathfrak{g}^{p\bullet}, Q_m) = \mathrm{Hom}_H(S_H^{1+p}(Z[1]), H[2]).$$

5.4.4. The considerations above hold for every graded $\mathcal{G}$-algebra H with unit. Now we will use the fact that $H = H(C^*(A;A))$ where A is a smooth k-algebra.

Namely, according to Lemma 5.4.1, H is smooth as a graded commutative algebra. Therefore, there is a natural isomorphism

$$Z \xrightarrow{\sim} \Omega[1]$$

where $\Omega = \Omega_{H/k}$ is the module of Kähler differentials. This implies that

$$E_1^{pq} = \begin{cases} \mathrm{Hom}_H(S_H^{1+p}(\Omega[2]), H[2]), & q = 0 \\ 0, & q \neq 0. \end{cases} \tag{20}$$

Let us calculate $\Omega_{H/k}$. The sequence of smooth morphisms of graded commutative algebras

$$k \to A \to H = S_A(T_A[-1])$$

gives rise to an isomorphism

$$\Omega \xrightarrow{\sim} H \otimes_A \Omega_{A/k} \oplus \Omega_{H/A} = H \otimes_A (T_A[-1] \oplus T_A^*) = H \otimes_A \omega \tag{21}$$

where $\omega = T_A[-1] \oplus T_A^*$. Note that ω is a finitely generated graded projective A-module.

The only non-vanishing cohomology in (20) can be rewritten as

$$E_1^{p0} = S_A^{1+p}(\omega[-1]) \otimes_A H[2] = S_H^{1+p}(\Omega[-1])[2] \tag{22}$$

since $\omega[2]^* = \omega[-1]$.

Note that E_1^{p0} embeds into

$$\mathfrak{g}^{p0} = \mathrm{Hom}(\mathbb{F}^{*1+p}_{\mathtt{COM}}(H[2]), H[2])$$

and the differential Q_ℓ on the latter is defined by the Lie algebra structure on $H[1]$. This allows one to identify the the differential Q_2 on (22) with the differential on the (shifted and truncated) de Rham complex of H.

5.4.5. Suppose now that A is a polynomial algebra over k. In this case de Rham complex of H is acyclic. Then the calculation in the previous subsection gives a quasi-isomorphism $\mathfrak{g} \xrightarrow{\sim} H/k[1]$. This implies that $\mathfrak{g}$ has no cohomology coming from the cohomology of $\mathfrak{g}_{\geq 1}$.

5.4.6. *Remark.* A calculation similar to the above proves that $\mathcal{G}$ is Koszul.

In fact, according to 3.1.10, one has to check that for each graded vector space V the natural map

$$V \to (\mathbb{F}^*_{\mathcal{G}^\perp}(\mathbb{F}_{\mathcal{G}}(V), Q)$$

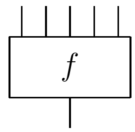
is a quasi-isomorphism.

Taking into account the formulas (17) and (18) and using, as in 5.4.3, the presentation of $\mathbb{F}^*_{\mathcal{G}^\perp}(H)$ by a bicomplex, one easily obtains the result.

5.4.7. Now Theorem 1.2 have been proven modulo Theorem 5.3.3. In the end of this section we describe the operad $\mathcal{B}_\infty$ naturally acting on the Hochschild complex $C(A; A)$ of any associative algebra A. This is the first step in the proof of Theorem 5.3.3 which is presented in Sections 6 and 7.

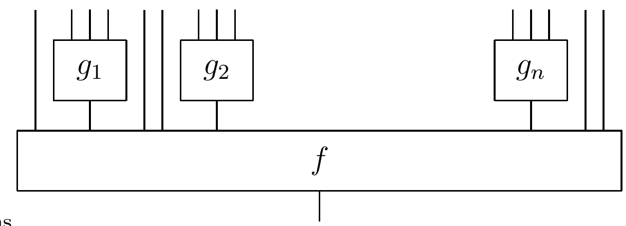
5.5. **Hochschild complex is a $\mathcal{B}_\infty$-algebra.** We shall now describe an operad which acts naturally on the Hochschild complex of any associative algebra. This operad is denoted $\mathcal{B}_\infty$. It has been invented by H.-I. Baues; its action on the Hochschild complex was defined in [GJ].

