DG COALGEBRAS AS FORMAL STACKS

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1. Introduction

1.1. In this paper we study the category of differential graded cocommutative coalgebras over a field k of characteristic zero.

The main motivation comes from formal deformation theory — we look for an object which might be called a formal moduli space (or, more precisely, a formal moduli dg stack). In our approach, which goes back to Drinfeld's letter [D], formal stacks are described by dg cocommutative coalgebras which are defined up to weak equivalence. We prefer working with coalgebras (instead of complete local algebras) in order to avoid superfluous finiteness conditions.

In the first part of the paper (Sections 3 – 7) we provide the category of unital (dg, unbounded) coalgebras $\mathtt{dgcu}(k)$ over a field k of characteristic zero with a structure of a simplicial closed model category — see 2.3. This structure generalizes the one defined by Quillen [Q2] in 1969 for 2-reduced unital coalgebras. The major difference is that our notion of weak equivalence is strictly stronger (see the example in 9.1.2) then that of quasi-isomorphism.

In the second part of the paper (Sections 8 – 10) we use unital dg coalgebras (defined up to homotopy) to represent formal deformation functors in characteristic zero.

Classical formal deformation theory [Sc] is described by a functor from artinian local rings to sets which is seldom representable.

We suggest describing formal deformation problems over a field k of characteristic zero by functors

$$F : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens} \tag{1}$$

from the category of non-positively graded artinian local dg k-algebras to the category of simplicial sets. These functors are represented (in a homotopy category sense) by unital coalgebras from $\mathtt{dgcu}(k)$ — see 8.1. Such a coalgebra plays the role of the coalgebra of distributions concentrated at a point and it is defined uniquely up to weak equivalence in $\mathtt{dgcu}(k)$.

In Section 8 we study properties of the functors (1) which appear as *nerves* of dg Lie algebras. In Section 9 we calculate some elementary examples. The most interesting example — that of deformations of a principal G-bundle — in considered in Section 10. The main result of this section, Theorem 10.4.4, claims that the deformations of a principal G-bundle P on a scheme X are represented by the standard complex of the Lie algebra $\mathbf{R}\Gamma(X, \mathfrak{g}_P)$. A similar description was given in [Ka] for the deformations of G-local systems on a topological space.

1.2. **Tangent Lie algebra.** We suggest interpreting a unital dg coalgebra as the coalgebra of distributions concentrated at a point of a would-be-a-space.

1.2.1. The Quillen functor $\mathcal{L} : \mathtt{dgcu}(k) \to \mathtt{dglie}(k)$ to the category of dg Lie algebras (see [Q2], App. B or 2.2 below) is interpreted as the tangent Lie algebra functor. Theorem 3.2 claims that the adjoint pair $(\mathcal{L}, \mathcal{C})$ of functors establishes an equivalence between the homotopy categories of $\mathtt{dgcu}(k)$ and of $\mathtt{dglie}(k)$. This means that a formal dg stack can be (uniquely up to weak equivalence) reconstructed from the homotopy type of its tangent Lie algebra.

1.2.2. Formal schemes define, of course, coalgebras concentrated in degree zero. The corresponding tangent Lie algebra is concentrated in strictly positive degrees. More generally, a formal stack $X \in \mathtt{dgcu}(k)$ satisfying $H^i(\mathcal{L}(X)) = 0$ for $i \leq 0$, is called *a formal space*. Equivalently, this means that X is weakly equivalent to a coalgebra concentrated in non-negative degrees. Formal spaces have also a description in terms of the functor on Artin rings they represent — see 1.3.2 below.

1.3. **Functors on Artin rings.** Thus, we consider unital coalgebras as "the most general" formal (dg) stacks concentrated at a point. It is reasonable to describe them as functors on formal spaces — as one defines stacks using functors on affine schemes. One can go even further and take into account that any formal space is a filtered colimit of finite dimensional ones which now take form A^* where $A \in \mathtt{dgart}^{\leq 0}(k)$.

1.3.1. Any formal stack $X \in \mathtt{dgcu}(k)$ gives rise to a *deformation functor*

$$\widetilde{X} : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

which is defined up to homotopy equivalence. This corresponds to the usual description of stacks as 2-functors from affine schemes to groupoids. The deformation functor enjoys nice exactness properties (see 8.1.3). Given the tangent Lie algebra $\mathfrak{g} = \mathcal{L}(X)$, the functor $\widetilde{X}$ can be described as the *nerve* $\Sigma_{\mathfrak{g}}$ of the dg Lie algebra $\mathfrak{g}$ (see [H1] and 8.1.1) defined by the formula

$$\Sigma_{\mathfrak{g}}((A, \mathfrak{m}))_n = \mathrm{MC}(\mathfrak{m} \otimes \Omega_n \otimes \mathfrak{g})$$

for $(A, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$, where $\mathrm{MC}(\_)$ denotes the collection of Maurer-Cartan elements of a dg Lie algebra and Ω_n is the algebra of polynomial differential forms on the standard n-simplex.

The nerve of a dg Lie algebra is homotopy equivalent to the Deligne groupoid (cf. [GM1], [H1]) if $(\mathfrak{m} \otimes \mathfrak{g})^i = 0$ for $i < 0$.

1.3.2. If X is a formal space, the restriction of $\widetilde{X}$ to the category $\mathtt{art}(k)$ of artinian k-algebras concentrated in degree zero, is a functor to discrete simplicial sets (i.e., essentially, to $\mathtt{Ens}$) — see 9.3.2.

This means in particular, that the restriction of $\widetilde{X}$ to $\mathtt{art}(k)$ is representable in the usual sense by $H^0(X)$.

1.4. **Rational spaces.** A very "non-geometric" class of unital coalgebras is the Quillen's category $\mathtt{dgcu}_2(\mathbb{Q})$ of 2-reduced unital coalgebras which is one of the models for simply connected rational homotopy types.

One can easily calculate the deformation functor defined by a simply connected rational homotopy type. Let $\mathfrak{g} \in \mathtt{dglie}(\mathbb{Q})$ be the Lie algebra model for it. This is the tangent Lie algebra of the corresponding unital coalgebra. One has $\mathfrak{g}^i = 0$ for $i \geq 0$.

Let $(A, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(\mathbb{Q})$. Then $\Sigma_{\mathfrak{g}}(A)$ is a simply-connected rational space and its homotopy type corresponds to the Lie algebra $\mathfrak{m} \otimes \mathfrak{g}$ — see 9.4.

1.5. **Coarse moduli.** Let $\mathfrak{g} \in \mathtt{dglie}(k)$. One might be willing to consider the functor $\Pi_{\mathfrak{g}} : (A, \mathfrak{m}) \mapsto \pi_0(\Sigma_{\mathfrak{g}}(A))$ on the category $\mathtt{art}(k)$ as the "coarse moduli" space for the deformation problem defined by $\mathfrak{g}$. Usually this functor is not representable (except for the case described in 1.3.2). However, it admits a hull in the sense of [Sc] [1] which can be easily constructed using a dg Lie subalgebra $\mathfrak{h}$ which is a 1-truncation of $\mathfrak{g}$ as in [GM2]. A 1-truncation $\mathfrak{h}$ being chosen, the hull of the functor $\Pi_{\mathfrak{g}}$ can be described as $H^0(\mathcal{C}(\mathfrak{h}))$ — see 9.3.4.

The choice of the truncation $\mathfrak{h}$ is not unique though the resulting coalgebra is unique up to non-canonical isomorphism by a general result of [Sc]. One might ask whether the 1-truncation $\mathfrak{h}$ of $\mathfrak{g}$ is unique up to quasi-isomorphism. This is obviously so in the case of [GM2] where $H^i(\mathfrak{g}) = 0$ for $i \leq 0$. We doubt this is true in general.

1.6. Two general ideas

Idea 1: Any reasonable formal deformation problem in characteristic zero can be described by Maurer-Cartan elements of an appropriate dg Lie algebra

Idea 2: Moduli spaces should admit a natural sheaf of dg commutative algebras as a structure sheaf

have been spelled out by different people during the last years (Drinfeld [D], Feigin, Deligne, Kontsevich [Ko]). In this paper we tried to show that these two claims are essentially equivalent to the one saying that any reasonable formal deformation problem can be described by a representable functor on dg artinian rings with values in simplicial sets.

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2. Preliminaries

In this section we fix some notation and definitions.

Throughout this section k is a commutative ring containing $\mathbb{Q}$.

2.1. **Unital coalgebras.** Let X be a cocommutative dg coalgebra X over k with comultiplication $\Delta : X \to X \otimes X$ and counit $\epsilon : X \to k$.

Recall that an element $u \in X$ is called *a group-like element* if

(1) $d(u) = 0$; (2) $\Delta(u) = u \otimes u$; (3) $\epsilon(u) = 1$.

[1] translated to the language of coalgebras

A choice of a group-like element $u \in X$ defines a decomposition

$$X = k \cdot u \oplus \overline{X}$$

where $\overline{X} = \ker(\epsilon)$. This defines an increasing filtration on X by the formula

$$X_n = \ker(X \xrightarrow{\Delta_n} X^{\otimes n+1} \to \overline{X}^{\otimes n+1}) \tag{2}$$

where Δ_n is the n-th iteration of Δ.

2.1.1. **Definition.** (see [HS2]) A pair (X, u) consisting of a dg cocommutative coalgebra X and a group-like element u is called *a unital coalgebra* if the filtration (2) is exhausting.

The group-like element defining a unital coalgebra X is called *the unit of* X and it is usually denoted by 1. The filtration (2) of a unital coalgebra is called *the canonical filtration*.

Note that if k is a field then unital coalgebras are just connected coalgebras of Quillen (see [Q2], App. B). In this case the unit element is unique and it is preserved by any coalgebra map.

2.1.2. The category of unital coalgebras over k will be denoted by $\mathtt{dgcu}(k)$. The morphisms in it are supposed to preserve the units (this is automatically fulfilled when k is a field).

2.1.3. Let Ω be a commutative dg algebra over k. Similarly to the above, one defines Ω-coalgebras as cocommutative coalgebras in the category of Ω-modules. Furthermore, one defines unital Ω-coalgebras as pairs $(X, 1)$ with an Ω-coalgebra X and a group-like element 1 of X such that the filtration (2) is exhausting. The category of unital Ω-coalgebras is denoted $\mathtt{dgcu}(\Omega)$.

2.2. **Quillen functors.** Recall the definition of the couple of adjoint functors

$$\mathcal{L} : \mathtt{dgcu}(k) \rightleftarrows \mathtt{dglie}(k) : \mathcal{C} \tag{3}$$

defined by Quillen in [Q2], App. B.