5.5.1. *Notation.* In this subsection A is any associative k-algebra and $\mathcal{C} = C^*(A;A)$. It is convenient to denote elements $f \in \mathcal{C}^n$ as boxes having n hands and one leg like this:



5.5.2. *Basic operation.* Let $f, g_1, \ldots, g_n \in \mathcal{C}$. Denote *the brace* $f\{g_1, \ldots, g_n\}$ by the following formula

(23) $$f\{g_1, \ldots, g_n\} = \sum_{\text{all possible insertions}}$$



Here the sum is taken over all possible order preserving insertions of legs of g_i into hands of f.

5.5.3. *Remark.* We have chosen to use pictures in (23) in order to avoid unpleasant signs in formulas. The signs reappear if one decides to write down the expression for $f\{g_1, \ldots, g_n\}(a_1 \otimes \ldots \otimes a_m)$, compare to [GJ], Formula (1) on p. 49.

5.5.4. The Lie bracket on $\mathcal{C}[1]$ is given explicitly, in terms of braces, by the formula
$$[f, g] = f\{g\} - (-1)^{|f||g|} g\{f\}.$$

5.5.5. **Definition.** A $\mathcal{B}_\infty$-algebra structure on a graded vector space X is given by a structure of dg bialgebra on $\mathbb{F}^*_{\mathtt{ASS}}(X[1])$ so that the coalgebra structure is the standard (cofree) one.

Let us check that $\mathcal{B}_\infty$-algebra structure is given by an operad (as usual, it will be denoted by $\mathcal{B}_\infty$).

The dg bialgebra structure on $\mathbb{F}^*_{\mathtt{ASS}}(X[1])$ is given by the following data.

- a differential $X[1]^{\otimes n} \to X[1]^{\otimes m}$ of degree 1. The differential is uniquely defined by its $m = 1$ part. We denote its $(n, 1)$-components by $m_n : X[1]^{\otimes n} \to X[2]$ (or, what is the same, $m_n : X^{\otimes n} \to X[2-n]$).
- a multiplication $X[1]^{\otimes p} \otimes X[1]^{\otimes q} \to X[1]^{\otimes r}$ of degree 0 — it is also uniquely defined by its $r = 1$ part. We denote the collection of $r = 1$ multiplications by

  $$m_{pq} : X[1]^{\otimes p} \otimes X[1]^{\otimes q} \to X[1]$$

  or, what is the same,

  $$m_{pq} : X^{\otimes p} \otimes X^{\otimes q} \to X[1-p-q].$$

Therefore, the $\mathcal{B}_\infty$-algebra structure is given by a collection of operations m_n, m_{pq} subject to some relations. This defines an operad $\mathcal{B}_\infty$ as the one generated by $m_n \in \mathcal{B}_\infty(n)^{2-n}$ and $m_{pq} \in \mathcal{B}_\infty(p+q)^{1-p-q}$ subject to some relations.

5.5.6. **WARNING.** The operad $\mathcal{B}_\infty$ is not obtained in any sense from a(ny) Koszul operad $\mathcal{B}$. Getzler and Jones are responsible for this notation.

5.5.7. *Action of $\mathcal{B}_\infty$ on $C^*(A; A)$.* We have to define the action of the operations m_n, m_{pq} on $\mathcal{C} = C^*(A; A)$ and to check the compatibilities. Here it is.

- m_1 is the differential in $\mathcal{C}$
- m_2 is the multiplication μ defined by (16)
- $m_i = 0$ for $i > 2$
- $m_{1k}(f \otimes g_1 \otimes \ldots \otimes g_k) = f\{g_1, \ldots, g_k\}$ where the brace operations are defined by formula (23)
- $m_{kl} = 0$ for $k > 1$.