2.2.1. Let $X \in \mathtt{dgcu}(k)$, $\overline{X} = \ker \epsilon$ in the standard notation. The dg Lie algebra $\mathcal{L}(X)$ is defined as follows. As a graded Lie algebra, this is the free Lie algebra $F(\overline{X}[-1])$. The differential in $\mathcal{L}(X)$ is the sum of two parts: the one generated by the differential of $\overline{X}[-1]$, and the second defined to be the only derivation of the free Lie algebra $F(\overline{X}[-1])$ whose restriction to $\overline{X}[-1])$ is given by the map

$$\Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1 : \overline{X} \to \overline{X} \otimes \overline{X}.$$

2.2.2. Let $\mathfrak{g} \in \mathtt{dglie}(k)$. The unital coalgebra $\mathcal{C}(\mathfrak{g})$ is defined as follows. As a unital graded coalgebra, this is the cofree cocommutative coalgebra $S(\mathfrak{g}[1]) = \oplus_{n \geq 0} S^n(\mathfrak{g}[1])$. The differential in $\mathcal{C}(\mathfrak{g})$ is the sum of two parts: the one generated by the differential in $\mathfrak{g}[1]$, and the second defined by its 1-component given by the Lie bracket $[,] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$.

2.2.3. Let $\mathfrak{g} \in \mathtt{dglie}(k)$. Recall that an element $x \in \mathfrak{g}^1$ is called a Maurer-Cartan element if $dx + \frac{1}{2}[x, x] = 0$. The set of Maurer-Cartan elements of $\mathfrak{g}$ is denoted by $\mathrm{MC}(\mathfrak{g})$.

2.2.4. If $X \in \mathtt{dgcu}(k), \mathfrak{g} \in \mathtt{dglie}(k)$, then the complex $\mathrm{Hom}(\overline{X}, \mathfrak{g})$ admits a natural structure of a dg Lie algebra given by the formula

$$[f, g] = \ell(f \otimes g)\Delta$$

where Δ is the comultiplication in X and ℓ is the bracket in $\mathfrak{g}$. In particular the set $\mathrm{MC}(X, \mathfrak{g}) := \mathrm{MC}(\mathrm{Hom}(\overline{X}, \mathfrak{g}))$ is defined.

2.2.5. **Theorem.** *The functors $\mathcal{L}$ and $\mathcal{C}$ (3) are adoint. More precisely, for $X \in \mathtt{dgcu}(k), \mathfrak{g} \in \mathtt{dglie}(k)$ one has natural bijections*

$$\mathrm{Hom}(\mathcal{L}(X), \mathfrak{g}) = \mathrm{Hom}(X, \mathcal{C}(\mathfrak{g})) = \mathrm{MC}(X, \mathfrak{g}).$$

See [Q2], App. B6.

2.3. **Simplicial closed model categories.** We use the axioms (CM1)–(CM5) of [Q2] for the definition of closed model category (denoted CMC).

A simplicial category $\mathcal{C}$ is a collection of objects $\mathrm{Ob}\,\mathcal{C}$ together with a collection of simplicial sets $\mathcal{H}om(X, Y)$ assigned to each pair (X, Y) of objects, with strictly associative compositions.

A simplicial category $\mathcal{C}$ with a CMC structure will be called a simplicial CMC if the following Quillen's axiom (SM7) [Q1] is fulfilled

(SM7) Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration in $\mathcal{C}$. Then the map of simplicial sets

$$\mathcal{H}om(B, X) \to \mathcal{H}om(A, X) \times_{\mathcal{H}om(A,Y)} \mathcal{H}om(B, X) \tag{4}$$

is a Kan fibration. If, moreover, either i or p is a weak equivalence, then (4) is an acyclic Kan fibration.

Note that we do not include Quillen's axiom (SM0) claiming the existence of cylinder and path objects — see [Q1] *— in our definition of simplicial closed model category.*

2.4. **Models for dg Lie algebras.** Recall [H2] that the category $\mathtt{dglie}(k)$ of dg Lie algebras over a commutative ring $k \supseteq \mathbb{Q}$ admits a simplicial model structure. More precisely, one has the following

Theorem. *(see* [H2]*, 4.1.1 and 4.8) The category $\mathtt{dglie}(k)$ admits a simplicial CMC structure with surjective maps as fibrations and quasi-isomorphisms as weak equivalences. The simplicial structure on $\mathtt{dglie}(k)$ is defined by the formula*

$$\mathcal{H}om_n(\mathfrak{g}, \mathfrak{h}) = \mathrm{Hom}_{\mathtt{dglie}(k)}(\mathfrak{g}, \Omega_n \otimes \mathfrak{h}). \tag{5}$$

where Ω_n is the algebra of polynomial differential forms on the standard n-simplex.

2.5. **Operad notation.** Throughout the paper we will sometimes use the language of operads — see e.g. [HS1]. In what follows $\mathtt{COM}$ denotes the operad for commutative algebras, $\mathtt{LIE}$ the one for Lie algebras and $\mathtt{LIE}_\infty$ denotes the standard Lie operad of [HS1], 4.1, governing "strongly homotopy Lie algebras". If $\mathcal{O}$ is an operad and A is an $\mathcal{O}$-algebra then $U(\mathcal{O}, A)$ denotes the corresponding enveloping algebra — see [HS1], 3.3.

We will use sometimes different base tensor categories. If $\mathcal{C}$ is a tensor (= symmetric monoidal) category, $\mathtt{Op}(\mathcal{C})$ denotes the category of operads over $\mathcal{C}$.

3. Simplicial closed model category structure on $\mathtt{dgcu}(k)$

Now we are ready to formulate the main results of the first part of the paper.

3.1. **Theorem.** *The category $\mathtt{dgcu}(k)$ of unital coalgebras over a field k of characteristic zero admits a simplicial CMC structure. Cofibrations in it are injective maps and weak equivalences are the maps f in $\mathtt{dgcu}(k)$) such that $\mathcal{L}(f)$ is a quasi-isomorphism. The simplicial structure on $\mathtt{dgcu}(k)$ is given by the condition*

$$\mathcal{H}om_n(X, Y) = \mathrm{Hom}_{\mathtt{dgcu}(\Omega_n)}(\Omega_n \otimes X, \Omega_n \otimes Y) \tag{6}$$

where as in (5) Ω_n is the algebra of polynomial differential forms on the standard n-simplex.

Note that the property of a morphism f of $\mathtt{dgcu}(k)$ to be a weak equivalence is strictly stronger then that of being a quasi-isomorphism — see the counter-example in 9.1.2.

3.2. **Theorem.** *The adjoint functors $\mathcal{L}, \mathcal{C}$ induce an equivalence of the corresponding homotopy categories*

$$\mathbf{L}\mathcal{L} : \mathrm{Ho}(\mathtt{dgcu}(k)) \rightleftarrows \mathrm{Ho}(\mathtt{dglie}(k)) : \mathbf{R}\mathcal{C}.$$

The proof of Theorems 3.1, 3.2 is given in Sections 3 – 7.

3.3. **Functors $\mathcal{C}$ and $\mathcal{L}$.** Let us study some basic properties of the adjoint functors $\mathcal{L}$ and $\mathcal{C}$ defined above. We start with the following lemma whose proof can be deduced from [H2], Sect. 6 (note that the general claim of [HS1], 3.6.12 contains an error).

3.3.1. **Lemma.** *(see* [H2]*, 6.8.5) Let $\mathfrak{g}$ be a dg Lie algebra over a commutative ring $k \supseteq \mathbb{Q}$. It can be obviously considered as a $\mathtt{LIE}_\infty$-algebra. Suppose that $\mathfrak{g}$ is k-flat. Then the natural map*

$$U(\mathtt{LIE}_\infty, \mathfrak{g}) \to U(\mathtt{LIE}, \mathfrak{g})$$

is a quasi-isomorphism.

Proof. We use the results of [H2], Sect. 6. The functor $U(\mathtt{LIE}_\infty, \_)$ carries quasi-isomorphisms of flat dg Lie algebras into quasi-isomorphisms, since $\mathtt{LIE}_\infty$ is a cofibrant operad — see [H2], 6.8.3.

The functor $U(\mathtt{LIE}, \_)$ carries quasi-isomorphisms of flat dg Lie algebras into quasi-isomorphisms by the PBW theorem (see [Q2], App. B). Choose a cofibrant resolution $P \to \mathfrak{g}$ of $\mathtt{LIE}_\infty$-algebra $\mathfrak{g}$ and let $\overline{P}$ be the Lie algebra which is the inverse image of P with respect to the map $\mathtt{LIE}_\infty \to \mathtt{LIE}$.

Now the lemma fommows from Comparison theorem [H2], 5.5.1 and from the fact that P and $\overline{P}$ are flat. $\square$

The following Proposition 3.3.2 has a very important filtered analog — see 4.4.3 below.

3.3.2. **Proposition.**

(1) The functor $\mathcal{C}$ preserves quasi-isomorphisms.

(2) The adjunction maps $i_X : X \to \mathcal{CL}(X)$ and $p_{\mathfrak{g}} : \mathcal{LC}(\mathfrak{g}) \to \mathfrak{g}$ are quasi-isomorphisms.

(3) The restriction of $\mathcal{L}$ to the subcategory $\mathtt{dgcu}^{\geq 0}(k)$ of non-negatively graded coalgebras preserves quasi-isomorphisms.

Proof. Step 1. Let us prove first that the map $p_{\mathfrak{g}} : \mathcal{LC}(\mathfrak{g}) \to \mathfrak{g}$ is a quasi-isomorphism. Consider the Lie algebra $\mathfrak{g}$ as an algebra over the operad $\mathtt{LIE}_\infty$ as above. According to Lemma 3.3.1, the natural map of the enveloping algebras $U(\mathtt{LIE}_\infty, \mathfrak{g}) \to U(\mathfrak{g})$ is a quasi-isomorphism. Now, one has an isomorphism $U(\mathtt{LIE}_\infty, \mathfrak{g}) = U(\mathcal{LC}(\mathfrak{g}))$ and then the adjunction map $p_{\mathfrak{g}}$ is quasi-isomorphism by the PBW theorem [Q2], App. B.

Step 2. Now we check that $\mathcal{C}$ preserves quasi-isomorphisms. The coalgebra $\mathcal{C}(\mathfrak{g})$ admits an increasing filtration natural in $\mathfrak{g}$ so that the associated graded pieces are $S^n(\mathfrak{g}[1])$. This clearly implies the claim.