One can directly check that the collection of operations m_n, m_{pq} defined above gives rise to a $\mathcal{B}_\infty$-algebra structure on $\mathcal{C}$.

6. Between $\mathcal{G}$ and $\mathcal{G}_\infty$

In this section we present an operad $\widetilde{\mathcal{B}}$ lying between $\mathcal{G}$ and $\mathcal{G}_\infty$: it admits a pair of maps

$$\mathcal{G}_\infty \to \widetilde{\mathcal{B}}, \quad \widetilde{\mathcal{B}} \to \mathcal{G}$$

so that the composition is the canonical map $\mathcal{G}_\infty \to \mathcal{G}$.

In the next section we will prove, using Etingof-Kazhdan theorem on quantization of Lie bialgebras [EK], that the operad $\widetilde{\mathcal{B}}$ is isomorphic to the operad $\mathcal{B}_\infty$ acting on the Hochschild complex of any associative algebra by 5.5.7. This will yield Theorem 5.3.3 and, therefore, Theorem 1.2.

6.1. **$\widetilde{\mathcal{B}}$-algebras.** A $\widetilde{\mathcal{B}}$-algebra structure on a graded vector space X is a dg Lie bialgebra structure on $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$ extending the standard free Lie coalgebra structure.

The Lie bracket on a Lie bialgebra $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$ is defined by its corestriction to the cogenerators $X[1]$. Therefore, it is given by a collection of maps

$$\ell_{mn} : \mathbb{F}^{*m}_{\mathtt{LIE}}(X[1]) \otimes \mathbb{F}^{*n}_{\mathtt{LIE}}(X[1]) \to X[1]$$

satisfying a collection of quadratic identities. The differential on $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$ is also defined by its corestriction to the cogenerators. This amounts to a collection

$$d_n : \mathbb{F}^{*n}_{\mathtt{LIE}}(X[1]) \to X[2]$$

satisfying some more quadratic identities — the one saying that $d^2 = 0$ and the other that d is the derivation of the Lie algebra structure given by ℓ_{mn}.

In particular, one has $d_1^2 = 0$ and this endows X with a structure of complex. The obvious maps $X[1] \to \mathbb{F}^*_{\mathtt{LIE}}(X[1]) \to X[1]$ are maps of complexes.

Since a $\widetilde{\mathcal{B}}$-structure on X is given by a collection of operations subject to some relations, there is an operad in the category of complexes which will be called in the sequel $\widetilde{\mathcal{B}}$ such that $\widetilde{\mathcal{B}}$-algebras are just algebras over $\widetilde{\mathcal{B}}$.

6.2. A map $\mathcal{O} \to \mathcal{O}'$ of operads endows a $\mathcal{O}'$-algebra with a canonical $\mathcal{O}$-algebra structure. The converse is also obviously true — in order to define a map of operads it is enough to endow any $\mathcal{O}'$-algebra with a canonical $\mathcal{O}$-algebra structure.

Let us construct a map $\widetilde{\mathcal{B}} \to \mathcal{G}$. For this we have to define canonically a $\widetilde{\mathcal{B}}$-algebra structure on each $\mathcal{G}$-algebra X. Recall that a $\mathcal{G}$-algebra X is endowed with a commutative multiplication $m : X^{\otimes 2} \to X$ and a Lie bracket $l : X[1]^{\otimes 2} \to X[1]$. The Harrison complex of the commutative algebra (X, m) is given by a differential on $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$. The Lie algebra structure on $X[1]$ can be uniquely extended to $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$ to get a Lie bialgebra. The Harrison differential will be a derivation with respect to the Lie algebra structure, so this construction defines a dg Lie bialgebra structure on $\mathbb{F}^*_{\mathtt{LIE}}(X[1])$.

The construction is obviously canonical and yields a morphism of operads $\widetilde{\mathcal{B}} \to \mathcal{G}$.