Step 3. Now we can prove that the map $i_X : X \to \mathcal{CL}(X)$ is a quasi-isomorphism for any $X \in \mathtt{dgcu}(k)$. In fact, the map i_X admits a natural splitting $q_X : \mathcal{CL}(X) \to X$ as a map of complexes: $Y = \mathcal{CL}(X)$ as a graded vector space takes form

$$Y = S(F(X[-1])[1])$$

where S is the symmetric algebra and F is the free Lie algebra functor. This defines the projection $q_X : Y \to F(X[-1])[1] \to X$ which is compatible with the differentials and splits $i : X \to Y$. Now consider the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{i_X} & \mathcal{CL}(X) & \xrightarrow{q_X} & X \\
{\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle \mathcal{CL}(i_X)} & & \downarrow{\scriptstyle i_X} \\
\mathcal{CL}(X) & \xrightarrow[i_{\mathcal{CL}(X)}]{} & \mathcal{CLCL}(X) & \xrightarrow[q_{\mathcal{CL}(X)}]{} & \mathcal{CL}(X)
\end{array}$$

The left square in it is commutative by the general nonsense of adjoint functors; the right square is commutative since q_X is functorial in X. The map $\mathcal{L}(i_X)$ is a quasi-isomorphism since it is split by the quasi-isomorphism $p_{\mathcal{L}(X)}$ by Step 1. Then, by Step 2, the map $\mathcal{CL}(i_X)$ is also a quasi-isomorphism, and therefore its retract i_X is a quasi-isomorphism as well.

Step 4. The claim (3) follows by a standard spectral sequence argument.

$\square$

4. Filtered world and graded world

In this section we prove a filtered analog of Lemma 3.3.1 and of Proposition 3.3.2 — see Proppositions 4.3.7, 4.4.3 below. For this we need a number of new categories and functors and a well-known Rees trick which allows one to reduce some filtered objects to graded objects over the polynomial ring — see 4.3.

4.1. **Filtered world.**

4.1.1. *Definitions.* Here k is a base commutative ring. A filtered k-module V is a collection $V=\{V_i\},\ i\in\mathbb{Z}$ with $V_i\subseteq V_{i+1}$ and $V=\cup V_i$. The category of filtered k-modules is denoted $\mathtt{modf}(k)$.

A filtered complex is a complex in $\mathtt{modf}(k)$. The category of filtered complexes will be denoted in the sequel $CF(k)$ instead of $C(\mathtt{modf}(k))$.

A morphism $f:X\to Y$ in $CF(k)$ is called *a filtered quasi-isomorphism* if for each $n\in\mathbb{Z}$ the map of the corresponding subcomplexes

$$f_n:X_n\to Y_n$$

is a quasi-isomorphism.

The category $\mathtt{modf}(k)$ admits a tensor structure given by the formula

$$(X\otimes Y)_n=\sum_{p+q=n}\mathrm{Im}(X_p\otimes Y_q\to X\otimes Y).$$

This tensor structure induces a tensor structure on $CF(k)$.

The functor $\#:CF(k)\to C(k)$ forgetting the filtration preserves the tensor structure.

4.1.2. There is an obvious functor

$$\tau:C(k)\to CF(k)$$

given by $\tau(X)_{-1}=0$; $\tau(X)_n=X$, for $n\geq 0$. The functor τ preserves the tensor structure. Thus τ induces a functor $\tau:\mathtt{Op}(C(k))\to\mathtt{Op}(CF(k))$. For an operad $\mathcal{O}\in\mathtt{Op}(C(k))$ we denote by $\mathtt{Algf}(\mathcal{O})$ (instead of $\mathtt{Alg}(\tau(\mathcal{O}))$) the category of filtered $\mathcal{O}$-algebras.

4.1.3. **Example.** We write $\mathtt{dglf}(k)$ instead of $\mathtt{Algf}(\mathtt{LIE})$ for the category of filtered dg Lie algebras. Explicitly, such an algebra is a filtered complex $\mathfrak{g}=\{\mathfrak{g}_i\}$ with a Lie bracket satisfying $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_{i+j}$. Similarly, we write $\mathtt{dgcf}$ for the category of filtered cocommutative coalgebras. Its objects are filtered complexes $X=\{X_i\}$ endowed with a cocommutative comultiplication satisfying

$$\Delta(X_n)\subseteq\sum_{p+q=n}X_p\otimes X_q.$$

4.1.4. Fix $\mathcal{O} \in \mathtt{Op}(C(k))$ and let $A \in \mathtt{Algf}(\mathcal{O})$. The filtration on A induces a natural filtration on the enveloping algebra $U(\mathcal{O}, A^\#)$ defined as follows. Recall (see [HS1], Sect. 3) that $U(\mathcal{O}, A^\#)$ is a quotient of the "$\mathcal{O}$-tensor algebra"

$$T(\mathcal{O}, A^\#) = \bigoplus_{n \geq 0} \mathcal{O}(n+1) \otimes_{\Sigma_n} (A^\#)^{\otimes n}.$$

We endow $T(\mathcal{O}, A^\#)$ with the tensor product filtration and $U(\mathcal{O}, A^\#)$ with the quotient filtration. One easily sees that this defines a filtered associative dg algebra which will be denoted in the sequel by $U(\mathcal{O}, A)$.

4.2. **Graded world.** Let R be a commutative graded k-algebra and let $R^\#$ be the underlying commutative algebra.

Denote by $\mathtt{modg}(R)$ the category of graded R-modules. It has an obvious tensor structure with the commutativity constraint

$$M \otimes_R N \to N \otimes_R M$$

given by the formula $m \otimes n \mapsto n \otimes m$ (no signs involved).

One has an obvious forgetful functor $\# : \mathtt{modg}(R) \to \mathtt{mod}(R^\#)$ preserving the tensor structure.

Denote $CG(R) = C(\mathtt{modg}(R))$. The forgetful functor defines a tensor functor

$$\# : CG(R) \to C(R^\#).$$

4.2.1. Tensoring by R defines a tensor functor

$$\tau : \mathtt{mod}(k) \to \mathtt{modg}(R).$$

This allows one, for any operad $\mathcal{O} \in \mathtt{Op}(C(k))$, to consider the category of $\tau(\mathcal{O})$-algebras. This latter will be denoted $\mathtt{Algg}(\mathcal{O}, R)$ or just $\mathtt{Algg}(\mathcal{O})$ (this will not lead to a confusion).

The enveloping algebra $U(\mathcal{O}, A)$ of $A \in \mathtt{Algg}(\mathcal{O})$ is defined in a standard way as in [HS1], Sect. 3, using the tensor structure on $CG(R)$.

We will need the following graded analog of Lemma 3.3.1.

4.2.2. **Proposition.** *Let $\mathfrak{g}$ be a flat graded dg Lie algebra over R. Then the natural map*

$$U(\mathtt{LIE}_\infty, \mathfrak{g}) \to U(\mathtt{LIE}, \mathfrak{g})$$

is a graded quasi-isomorphism.

Proof. Since the forgetful functor $\# : \mathtt{modg}(R) \to \mathtt{mod}(R^\#)$ is exact, and since a map $f : X \to Y$ is a graded quasi-isomorphism if and only if $f^\#$ is a quasi-isomorphism, the result immediately follows from Lemma 3.3.1. $\square$

4.3. **Rees functor.**

From now on k is a field of characteristic zero and $R = k[t]$ with $\deg(t) = 1$*.*

The Rees functor

$$\rho : \mathtt{modf}(k) \longrightarrow \mathtt{modg}(R)$$

is defined by the formula

$$\rho(V) = \sum V_i t^i \subseteq \tau(V) = V \otimes R.$$

4.3.1. **Lemma.** *1. Rees functor preserves the tensor structure.*

2. One has $\rho\tau = \tau$ (two different τ, from 4.1 and from 4.2, are involved).

Proof. Straighforward. □

4.3.2. **Corollary.** *The Rees functor induces a functor*

$$\rho : \mathtt{Algf}(\mathcal{O}) \to \mathtt{Algg}(\mathcal{O}).$$

4.3.3. The Rees functor ρ identifies the category $\mathtt{modf}(k)$ with the full subcategory of $\mathtt{modg}(R)$ consisting of graded torsion-free (=flat) R-modules. The functor ρ admits a left adjoint functor $\phi : \mathtt{modg}(R) \to \mathtt{modf}(k)$ defined by the formulas

$$\phi(M) = \lim_{\to} M_n = M/(1-t)M;\ \phi(M)_n = \mathrm{Im}(M_n \to \phi(M)).$$

4.3.4. **Proposition.** *Let $\mathcal{O} \in \mathtt{Op}(C(k))$, $A \in \mathtt{Algf}(\mathcal{O})$. The filtered enveloping algebra of A can be calculated by the formula*

$$U(\mathcal{O}, A) = \phi(U(\mathcal{O}, \rho(A))).$$

Proof. The total space of $\phi(U(\mathcal{O}, \rho(A)))$ is equal to

$$U(\mathcal{O}, \rho(A)) \otimes_R R/(1-t)R = U\left(\mathcal{O}, \rho(A) \otimes_R R/(1-t)R\right) = U(\mathcal{O}, A).$$

To identify the filtration, recall that $U(\mathcal{O}, \rho(A))_n$ is the image of the n-th component of the tensor algebra $T(\mathcal{O}, \rho(A))$ which is an image of

$$\bigoplus_{i_1+\ldots+i_k=n} \mathcal{O}(k+1) \otimes A_{i_1} \otimes \ldots \otimes A_{i_k}.$$

This coincides with the definition of the filtration on $U(\mathcal{O}, A)$ as in 4.1.4.

□

4.3.5. **Corollary.** *Let $\mathcal{O} = \mathtt{LIE}$ or $\mathtt{LIE}_\infty$. Then for any $A \in \mathtt{Algf}(\mathcal{O})$ one has*

$$\rho(U(\mathcal{O}, A)) = U(\mathcal{O}, \rho(A)).$$

Proof. Having in mind Proposition 4.3.4, it is enough to check that $U(\mathcal{O}, \rho(A))$ is torsion-free for $\mathcal{O} = \mathtt{LIE}$ or $\mathtt{LIE}_\infty$.

We can forget the differentials in our dg objects. If $\mathcal{O} = \mathtt{LIE}$ the claim follows from the PBW theorem. In the second case $\mathcal{O} = \mathtt{LIE}_\infty$ is free as a graded operad, so that the corresponding enveloping algebra is a tensor algebra which has no torsion. □

4.3.6. Note the following nice (surely well-known) generalization of the PBW theorem.

Corollary. *Let $\mathfrak{g}$ be a filtered Lie algebra. Then the associated graded of the filtered enveloping algebra $U(\mathfrak{g})$ is isomorphic to the enveloping algebra of the associated graded Lie algebra.*

Proof. The passage to the associated graded module is the composition of the Rees functor with the base change with respect to $R \to R/(t)$. □

4.3.7. Comparing Corollary 4.3.5 with Proposition 4.2.2 we get the following filtered version of 3.3.1.

Proposition. *Let $\mathfrak{g}$ be a filtered dg Lie algebra over k. The natural map*

$$U(\mathtt{LIE}_\infty, \mathfrak{g}) \to U(\mathtt{LIE}, \mathfrak{g})$$

is a filtered quasi-isomorphism.

4.3.8. Note also the following filtered version of the PBW theorem.

Lemma. *Let $\mathfrak{g}$ be a filtered dg Lie algebra over k. The symmetrization map $S(\mathfrak{g}) \to \mathrm{U}(\mathtt{LIE}, \mathfrak{g})$ is an isomorphism of filtered complexes.*

Proof. Use the usual PBW theorem for the dg R-Lie algebra $\rho(\mathfrak{g})$. □

4.4. **A filtered version of Proposition 3.3.2.** Let $\mathfrak{g}$ be a filtered dg Lie algebra. The coalgebra $\mathcal{C}(\mathfrak{g})$ endowed with the induced filtration is a filtered unital coalgebra. *Note that filtered unital coalgebras admit two filtrations, the one being the given filtration, and the second being defined by the unit.* In the same way the functor $\mathcal{L}$ sends filtered unital coalgebras to filtered Lie algebras.

4.4.1. **Definition.** 1. A unital filtered coalgebra $X = \{X_i\} \in \mathtt{dgcf}(k)$ is called *admissible* (or, in other words, $\{X_i\}$ is an admissible filtration on X) if $X_{-1} = 0$ and $X_0 = k \cdot 1$.

2. A filtered dg Lie algebra $\mathfrak{g} = \{\mathfrak{g}_i\} \in \mathtt{dglf}(k)$ is admissible if $\mathfrak{g}_0 = 0$.

4.4.2. *Note.* If X is an admissible coalgebra, one has

$$\Delta(x) - 1 \otimes x - x \otimes 1 \in X_n$$

whenever $x \in X_{n+1}$. This follows from the formula

$$(1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta = \mathrm{id}\,.$$

The category of admissible filtered coalgebras is denoted by $\mathtt{dgca}(k)$, and the category of admissible dg Lie algebras is $\mathtt{dgla}(k)$.

4.4.3. **Proposition.** *1. The functors $\mathcal{L}$ and $\mathcal{C}$ define a pair of adjoint functors*

$$\mathcal{L} : \mathtt{dgca}(k) \rightleftarrows \mathtt{dgla}(k) : \mathcal{C}.$$

2. The functor $\mathcal{C}$ preserves filtered quasi-isomorphisms.

3. The adjunction maps $i_X : X \to \mathcal{C}\mathcal{L}(X)$ and $p_{\mathfrak{g}} : \mathcal{L}\mathcal{C}(\mathfrak{g}) \to \mathfrak{g}$ are filtered quasi-isomorphisms.

Proof. For $X \in \mathtt{dgca}(k)$ and $\mathfrak{g} \in \mathtt{dgla}(k)$ the sets $\mathrm{Hom}(\mathcal{L}(X), \mathfrak{g})$ and $\mathrm{Hom}(X, \mathcal{C}(\mathfrak{g}))$ coincide with the collection of filtration preserving maps from $\mathrm{MC}(X, \mathfrak{g})$. This proves the first claim.

Now we can proceed as in the proof of 3.3.2.

Step 1. Let $\mathfrak{g} \in \mathtt{dgla}(k)$. To check that the adjunction map $p_{\mathfrak{g}} : \mathcal{L}\mathcal{C}(\mathfrak{g}) \to \mathfrak{g}$ is a filtered quasi-isomorphism, note that $U(\mathtt{LIE}_\infty, \mathfrak{g}) = U(\mathtt{LIE}, \mathcal{L}\mathcal{C}(\mathfrak{g}))$ as filtered algebras so that Proposition 4.3.7 and Lemma 4.3.8 give what we need.

Step 2. Exactly as in the proof of 3.3.2, $\mathcal{C}$ preserves filtered quasi-isomorphisms since the coalgebra $\mathcal{C}(\mathfrak{g})$ admits an increasing filtration natural in $\mathfrak{g}$ with the associated graded pieces $S^n(\mathfrak{g}[1])$.

Step 3. Consider the diagram of Step 3 of the proof of 3.3.2. Since all the maps involved preserve the filtrations, and since the map $p_{\mathcal{L}(X)}$ is a filtered quasi-isomorphism by Step 1, the map $\mathcal{L}(i_X)$ and, therefore, its retract i_X, are also filtered quasi-isomorphisms.

This proves Proposition 4.4.3. ☐

Filtered quasi-isomorphisms of admissible coalgebras are useful because of the following

4.4.4. **Proposition.** *Let $f : X \to Y$ be a filtered quasi-isomorphism of admissible coalgebras. Then $\mathcal{L}(f)$ is a quasi-isomorphism.*

Proof. Since the functor $\mathcal{L}$ commutes with colimits and passing to cohomology commutes with filtered colimits, the claim can be proven by induction. Suppose, by the inductive hypothesis, that the map $\mathcal{L}(f_n) : \mathcal{L}(X_n) \to \mathcal{L}(Y_n)$ is a quasi-isomorphism. Denote $M = X_{n+1}/X_n$ and $N = Y_{n+1}/Y_n$.

The short exact sequences $X_n \to X_{n+1} \to M$ and $Y_n \to Y_{n+1} \to N$ split as sequences of graded k-vector spaces. Choose compatible splittings $a : M \to X_{n+1}$ and $b : N \to Y_{n+1}$. The maps $d(a) : M \to X_{n+1}$ and $d(b) : N \to Y_{n+1}$ can be carried through unique maps $\alpha : M[-1] \to X_n$

and $\beta : N[-1] \to Y_n$. We will denote by the same letters the compatible maps of complexes $\alpha : M[-2] \to \mathcal{L}(X_n)$ and $\beta : N[-2] \to \mathcal{L}(Y_n)$.

Recall the following notation from [H2], Sect. 1. Let $\mathcal{O}$ be a dg operad over k, X be an $\mathcal{O}$-algebra, M be a complex of k-modules and $f : M \to X$ be a map of complexes. Then an $\mathcal{O}$-algebra $X\langle M, f\rangle$ is defined to be the colimit of the diagram

$$X \leftarrow F(X) \to F(Y)$$

where $F(\_)$ is the free $\mathcal{O}$-algebra functor, Y is the cone of the map $f : M \to X$, the left arrow is induced by the $\mathcal{O}$-algebra structure on X and the right arrow is induced by the canocal map $X \to Y$ of complexes.

Using Note 4.4.2 one can easily observe that $\mathcal{L}(X_{n+1}) = \mathcal{L}(X_n)\langle M[-2], \alpha\rangle$ and $\mathcal{L}(Y_{n+1}) = \mathcal{L}(Y_n)\langle N[-2], \beta\rangle$. Since the complexes M and N are quasi-isomorphic and the Lie algebras $\mathcal{L}(X_n)$, $\mathcal{L}(Y_n)$ are cofibrant, Proposition 4.4.4 follows from the following lemma in the spirit of [H2], 5.3.3. □

4.4.5. **Lemma.** *Let $\mathcal{O}$ be an operad in $C(k)$. Suppose a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\beta} & Y \end{array}$$

is given so that $f : X \to Y$ is a quasi-isomorphism of cofibrant $\mathcal{O}$-algebras and $g : M \to N$ is a quasi-isomorphism of complexes. Then the induced map of $\mathcal{O}$-algebras $X\langle M, \alpha\rangle \to Y\langle N, \beta\rangle$ is a quasi-isomorphism.

Proof. One easily reduces the claim to the case $M = N = k \cdot e$ is generated by an only element e. Since X and Y are cofibrant, f is homotopy equivalence. This means that there exists a map $g : Y \to X$ homotopically inverse to f. Let $x = \alpha(e), y = \beta(e) = f(x)$. The difference $x - g(y)$ is obviously a boundary in the complex X; write it as $x - g(y) = du,\ u \in X$.

Choose a path diagram $X \xrightarrow{i} X^I \rightrightarrows X$ so that i is a standard acyclic cofibration (i.e. X^I has form $X \coprod F(V)$ with contractible V) and a map $\Phi : X \to X^I$ which realizes a homotopy between id_X and $gf : X \to X$. The map

$$i' : X\langle e; de = x\rangle \to X^I\langle e; de = i(x)\rangle$$

is a quasi-isomorphism since $X^I = X \coprod F(V)$. Since $i(x)$ and $\Phi(x)$ represent one and the same cohomology class in X^I, there is an obvious isomorphism between $X^I\langle e; de = i(x)\rangle$ and $X^I\langle e; de = \Phi(x)\rangle$. This implies that the arrows p'_i in the diagram below are quasi-isomorphisms.

$$\begin{array}{ccccc} & & & \overset{p'_1}{\nearrow} & X\langle e; de = x + du\rangle \\ X\langle e; de = x\rangle & \xrightarrow{\Phi'} & X^I\langle e; de = \Phi(x)\rangle & & \\ & & & \underset{p'_2}{\searrow} & X\langle e; de = x\rangle \end{array}$$

Thus, the map Φ', and therefore the composition

$$p_1'\Phi' : X\langle e; de = x\rangle \to X\langle e; de = x + du\rangle$$

induced by the map $gf : X \to X$, is a quasi-isomorphism. The same is true for the morphism $fg : Y \to Y$. This implies that the map

$$f' : X\langle e; e = x\rangle \to Y\langle e; e = y\rangle$$

is a quasi-isomorphism. □

5. Proofs of the theorems

5.1. The proof of Theorem 3.1 is based on the following Key Lemma whose proof we postpone until Section 6.

5.1.1. **Key Lemma.** *Given $X \in \mathtt{dgcu}(k)$, let $f : \mathfrak{g} \to \mathcal{L}(X)$ be a surjective map in $\mathtt{dglie}(k)$. Consider the cartesian diagram*

$$\begin{array}{ccc} Z & \xrightarrow{j} & \mathcal{C}(\mathfrak{g}) \\ \downarrow & & \downarrow{\scriptstyle \mathcal{C}(f)} \\ X & \xrightarrow{i_X} & \mathcal{CL}(X) \end{array}$$

in $\mathtt{dgcu}(k)$. Then $\mathcal{L}(j) : \mathcal{L}(Z) \to \mathcal{LC}(\mathfrak{g})$ is a quasi-isomorphism.

The lemma is very close to [Q2], II.5.6. Its proof in our context uses the technicalities of Section 4.

5.2. **Proof of Theorem 3.1.** To prove Theorem 3.1 we use Lemma 5.1.1 and follow the proof of Theorem II.5.2 of [Q2].

5.2.1. *Limits and colimits in* $\mathtt{dgcu}(k)$. The existence of colimits in $\mathtt{dgcu}(k)$ is obvious; the functor $\# : \mathtt{dgcu}(k) \to C(k)$ defined by the formula $\#(X) = \overline{X}$, commutes with colimits.

Finite products in $\mathtt{dgcu}(k)$ correspond to tensor products of the underlying complexes; also kernels of a pair of maps in $\mathtt{dgcu}(k)$ are the same as in $C(k)$. This proves the property (CM1) — see [Q2], II.1.

5.2.2. *Cofibrations in* $\mathtt{dgcu}(k)$.