6.3. Let us now construct a map $\mathcal{G}_\infty \to \widetilde{\mathcal{B}}$.

Let X be a $\widetilde{\mathcal{B}}$-algebra. This means that a dg Lie bialgebra structure on $\mathfrak{g} = \mathbb{F}^*_{\mathtt{LIE}}(X[1])$ is given. In particular, $\mathfrak{g}$ is a dg Lie algebra and this defines a differential on $\mathbb{F}^*_{\mathtt{COM}}(\mathfrak{g}[1])$. The latter complex is by formula (17) just $\mathbb{F}^*_{\mathcal{G}^\perp}(X)[2]$. Differential on it gives a $\mathcal{G}_\infty$-structure on X.

Thus the map $\mathcal{G}_\infty \to \widetilde{\mathcal{B}}$ is constructed.

7. Equivalence of $\widetilde{\mathcal{B}}$ with $\mathcal{B}_\infty$

In this section we prove that the operads $\widetilde{\mathcal{B}}$ and $\mathcal{B}_\infty$ are isomorphic. The isomorphism is obtained using Etingof-Kazhdan theorems [EK] on quantization of Lie bialgebras. In particular, it will depend on the choice of associator, as in [EK].

7.1. For some technical reasons, it is more convenient to use coalgebras over $\widetilde{\mathcal{B}}$ and $\mathcal{B}_\infty$ instead of algebras. Our aim is to prove that any $\widetilde{\mathcal{B}}$-coalgebra admits a natural $\mathcal{B}_\infty$-coalgebra structure and vice versa.

Note

7.1.1. **Lemma.** *$\widetilde{\mathcal{B}}$-coagebra structure on a graded vector space X is given by a structure of dg Lie bialgebra on*

$$\widehat{\mathbb{F}}_{\mathtt{LIE}}(X[1])=\prod_{n=0}^{\infty}\mathbb{F}^n_{\mathtt{LIE}}(X[1]).$$

7.1.2. **Lemma.** *$\mathcal{B}_\infty$-coagebra structure on a graded vector space X is given by a structure of dg bialgebra on*

$$\widehat{\mathbb{F}}_{\mathtt{ASS}}(X[1])=\prod_{n=0}^{\infty}X[1]^{\otimes n}.$$

Now we wish to use [EK] in order to pass from one structure above to the other. The idea is the following. One can interpret completions of free algebras $\mathbb{F}_{\mathtt{LIE}}(V)$ and $\mathbb{F}_{\mathtt{ASS}}(V)$ as equivariant $k[[h]]$-algebras $\mathbb{F}_{\mathtt{LIE}}(V)[[h]]$ and $\mathbb{F}_{\mathtt{LIE}}(V)[[h]]$. This is the situation Etingof-Kazhdan theory applies.

7.2. **Etingof-Kazhdan theory.** Let $\mathtt{Locc}(k)$ be the category of local complete k-algebras with residue field k.

Let $\mathcal{A}$ be an abelian k-linear tensor category. For each $R\in\mathtt{Locc}(k)$ we denote by $\mathcal{A}(R)$ the category with the same objects as $\mathcal{A}$ and with the morphisms defined by the formula

$$\mathrm{Hom}_{\mathcal{A}(R)}(X,Y)=\mathrm{Hom}_{\mathcal{A}}(X,Y)\otimes R.$$

The object of $\mathcal{A}(R)$ corresponding to an object $X\in\mathcal{A}$ is denoted X_R. In the other direction, for $Y\in\mathcal{A}(R)$ we write $\overline{Y}$ for the corresponding object of $\mathcal{A}$. The assignments $X\mapsto X_R$ and $Y\mapsto\overline{Y}$ define a pair of functors between $\mathcal{A}$ and $\mathcal{A}(R)$.

Let $\mathtt{LBA}_0(R)$ be the category of Lie bialgebras $(\mathfrak{g},[\quad],\delta)$ in $\mathcal{A}(R)$ whose cobracket δ vanishes modulo the maximal ideal $\mathfrak{m}$ of R. Let $\mathtt{HA}_0$ denote the category of Hopf algebras in $\mathcal{A}(R)$ whose reduction modulo $\mathfrak{m}$ is isomorphic to the enveloping algebra of a Lie algebra in $\mathcal{A}$.