Lemma. *Let $f : X \to Y$ be a cofibration (resp., an acyclic cofibration) in $\mathtt{dgcu}(k)$. Then $\mathcal{L}(f)$ is a cofibration (resp., an acyclic cofibration) in $\mathtt{dglie}(k)$.*

Proof. Let $f : X \to Y$ be injective, $\{Y_i\}$ be the canonical filtration of Y, $Z_i = f(X) + Y_i \subseteq Y$ are subcoalgebras in Y. Since $Y = \varinjlim Z_i$ and $\mathcal{L}$ commutes with colimits, it is enough to check that $\mathcal{L}(Z_i) \to \mathcal{L}(Z_{i+1})$ is a cofibration. Now, Z_i/Z_{i+1} is primitive, so any subcomplex of Z_{i+1} containing Z_i is a subcoalgebra. Therefore without loss of generality we can suppose $Z_{i+1} = Z_i \oplus k \cdot x$ with $dx = z \in Z_i$.

In this case the Lie algebra $\mathcal{L}(Z_{i+1})$ is a standard cofibration (see [H2], 2.2.2, 2.2.3) over $\mathcal{L}(Z_i)$ generated by the element $x[-1] \in Z_{i+1}[-1] \subseteq \mathcal{L}(Z_{i+1})$ corresponding to $x \in Z_{i+1}$.

The claim about acyclic cofibrations follows from the above and from the definition of weak equivalences in $\mathtt{dgcu}(k)$. $\square$

5.2.3. *Fibrations in* $\mathtt{dgcu}(k)$*.*

Lemma. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be a surjective map (resp., a surjective quasi-isomorphism) in* $\mathtt{dglie}(k)$*. Then $\mathcal{C}(f)$ is a fibration (resp., an acyclic fibration) in* $\mathtt{dgcu}(k)$*.*

Proof. If f is surjective, $\mathcal{C}(f)$ is a fibration by Lemma 5.2.2 and the adjointness of $\mathcal{L}$ and $\mathcal{C}$. If, moreover, f is a quasi-isomorphism, $\mathcal{C}(f)$ is a weak equivalence by Proposition 3.3.2(2). $\square$

5.2.4. The properties (CM2), (CM3) are obvious. Also the lifting property (CM4)(ii) is valid by definition of fibrations in $\mathtt{dgcu}(k)$.

(CM5)(i). Given a map $f : X \to Y$ in $\mathtt{dgcu}(k)$ let $\mathcal{L}(f) = pi$ be a decomposition of $\mathcal{L}(f)$ with a cofibration $i : \mathcal{L}(X) \to \mathfrak{g}$ and an acyclic fibration $p : \mathfrak{g} \to \mathcal{L}(Y)$.

Let $Z = Y \times_{\mathcal{C}\mathcal{L}(Y)} \mathcal{C}(\mathfrak{g})$. According to Lemma 5.1.1 and Proposition 3.3.2, the map $Z \to Y$ is a weak equivalence. It is also a fibration since it is obtained by a base change from the fibration $\mathcal{C}(p)$ in $\mathtt{dgcu}(k)$. Now, the induced map $X \to Z$ being injective, we get (CM5)(i).

(CM4)(i). If $f : X \to Y$ is an acyclic fibration, the already proven property (CM5)(i) gives a decomposition $f = qj$ where j is an acyclic cofibration and q is obtained by a base-change from a map $\mathcal{C}(p)$ with p an acyclic fibration in $\mathtt{dglie}(k)$. Adjointness of $\mathcal{L}$ and $\mathcal{C}$ immediately gives that $\mathcal{C}(p)$ has a RLP with respect to all cofibrations. Therefore q admits the same property. Since j admits a LLP with respect to f, we get that f is a retract of q and therefore, it also satisfies RLP with respect to all cofibrations.

(CM5)(ii). Let $f : X \to Y$ and let $\mathcal{L}(f) = pi$ be a decomposition with a fibration $p : \mathfrak{g} \to \mathcal{L}(Y)$ and an acyclic cofibratiton $i : \mathcal{L}(X) \to \mathfrak{g}$. According to Lemma 5.1.1 the map $j : Z \to \mathcal{C}(\mathfrak{g})$ is a weak equivalence and the map $Z \to Y$ is a fibration where $Z = Y \times_{\mathcal{C}\mathcal{L}(Y)} \mathcal{C}(\mathfrak{g})$. This immediately implies that the map $X \to Z$ is an acyclic cofibration.

Therefore, we proved $\mathtt{dgcu}(k)$ admits a CMC structure. The simplicial structure on $\mathtt{dgcu}(k)$ and the proof of the axiom (SM7) will be provided in Section 7.

5.3. **Proof of Theorem 3.2.** Now Theorem 3.2 follows immediately from the general Theorem II.1.4 of [Q2] and from Proposition 3.3.2.

6. Proof of Key Lemma 5.1.1

In this section we prove Key Lemma 5.1.1.

Endow X with the canonical filtration and $\mathcal{L}(X)$ with the induced filtration. Let $\mathfrak{a} = \mathrm{Ker}(f : \mathfrak{g} \to \mathcal{L}(X))$. Define an admissible filtration on $\mathfrak{g}$ by setting $\mathfrak{g}_n = f^{-1}(\mathcal{L}(X)_n)$ for $n > 0$. This induces admissible filtrations on $\mathcal{C}(\mathfrak{g})$ and on $\mathcal{C}(\mathcal{L}(X))$.

According to Proposition 4.4.3, $i_X : X \to \mathcal{CL}(X)$ is a filtered quasi-isomorphism. Define a filtration on Z by the formula $Z_n = j^{-1}(\mathcal{C}(\mathfrak{g})_n)$.

According to Proposition 4.4.4, it is enough to check that $j : Z \to \mathcal{C}(\mathfrak{g})$ is a filtered quasi-isomorphism.

Let us describe more explicitly the filtrations involved. Forget about the differentials. Choose a graded Lie algebra splitting $s : \mathcal{L}(X) \to \mathfrak{g}$ of f. This defines isomorphisms (not preserving the differentials)

$$\mathcal{C}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{CL}(X) \otimes \mathcal{C}(\mathfrak{a}) \tag{7}$$

and

$$Z \xrightarrow{\sim} X \otimes \mathcal{C}(\mathfrak{a}) \tag{8}$$

of filtered coalgebras, the filtration on $\mathcal{C}(\mathfrak{a})$ being the standard one. In fact, the first isomorphism obviously preserves filtrations, and the second one preserves the filtrations because of the equality $X_n = i_X^{-1}(\mathcal{CL}(X)_n)$.

The isomorphism (8) can be rewritten as

$$Z_n \xrightarrow{\sim} \bigoplus_r X_{n-r} \otimes S^r(\mathfrak{a}[1]) \tag{9}$$

and similarly

$$\mathcal{C}(\mathfrak{g})_n \xrightarrow{\sim} \bigoplus_r \mathcal{CL}(X)_{n-r} \otimes S^r(\mathfrak{a}[1]). \tag{10}$$

Unfortunately, the isomorphisms (9), (10) are not compatible with the differentials. To overcome this minor difficulty, we define a double filtration on the complexes involved so that the associated graded pieces are already isomorphic as complexes. We will write formulas only for the filtration on Z and on Z_n, the formulas for $\mathcal{C}(\mathfrak{g})$ being obtained by substitution of X_n with $\mathcal{CL}(X)_n$. Here are the formulas.

$$F_p^q = \bigoplus_{r \geq q} X_p \otimes S^r(\mathfrak{a}[1]) \tag{11}$$

$$F_p^q(Z_n) = F_p^q \cap Z_n = \bigoplus_{n \geq r \geq q} X_{\min(p,n-r)} \otimes S^r(\mathfrak{a}[1]) \tag{12}$$

The filtrations are increasing on p and decreasing on q. The filtration (12) is finite. Its (p,q)-graded piece vanishes for $p+q > n$ and is otherwise isomorphic to $X_p/X_{p+1} \otimes S^q(\mathfrak{a}[1])$ as a complex.

Associated graded pieces of the corresponding filtration for $\mathcal{C}(\mathfrak{g})$ have the form

$$\mathcal{CL}(X)_p/\mathcal{CL}(X)_{p+1} \otimes S^q(\mathfrak{a}[1]).$$

The (p,q)-graded piece of the map $j_n : Z_n \to \mathcal{C}(\mathfrak{g})_n$ has the form

$$X_p/X_{p+1} \otimes S^q(\mathfrak{a}[1]) \longrightarrow \mathcal{CL}(X)_p/\mathcal{CL}(X)_{p+1} \otimes S^q(\mathfrak{a}[1])$$

which obviously a quasi-isomorphism by Proposition 4.4.3.

Key Lemma is proven.

7. Simplicial structure on $\mathtt{dgcu}(k)$

In this section we define a simplicial structure on the category $\mathtt{dgcu}(k)$ of dg unital coalgebras over a field k of characteristic zero and check the axiom (SM7) — see Introduction.

7.1. **Functional spaces for unital coalgebras.** Recall (cf. [BG]) that the functor of polynomial differential forms

$$\Omega : \Delta^{\mathrm{op}}\mathtt{Ens} \to \mathtt{dga}(k) \tag{13}$$

is the one defined uniquely by its values on the standard simplices

$$\Omega(\Delta^n) = \Omega_n = k[t_0, \dots, t_n, dt_0, \dots, dt_n]/(\sum t_i - 1, \sum dt_i)$$

and by the property that Ω commutes with colimits.

7.1.1. For any commutative dg algebra $\Omega \in \mathtt{dga}(k)$ tensoring by Ω defines a functor

$$\Omega \otimes \_ : \mathtt{dgcu}(k) \to \mathtt{dgcu}(\Omega). \tag{14}$$

Therefore, the following definition makes sense.

7.1.2. **Definition.** Let $X, Y \in \mathtt{dgcu}(k)$. The simplicial set $\mathcal{H}om(X, Y)$ is defined by the formula

$$\mathcal{H}om(X, Y)_n = \mathrm{Hom}_{\mathtt{dgcu}(\Omega_n)}(\Omega_n \otimes X, \Omega_n \otimes Y),$$

the faces and the degeneracies being defined in an obvious way.

Note the following

7.1.3. **Lemma.** *The functor (14) commutes with colimits and with finite limits.*

Proof. The claim about colimits is obvious. In order to prove that (14) commutes with finite limits we check separately the case of a product of two coalgebras and that of equalizer of a pair of maps. In these two cases the result follows from the description of limits in 5.2.1. □

Lemma 7.1.3 yields the following

7.1.4. **Corollary.** *1. The functor $\mathcal{H}om(X, \_) : \mathtt{dgcu}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$ commutes with finite limits.*

2. The functor $\mathcal{H}om(\_, Y) : \mathtt{dgcu}(k)^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathtt{Ens}$ carries arbitrary colimits to limits.

One has the following standard fact.

7.1.5. **Lemma.** *(see* [BG]*, Lemma 5.2,* [H2]*, 4.8.3) There is a natural in* $S \in \Delta^{\mathrm{op}}\mathtt{Ens}$ *morphism*

$$\Phi(S) : \mathrm{Hom}_{\mathtt{dgcu}(\Omega(S))}(\Omega(S) \otimes X, \Omega(S) \otimes Y) \xrightarrow{\sim} \mathrm{Hom}(S, \mathcal{H}om(X, Y))$$

which is a bijection provided S *is finite.*

Proof. The proof of [H2], 4.8.3 is applicable here. □

7.1.6. **Lemma.** *The adjoint functors* $\mathcal{C}$ *and* $\mathcal{L}$ *induce an isomorphism*

$$\mathcal{H}om(X, \mathcal{C}(\mathfrak{g})) \xrightarrow{\sim} \mathcal{H}om(\mathcal{L}(X), \mathfrak{g})$$

of simplicial sets for every $X \in \mathtt{dgcu}(k)$*,* $\mathfrak{g} \in \mathtt{dglie}(k)$*.*

Proof. Repeats the standard argument of Theorem 2.2.5 substituting the base category $C(k)$ with $\mathtt{mod}(\Omega_n)$. □

7.2. Property (SM7).

7.2.1. **Proposition.** *Let* $i : A \to B$ *be a cofibration and* $p : X \to Y$ *be a fibration in* $\mathtt{dgcu}(k)$*. Then the map of simplicial sets*

$$\pi(i, p) : \mathcal{H}om(B, X) \to \mathcal{H}om(A, X) \times_{\mathcal{H}om(A,Y)} \mathcal{H}om(B, Y) \qquad (15)$$

is a Kan fibration. If, moreover, either i *or* p *is a weak equivalence, then* $\pi(i, p)$ *is an acyclic Kan fibration.*

7.2.2. In what follows we will say that a pair of maps $(i : A \to B, p : X \to Y)$ satisfies (SM7) if the map $\pi(i, p)$ from (15) is a Kan fibration, acyclic if one of (i, p) is a weak equivalence. To prove Proposition 7.2.1, we will show step by step that any pair $(i : A \to B, p : X \to Y)$ such that i is a cofibration and p is a fibration, satisfies (SM7).

Step 1. Suppose that $p = \mathcal{C}(f)$ where $f : \mathfrak{g} \to \mathfrak{h}$ is a surjective map of dg Lie algebras. Then any pair (i, p) satisfies (SM7) by 7.1.6, 5.2.2 and [H2], 4.8.4.

Step 2. Suppose that a pair (i, p) satisfies (SM7) and let a map $q : Z \to T$ be obtained from $p : X \to Y$ by a base change $a : T \to Y$.

Using Corollary 7.1.4, we easily see that $\pi(i, q)$ is obtained by a base change from $\pi(i, p)$. Therefore the pair (i, q) also satisfies (SM7).

Step 3. Suppose that a pair (i, p) satisfies (SM7) and let a map $q : Z \to T$ be a retract of $p : X \to Y$. Then the map $\pi(i, q)$ is a retract of $\pi(i, p)$ and therefore, the pair (i, q) also satisfies (SM7).

Now Proposition 7.2.1 follows from the following lemma.

7.2.3. **Lemma.** *1. Any fibration in* $\mathtt{dgcu}(k)$ *can be obtained, using the operations of retraction and base change, from a map* $\mathcal{C}(f)$ *where* f *is a surjective map of dg Lie algebras.*

2. Any acyclic fibration in $\mathtt{dgcu}(k)$ *can be obtained, using the operations of retraction and base change, from a map* $\mathcal{C}(f)$ *where* f *is a surjective quasi-isomorphism of dg Lie algebras.*

Proof. 1. Let $f : X \to Y$ be a fibration. Using the maps $i : Y \to \mathcal{CL}(Y)$ and $\mathcal{CL}(f) : \mathcal{CL}(X) \to \mathcal{CL}(Y)$, define $Z = Y \times_{\mathcal{CL}(Y)} \mathcal{CL}(X)$ and let $j : Z \to \mathcal{CL}(X), k : X \to Z$ and $g : Z \to Y$ be the obviously defined maps. According to Key Lemma, j is an acyclic cofibration, and therefore, k is an acyclic cofibration as well. Since f is a fibration, the map k splits over g and this gives a presentation of f as a retract of g which is obtained by a base change from $\mathcal{C}(\mathcal{L}(f))$.

2. If, moreover, $f : X \to Y$ is an acyclic fibration, then $\mathcal{L}(f)$ is a surjective quasi-isomorphism. This proves the second assertion of the lemma.

□

8. The nerve of a dg Lie algebra

8.1. **The nerve and its properties.** Let $X \in \mathtt{dgcu}(k)$. Choose a fibrant resolution $X \to F$ and define a functor

$$\widetilde{X} : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

by the formula

$$\widetilde{X}(A) = \mathcal{H}om(A^*, F)$$

where A^* is the unital coalgebra with the unit $A \to k$. The resulting functor $\widetilde{X}$ does not depend, up to homotopy, on the choice of the resolution. One can get a specific representative for $\widetilde{X}$ as follows.

Let $\mathfrak{g} = \mathcal{L}(X)$ be the tangent Lie algebra of X. Choose $\mathcal{C}(\mathfrak{g})$ to be a fibrant resolution of X. This allows one to easily express the functor $\widetilde{X}$ through the tangent Lie algebra $\mathfrak{g}$.

8.1.1. **Definition.** Let $\mathfrak{g} \in \mathtt{dglie}(k)$. The nerve of $\mathfrak{g}$ is the functor

$$\Sigma_{\mathfrak{g}} : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

defined by the formula

$$\Sigma_{\mathfrak{g}}((A, \mathfrak{m}))_n = \mathrm{MC}(\mathfrak{m} \otimes \Omega_n \otimes \mathfrak{g}).$$

One has immediately the following

8.1.2. **Proposition.** *For* $X \in \mathtt{dgcu}(k)$ *the functor* $\widetilde{X} : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$ *is homotopy equivalent to the nerve* $\Sigma_{\mathcal{L}(X)}$.

Proof. According to Lemma 7.1.6, one has for any $\mathfrak{g} \in \mathtt{dglie}(k)$ and $A \in \mathtt{dgart}^{\leq 0}(k)$

$$\Sigma_{\mathfrak{g}}(A)_n = \mathrm{MC}(\Omega_n \otimes \mathfrak{m} \otimes \mathfrak{g}) = \mathrm{Hom}_{\mathtt{dgcu}(\Omega_n)}(\Omega_n \otimes A^*, \Omega_n \otimes \mathcal{C}(\mathfrak{g})) = \mathcal{H}om_n(A^*, \mathcal{C}(\mathfrak{g})). \tag{16}$$

□

8.1.3. **Proposition.** *1. A quasi-isomorphism $f : \mathfrak{g} \to \mathfrak{h}$ of dg Lie algebras induces for every $A \in \mathtt{dgart}^{\leq 0}(k)$ a homotopy equivalence*

$$\Sigma_f : \Sigma_{\mathfrak{g}}(A) \to \Sigma_{\mathfrak{h}}(A).$$

2. For each $\mathfrak{g} \in \mathtt{dglie}(k)$ the functor $\Sigma_{\mathfrak{g}}$ carries quasi-isomorphisms in $\mathtt{dgart}^{\leq 0}(k)$ to homotopy equivalences.

3. For each $\mathfrak{g} \in \mathtt{dglie}(k)$ the functor $\Sigma_{\mathfrak{g}}$ carries surjective maps to Kan fibration. In particular, $\Sigma_{\mathfrak{g}}(A)$ is always a Kan simplicial set.

4. $\Sigma_{\mathfrak{g}}$ commutes with finite projective limits.

Proof. The claims 1, 3, 4 follow from Proposition 7.2.1 and Lemma 7.1.6. By Proposition 3.3.2, (3), a quasi-isomorphism $f : A \to B$ in $\mathtt{dgart}^{\leq 0}(k)$ defines a weak equivalence $f^* : B^* \to A^*$ in $\mathtt{dgcu}(k)$. Then the induced map $\Sigma_{\mathfrak{g}}(f)$ is a weak equivalence of Kan simplicial sets, hence a homotopy equivalence. □

8.2. **Some calculations.** Here we provide some explicit calculations which help to better understand what the nerve of a dg Lie algebra looks like. We will use them below in Section 10.

In this subsection $\mathfrak{g}$ is a nilpotent dg Lie algebra (it replaces $\mathfrak{m} \otimes \mathfrak{g}$ of 8.1.1), and we denote by $\Sigma(\mathfrak{g})$ the simplicial set

$$\Sigma(\mathfrak{g})_n = \mathrm{MC}(\mathfrak{g}_\bullet)$$

where the simplicial dg Lie algebra $\mathfrak{g}_\bullet = \{\mathfrak{g}_{(n)}\}$ is defined by the formula $\mathfrak{g}_{(n)} = \Omega_n \otimes \mathfrak{g}$.

8.2.1. *Deligne groupoid.* Recall (cf. [GM1]) that for a nilpotent dg Lie algebra $\mathfrak{g}$ one defines *Deligne groupoid* $\Gamma(\mathfrak{g})$ as follows.

The Lie algebra $\mathfrak{g}^0$ acts on the set $\mathrm{MC}(\mathfrak{g})$ of the Maurer-Cartan elements of $\mathfrak{g}$ by vector fields:

$$\rho(y)(x) = dy + [x, y] \text{ for } y \in \mathfrak{g}^0, x \in \mathfrak{g}^1.$$

This defines the action of the nilpotent group $G = \exp(\mathfrak{g}^0)$ on the set $\mathrm{MC}(\mathfrak{g})$. Then the groupoid $\Gamma(\mathfrak{g})$ is defined by the formulas

$$\mathrm{Ob}\,\Gamma = \mathrm{MC}(\mathfrak{g})$$
$$\mathrm{Hom}_\Gamma(x, x') = \{g \in G | x' = g(x)\}.$$

8.2.2. *Maurer-Cartan elements of* $\mathfrak{g}_{(1)}$. Let us explicitly describe the set $\mathrm{MC}(\mathfrak{g}_{(1)})$. Since $\mathfrak{g}_{(1)} = k[t, dt] \otimes \mathfrak{g}$ we will iterate the calculation to get the description of $\mathrm{MC}(\mathfrak{g}_{(n)})$.

Write an element $z \in \mathfrak{g}^1_{(1)}$ in the form

$$z = x + dt \cdot y$$

with $x \in \mathfrak{g}^1[t]$, $y \in \mathfrak{g}^0[t]$. Then the Maurer-Cartan equation is easily seen to be equivalent to the differential equation

$$\frac{dx(t)}{dt} = dy(t) + [x(t), y(t)]$$
$$x(0) = x_0$$

where x_0 is an element of $\mathrm{MC}(\mathfrak{g})$.