The following theorem can be found in [EK].

7.2.1. **Theorem.** *There is an equivalence of categories*

$$Q : \mathtt{LBA}_0(R) \to \mathtt{HA}_0(R) \tag{24}$$

satisfying the following properties (see also explanations below)

1. $\overline{Q(\mathfrak{g})} = U(\overline{\mathfrak{g}})$
2. $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *measures the deviation of the coproduct in* $Q(\mathfrak{g})$ *from being cocommutative.*
3. Q *is given by universal formulas.*

7.2.2. The second property of the functor Q mentioned in Theorem 7.2.1 means the following. Denote by $i : \mathfrak{g} \to Q(\mathfrak{g})$ the image under the functor $\otimes R$ of the obvious embedding $\overline{\mathfrak{g}} \to U(\overline{\mathfrak{g}})$. The property 2 claims that the map from $\mathfrak{g}$ to $Q(\mathfrak{g}) \otimes Q(\mathfrak{g})$ given by the difference

$$(\Delta - \Delta') \circ i - (i \otimes i) \circ \delta$$

vanishes modulo $\mathfrak{m}^2$.

Here Δ is the coproduct in $Q(\mathfrak{g})$ and Δ' is the coproduct composed with the commutativity constraint.

7.2.3. The third property means the following. As an object of $\mathcal{A}(R)$, $Q(\mathfrak{g})$ is just the symmetric algebra $S(\mathfrak{g}) = \oplus S^n(\mathfrak{g})$. Therefore, the Hopf algebra structure on $Q(\mathfrak{g})$ is given by a collection of maps $m_{pqr} : S^p(\mathfrak{g}) \otimes S^q(\mathfrak{g}) \to S^r(\mathfrak{g})$ and $\Delta_{pqr} : S^p(\mathfrak{g}) \to S^q(\mathfrak{g}) \otimes S^r(\mathfrak{g})$. Universality condition means that the maps m_{pqr}, Δ_{pqr} are described as universal polynomials on the bracket and cobracket in $\mathfrak{g}$.

7.2.4. It is convenient to define $\mathtt{LBA}_0$ to be the category of pairs $(R, \mathfrak{g})$ where $R \in \mathtt{Locc}(k)$ and $\mathfrak{g} \in \mathtt{LBA}_0(R)$. In the same fashion one defines the category $\mathtt{HA}_0$. Since the functors $Q : \mathtt{LBA}_0(R) \to \mathtt{HA}_0(R)$ are given by the universal formulas, they form a functor $Q : \mathtt{LBA}_0 \to \mathtt{HA}_0$ which is also an equivalence of categories. Reduction modulo the maximal ideal defines a commutative diagram of functors

$$\begin{array}{ccc}
\mathtt{LBA}_0 & \xrightarrow[Q]{} & \mathtt{HA}_0 \\
\downarrow & & \downarrow \\
\mathtt{Lie}(k) & \xrightarrow[U]{} & \mathtt{HA}_0(k)
\end{array}$$

where $\mathtt{Lie}(k)$ is the category of Lie algebras over k and U is the enveloping algebra functor.

7.2.5. The equivalence of categories $Q : \mathtt{LBA}_0 \to \mathtt{HA}_0$ gives rise to an equivalence Q^G : $\mathtt{LBA}_0^G \to \mathtt{HA}_0^G$ between the categories of objects endowed with a G-action, G being a group.

Let now $\mathcal{A}$ be the category of complexes of k-modules. Let $G = k^*$ be the multiplicative group, $R = k[[h]]$. Let k^* act on R by the formula $\lambda(h) = \lambda^{-1}h$. Let $V \in \mathcal{A}$. Let k^* act on V by the formula $\lambda(v) = \lambda \cdot v$. This action extends to a k^*-action on $\mathbb{F}_{\mathtt{LIE}}(V)$ and $\mathbb{F}_{\mathtt{ASS}}(V)$, as well as to an action on $\mathbb{F}_{\mathtt{LIE}}(V)[[h]]$ and $\mathbb{F}_{\mathtt{ASS}}(V)[[h]]$.