An element $y \in \mathfrak{g}[t]$ defines a unique polynomial path $g(t)$ in the Lie group $G = \exp(\mathfrak{g}^0)$ satisfying the differential equation $\dot{g}(t) = g(t)(y(t))$ with the initial condition $g(0) = 1$.

Let $\mathfrak{k}$ be the Lie subalgebra $t\mathfrak{g}^0[t] \subseteq \mathfrak{g}^0_{(1)}$ and let $K = \exp(\mathfrak{k})$. The above consideration proves the following

8.2.3. **Lemma.** *An element x of $\mathrm{MC}(\mathfrak{g}_{(1)})$ can be uniquely represented in the form*

$$x = g(x_0)$$

where $x_0 \in \mathrm{MC}(\mathfrak{g}) \subseteq \mathrm{MC}(\mathfrak{g}_{(1)})$, $g \in K \subseteq \exp(\mathfrak{g}^0_{(1)})$ and the action of $\exp(\mathfrak{g}^0_{(1)})$ on the set $\mathrm{MC}(\mathfrak{g}_{(1)})$ is defined as in 8.2.1.

8.2.4. Iteratively using Lemma 8.2.3 we obtain the following descriptioin of the set of Maurer-Cartan elements of $\mathfrak{g}_{(n)}$.

Let $\mathfrak{k}_i = t_i\mathfrak{g}^0_{(i-1)}[t_i]$ for $i > 0$ be the Lie subalgebra of $\mathfrak{g}^0_{(n)}$. Denote $K_i = \exp(\mathfrak{k}_i)$. This is a subgroup of $\exp(\mathfrak{g}^0_{(n)})$.

Lemma. *Let $0 \le i \le j \le n$. Then K_i normalizes K_j.*

Proof. This immediately follows from the inclusion

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_j.$$

□

Define $G_n = \exp(\mathfrak{g}^0_{(n)})$ and let $T_n = K_n \cdot K_{n-1} \cdots K_1$ be the subgroup in G_n. The lemma above implies that this group is the exponent of the Lie algebra $\oplus_{i\ge 1}\mathfrak{k}_i$. Then one has the following

8.2.5. **Proposition.** *Any element of $\mathrm{MC}(\mathfrak{g}_{(n)})$ can be uniquely presented as $g(x_0)$ where $x_0 \in \mathrm{MC}(\mathfrak{g})$ and $g \in T_n$.*

Note that the simplicial group $G_\bullet = \{G_n\}$ acts on the nerve $\Sigma(\mathfrak{g})$. Proposition 8.2.5 implies that the restriction

$$G_\bullet \times \mathrm{MC}(\mathfrak{g}) \to \Sigma(\mathfrak{g}) \tag{17}$$

is surjective. One has the following stronger

8.2.6. **Corollary.** *The map (17) admits a pseudo-cross section (see* [M], §*18).*

Proof. The pseudo-cross section is given as the composition

$$\Sigma_n(\mathfrak{g}) \xrightarrow{\sim} T_n \times \mathrm{MC}(\mathfrak{g}) \to G_n \times \mathrm{MC}(\mathfrak{g}).$$

This map obviously commutes with the faces d_i for $i > 0$ and with all degeneracies. □

8.2.7. *Note.* Using the explicit description of $\Sigma(\mathfrak{g})$ above, one can easily obtain the property (3) of 8.1.3 independently of Theorems 3.1, 3.2.

In this section we provide some examples, definitions, calculations and remarks.

9.1. **Homology of Lie algebras.** Let us give another description of the homology functor $\# \circ \mathcal{C} : \mathtt{dglie}(k) \to C(k)$ where the functor $\# : \mathtt{dgcu}(k) \to C(k)$ is given by the formula $\#(C) = \overline{C}$.

9.1.1. **Proposition.** *The composition $\# \circ \mathcal{C}$ is the left derived functor of the functor $\mathtt{Ab} : \mathtt{dglie}(k) \to C(k)$ defined as*

$$\mathtt{Ab}(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Proof. It is sufficient to check that the map of complexes $\overline{\mathcal{C}(\mathfrak{g})} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a quasi-isomorphism provided $\mathfrak{g}$ is standard cofibrant. Consider $\mathcal{C}(\mathfrak{g})$ as a bicomplex so that the horizontal differential d' is defined by the Lie bracket in $\mathfrak{g}$ and the vertical differential d'' is induced by the differential in $\mathfrak{g}$. Forget for a moment the differential in $\mathfrak{g}$. Then

$$\mathfrak{g} = F(V) = \oplus F^n(V) \tag{18}$$

is a free Lie algebra over a graded vector space V. The differential d' preserves the grading which comes from the presentation (18). Since $H(\mathcal{C}, d') = V$ and each homogeneous component of $(\mathcal{C}, d')$ is bounded, the proposition follows. □

Proposition 9.1.1 allows one to construct easily an example of acyclic coalgebra X such that $\mathcal{L}(X)$ has non-trivial cohomology (another example is given in Kontsevich's lectures [Ko]).

9.1.2. **Example.** Let $\mathfrak{g}$ be the cofibrant Lie algebra having generators e, f, h of degree 0, x, y, z of degree -1 with the differential given by

$$de = dh = df = 0; dx = [h, e] - 2e; dy = [h, f] + 2f; dz = [e, f] - h.$$

According to Proposition 9.1.1, $\mathcal{C}(\mathfrak{g})$ is acyclic though $\mathfrak{g}$ is not (one has $H^0(\mathfrak{g}) = \mathfrak{sl}_2(k)$). One can look at this the other way around putting $X = \mathcal{C}(\mathfrak{g})$ and obtaining a non-contractible $\mathcal{L}(X)$. This counter-example means that there are quasi-isomorphisms in $\mathtt{dgcu}(k)$ which are not weak equivalences.

9.2. **Infinitesimals.** Look at the first infinitesimal deformation corresponding to a dg Lie algebra $\mathfrak{g}$.

9.2.1. *Dual numbers.* For each $n = 0, 1, \dots$ define

$$A_n = k[\epsilon; \deg \epsilon = -n]/(\epsilon^2) \in \mathtt{dgart}^{\leq 0}(k).$$

This is a k-vector space object in the category $\mathtt{dgart}^{\leq 0}(k)$.

Let us calculate the simplicial vector space $\Sigma_{\mathfrak{g}}(A_n)$. Its i-simplices are the Maurer-Cartan elements of $\epsilon \cdot \Omega_i \otimes \mathfrak{g}$ which is a dg Lie algebra with zero multiplication. Therefore,

$$\Sigma_{\mathfrak{g}}(A_n)_i = Z^0(\Omega_i \otimes \mathfrak{g}[1 + n]).$$

Using the Dold-Puppe equivalence of categories and the fact that the cosimplicial complex $\{\Omega_i\}_{i\in\mathbb{N}}$ is homotopy equivalent to the cosimplicial complex of cochains $\{C^*(\Delta^i)\}_{i\in\mathbb{N}}$, we obtain the following

9.2.2. **Proposition.** *$\Sigma_{\mathfrak{g}}(A_n)$ is homotopy equivalent to the simplicial abelian group corresponding to the complex $\tau^{\leq 0}(\mathfrak{g}[1+n])$.*

Note the following

9.2.3. **Corollary.** *If a map $f:\mathfrak{g}\to\mathfrak{h}$ in $\mathtt{dglie}(k)$ induces an equivalence of the nerve functors, then f is itself a quasi-isomorphism.*

9.3. **Formal spaces.**

9.3.1. **Definition.** A formal stack $X\in\mathtt{dgcu}(k)$ is called a *formal space* if it is weakly equivalent to a coalgebra $Y\in\mathtt{dgcu}(k)$ satisfying $Y^i=0$ for $i<0$.

An equivalent condition: X is a formal space if $H^i(\mathcal{L}(X))=0$ for $i\leq 0$. According to 9.2.2, this property is equivalent to the one saying that $\Sigma_{\mathfrak{g}}(A_0)$ is discrete. By an obvious artinian induction we get

9.3.2. **Lemma.** *$X\in\mathtt{dgcu}(k)$ is a formal space iff for each $A\in\mathtt{art}(k)$ the simplicial set $\widetilde{X}(A)$ is discrete.*

9.3.3. Now the two ideas mentioned in the Introduction about formal deformations in characteristic zero can be formulated as follows.

Any formal deformation problem in characteristic zero can be described by a representable functor

$$F:\mathtt{dgart}^{\leq 0}(k)\to\Delta^{\mathrm{op}}\mathtt{Ens}.$$

Classical deformation problems are often not representable, since in the classical picture we see only the π_0 of the genuine deformation functor.

Definition. Let $X\in\mathtt{dgcu}(k)$. The classical part of $\widetilde{X}$ is the functor

$$\widetilde{X}_{cl}:\mathtt{art}(k)\to\mathtt{Ens}$$

defined by the formula $\widetilde{X}_{cl}(A)=\pi_0(\widetilde{X}(A))$.

Let $\mathfrak{g}=\mathcal{L}(X)$. Suppose first that X is a formal space. Put $Y=H^0(\mathcal{C}(\mathfrak{g}))$. Then for any $A\in\mathtt{art}(k)$ one has

$$X_{cl}(A)=\pi_0(\widetilde{X}(A))=\widetilde{X}(A)=\mathrm{Hom}(A^*,X)=\mathrm{Hom}(A^*,Y).$$

This means that the classical part $\widetilde{X}_{cl}$ of a formal space X is representable by the coalgebra Y. For a general $X\in\mathtt{dgcu}(k)$ one should not expect representability of the classical part. However, the functor $\widetilde{X}_{cl}$ admits a hull in the sense of [Sc]. In fact, choose a complement V in $\mathfrak{g}^1$ to the vector subspace $\mathrm{Im}(d:\mathfrak{g}^0\to\mathfrak{g}^1)$ and define a 1-truncation $\mathfrak{h}$ of $\mathfrak{g}$ by the formulas

$$\mathfrak{h}^i=\begin{cases}0 & , \quad i\leq 0\\ V & , \quad i=1\\ \mathfrak{g}^i & , \quad i>1.\end{cases} \tag{19}$$

Put $Y=H^0(\mathcal{C}(\mathfrak{h}))$ and define $h_Y(A)=\mathrm{Hom}(A^*,Y)$.

9.3.4. **Lemma.** *The injection $\mathfrak{h}\to\mathfrak{g}$ induces a smooth morphism of functors $h_Y\to\widetilde{X}_{cl}$ which is isomorphism on the tangent spaces.*

Proof. This claim essentially belongs to Goldman-Millson [GM1], [GM2] (who considered however only the case of formal spaces). The tangent spaces to h_Y and to $\widetilde{X}_{cl}$ are isomorphic to $H^1(\mathfrak{h})$ and to $H^1(\mathfrak{g})$ respectively. To check the smoothness it is enough to prove that for any surjection $f:(B,\mathfrak{n})\to(A,\mathfrak{m})$ in $\mathtt{art}(k)$ whose kernel I is annihilated by $\mathfrak{n}$, the map

$$h_Y(B)\to h_Y(A)\times_{\widetilde{X}_{cl}(A)}\widetilde{X}_{cl}(B)$$

is surjective.