Theorem 7.2.1 implies the following

7.2.6. **Corollary.** *The functor Q establishes an equivalence between the following categories:*

1. *Lie bialgebras $(R, \mathfrak{g}) \in \mathtt{LBA}_0$ endowed with a k^*-action compatible with the specified above k^*-action on $R = k[[h]]$ and on $\overline{\mathfrak{g}} = \mathbb{F}_{\mathtt{LIE}}(V)$.*
2. *Associative bialgebras $(R, H) \in \mathtt{HA}_0$ endowed with a k^*-action compatible with the specified above action on $R = k[[h]]$ and on $\overline{H} = \mathbb{F}_{\mathtt{ASS}}(V)$.*

7.3. **Theorem.** *There exists an isomorphism between the operads $\widetilde{\mathcal{B}}$ and $\mathcal{B}_\infty$.*

Theorem 7.3 is proven in 7.3.1–7.3.3 below.

7.3.1. Put $\mathfrak{g} = \mathbb{F}_{\mathtt{LIE}}(V)$. A $k[[h]]$-Lie bialgebra structure on $\mathfrak{g}[[h]]$ is given by a collection of maps

$$\delta^r_{pq} : V \to \mathbb{F}^p_{\mathtt{LIE}}(V) \otimes \mathbb{F}^q_{\mathtt{LIE}}(V)$$

such that the cobracket $\delta : \mathfrak{g}[[h]] \to \mathfrak{g}[[h] \otimes \mathfrak{g}[[h]]$ restricted to $\mathfrak{g}$ is given by the formula

$$\mathfrak{g} = \sum_{p,q,r} \delta^r_{pq} \cdot h^r.$$

Define an action of k^* on $\mathfrak{g}[[h]]$ as in 7.2.5. The cobracket δ of $\mathfrak{g}[[h]]$ is equivariant if and only if it satisfies the property

$$\delta^r_{pq} = 0 \text{ for } r \neq p + q - 1. \tag{25}$$

One can easily identify dg Lie bialgebra structures on $\mathfrak{g}[[h]]$ satisfying (25) with dg Lie bialgebra structures on $\widehat{\mathfrak{g}}$.

7.3.2. Similarly, dg bialgebra structures on $\widehat{\mathbb{F}}_{\mathtt{ASS}}(V)$ can be identified with equivariant bialgebra structures on $\mathbb{F}_{\mathtt{ASS}}(V)[[h]]$.

7.3.3. We use Corollary 7.2.6 of the equivalence Q from Etingof-Kazhdan Theorem 7.2.1.

Let X be a complex, $V=X[1]$. $\widetilde{\mathcal{B}}$-coalgebra structure on X is given by a structure of dg Lie bialgebra on $\widehat{\mathfrak{g}}=\widehat{\mathbb{F}}_{\mathtt{LIE}}(V)$ which is the same as an equivariant dg Lie bialgebra structure on $\mathfrak{g}[[h]]$.

According to Corollary 7.2.6, this defines canonically an equivariant dg Hopf algebra $(H,m,\Delta)\in\mathtt{HA}_0$.

The canonical map $i:V[[h]]\to H$ given by the composition

$$i:V\to\mathfrak{g}\to\overline{H}$$

induces an algebra homomorphism $F(i):\mathbb{F}_{\mathtt{ASS}}(V)[[h]]\to H$. It is isomorphism since its reduction modulo h $\overline{F(i)}$ is the identity map. This defines canonically an equivariant bialgebra structure on $\mathbb{F}_{\mathtt{ASS}}(V)[[h]]$ which is the same as a dg bialgebra structure on $\widehat{\mathbb{F}}_{\mathtt{ASS}}(V)$. Theorem is proven.

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