Suppose $x\in\mathrm{MC}(\mathfrak{m}\otimes V)$ and $y\in\mathrm{MC}(\mathfrak{n}\otimes\mathfrak{g}^1)$ have the same image in $\widetilde{X}_{cl}(A)$. Then, first of all, one can substitute the element y by an equivalent one, so that the images of x and of y in $\mathrm{MC}(\mathfrak{m}\otimes\mathfrak{g}^1)$ coincide. Then the element y belongs to $\mathfrak{n}\otimes V$, up to an element from $I\otimes\mathrm{Im}(d:\mathfrak{g}^0\to\mathfrak{g}^1)$ which can be killed by the action of $I\otimes\mathfrak{g}^0$. After this correction, the element y already belongs to $\mathfrak{n}\otimes V$ and it automatically satisfies the Maurer-Cartan equation. □

Therefore, the coalgebra Y (or, its dual complete local algebra) is a hull of $\widetilde{X}_{cl}$.

It would be nice to prove the uniqueness of Y. For this it would be enough to prove that $\mathfrak{h}$ does not depend, up to a quasi-isomorphism, on the choice of 1-truncation. This is of course so if (as in [GM2]) one supposes that $H^0(\mathfrak{g})=0$. Unfortunately, we doubt this is true in general.

Nevertheless, a general claim of [Sc] implies that the hull Y is unique up to a non-canonical isomorphism provided $H^1(\mathfrak{g})$ is finite-dimensional.

9.4. **Simply connected rational spaces.** Let S be a simply connected rational space. According to [Q2], S has a dg Lie algebra model $\mathfrak{g}$ which satisfies $\mathfrak{g}^i=0$ for $i\geq -1$ (we keep using degree +1 differentials). Therefore, S should define a formal deformation in our general definition. It looks a bit strange, since "usual" deformations are described by non-negatively graded Lie algebras. The classical part of such a deformation is trivial. However, one can easily calculate the deformation functor corresponding to S — in terms of dg Lie algebra models.

9.4.1. **Proposition.** *Let S have a finite $\mathbb{Q}$-type. For any $A\in\mathtt{dgart}^{\leq 0}(\mathbb{Q})$ the simplicial set $\Sigma_{\mathfrak{g}}(A)$ is simply connected and rational. Its Lie algebra model is given by $\mathfrak{m}\otimes\mathfrak{g}$.*

Proof. Put $\mathfrak{h}=\mathfrak{m}\otimes\mathfrak{g}$. We wish to check that $\mathfrak{h}$ is a Lie algebra model for the simplicial set $\Sigma(\mathfrak{h})$. But this is clear: the coalgebra model of $\mathfrak{h}$ is $\mathcal{C}(\mathfrak{h})$, so the dg algebra model of $\mathfrak{h}$ is the dual complex $\mathcal{C}^*(\mathfrak{h})$ and the corresponding simplicial set is given according to [BG], Thm. 9.4, by the formula

$$n \mapsto \mathrm{Hom}(\mathcal{C}^*(\mathfrak{h}), \Omega_n) = \mathrm{MC}(\Omega_n \otimes \mathfrak{h}) = \Sigma_n(\mathfrak{h})$$

since $\mathfrak{h}^i$ are finitely dimensional. □

9.5. **Example: Intersection of subschemes.** A typical example of a formal space which is not a formal scheme is given by a non-transversal intersection of subschemes. Let $X, Y \subseteq Z$ be closed subschemes in a noetherian scheme Z, $z \in X \cap Y$. We wish to describe the intersection of X and Y in Z near z. Let A, B, C be the local rings of X, Y, Z respectively.

These rings (or the corresponding dual coalgebras A^*, B^*, C^*) represent functors

$$F_A, F_B, F_C : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

where $k = k(z)$ is the base field.

The functors F_A, F_B, F_C being defined up to homotopy equivalence, the best thing we can do is to consider their homotopy fibre product.

Thus, define the homotopy intersection of X and Y at $z \in Z$ to be the homotopy fibre product functor F of F_A and F_B over F_C. In order to calculate it, one has to substitute a map $F_A \to F_C$ (or the other one) with a fibration and take the usual fibre product.

For this it suffices to take a cofibrant resolution $\widetilde{A}$ for the C-algebra A and substitute F_A with $F_{\widetilde{A}}$. The result will be given by the dg algebra $\widetilde{A} \otimes_C B$ defined canonically in the corresponding homotopy category, concentrated in the nonpositive degrees. Its cohomology is given by the formula

$$H^i(\widetilde{A} \otimes_C B) = \mathrm{Tor}^C_{-i}(A, B)$$

— exactly as one could expect.

9.5.1. *Remark.* It is unclear how to define a global object corresponding, say, to the intersection of two subschemes in a non-affine scheme. One should probably use a technique suggested by Hirschowitz-Simpson in [HiSi].

9.6. **Example: Quotient by a group action.** Let an algebraic group G over k acts on a dg Lie algebra $\mathfrak{h}$. Then G acts on each simplicial set $\Sigma_{\mathfrak{h}}(A)$. Define a functor

$$F : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

as the homotopy quotient $F(A) = \Sigma_{\mathfrak{h}}(A)/\widehat{G}_1(A)$ where $\widehat{G}_1$ is the formal completion of G at 1 so that $\widehat{G}_1(A) = \exp(\mathfrak{m} \otimes \mathfrak{g})$.

9.6.1. **Proposition.** *The functor F is homotopy equivalent to the nerve of the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$.*

Proof. According to Proposition 8.1.3, (3), (4), one has a fibration

$$f : \Sigma_{\mathfrak{g} \ltimes \mathfrak{h}}(A) \to \Sigma_{\mathfrak{g}}(A)$$

with fibre $\Sigma_{\mathfrak{h}}(A)$. On the other hand, the map (17) gives in our case a fibration

$$\pi : G_\bullet(A) \to \Sigma_{\mathfrak{g}}(A)$$

with fibre $\widehat{G}_1(A)$ and a contractible total space. Then the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow[\widehat{G}_1(A)]{} & \Sigma_{\mathfrak{g}\ltimes\mathfrak{h}}(A) \\ {\scriptstyle \Sigma_{\mathfrak{h}}(A)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma_{\mathfrak{h}}(A)} \\ G_\bullet(A) & \xrightarrow[\widehat{G}_1(A)]{} & \Sigma_{\mathfrak{g}}(A) \end{array}$$

(all the arrows being fibrations marked by the corresponding fibres) presents $\Sigma_{\mathfrak{g}\ltimes\mathfrak{h}}(A)$ as a homotopy quotient of $\Sigma_{\mathfrak{h}}(A)$ modulo $\widehat{G}_1(A)$. □

10. Example: Moduli of G-torsors

10.1. In this section we construct a formal stack of moduli of principal G-bundles.

Let G be an algebraic group over a field k of characteristic zero, S be a scheme over k, P be an S-torsor under G. We wish to study formal deformations of P. For this we have to define a deformation functor

$$F_P : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens} \tag{20}$$

naturally generalizing the standard functor of formal deformations $\mathtt{art}(k) \to \mathtt{Grp}$.

We will proceed as follows. First of all, we define in 10.2.1 torsors with an (affine) dg base. Then (in 10.2.2) we construct a functor

$$\mathcal{D}_P : \mathtt{dgart}^{\leq 0}(k) \to \mathtt{sGrp} \tag{21}$$

from artinian dg algebras to simplicial groupoids describing the formal deformations of a torsor in the affine case.

The nerve (see 11.3) of this simplicial groupoid gives a functor

$$\overline{F}_P : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens} \tag{22}$$

which we call *the deformation functor* for a torsor P over an affine base.

Deformation functor (28) of a torsor P over an arbitrary scheme is defined in 10.4.3 as an appropriate homotopy limit of $\overline{F}_P$ over affine bases.

Our aim is to present a dg Lie algebra governing the formal deformations of a torsor over an arbitrary scheme.

First of all, we explicitly calculate in 10.2.4– 10.2.6 the functor (22) describing deformations of a trivial torsor over an affine base.

Then, using faithfully flat descent, we deduce that for an affine base the functors $\overline{F}_P$ and F_P are homotopically equivalent. This easily implies the main result Theorem 10.4.4.

10.1.1. We fix some notation. Recall that $\mathtt{dga}(k) = \mathtt{Alg}(\mathtt{COM}(k))$ is the category of commutative dg algebras over k.

Let R be the Hopf algebra of regular functions on G. Then $\mathfrak{g} = \mathrm{Der}(R, R)$ is the Lie algebra of G.

10.2. **The affine case.**

10.2.1. **Definition.** Let $B \in \mathtt{dga}(k)$. A B-torsor under G is a morphism $x : B \to X$ together with an associative (co)multiplication map $\mu : X \to X \otimes R$ satisfying the following properties:

(0) $\mu x = (\mathrm{id}_X \otimes 1)x$

(1) (pseudo-torsor) The multiplication μ together with $\mathrm{id}_X \otimes 1$ induce an isomorphism $X \otimes_B X \to X \otimes R$.

(2) (local triviality) The map x is faithfully flat (this property does not depend on the differential, see [H3]).

10.2.2. The definition above gives rise to a stack of groupoids on $\mathtt{dga}(k)$ in the topology generated by the faithfully flat maps. This is a (2-) functor $\mathcal{C} : \mathtt{dga}(k) \to \mathtt{Grp}$ such that for $B \in \mathtt{dga}(k)$ $\mathcal{C}(B)$ is the groupoid of B-torsors under G.

For a fixed B-torsor P one defines a fibred category $\mathcal{C}(P)$ over $\mathtt{dgart}^{\leq 0}(k)$ by the formula

$$\mathcal{C}(P, A) = \{A \otimes B\text{-torsors } \widetilde{P} \text{ with a trivialization } \widetilde{P} \otimes_{A\otimes B} B \xrightarrow{\sim} P\}$$

Now we are ready to define the functor (21).

Definition. Let $B \in \mathtt{dga}(k)$ and let P be a G-torsor over B. The functor

$$\mathcal{D}_P : \mathtt{dgart}^{\leq 0}(k) \to \mathtt{sGrp}$$

is defined by the formulas

$$\begin{aligned} &\mathrm{Ob}\,\mathcal{D}_P(A) = \mathrm{Ob}\,\mathcal{C}(P, A); \\ &\mathcal{H}om_{\mathcal{D}_P(A)}(x, y)_n = \mathrm{Hom}_{\mathcal{C}(P_n, A)}(x_n, y_n). \end{aligned}$$

for $A \in \mathtt{dgart}^{\leq 0}(k)$. Here P_n is the torsor $\Omega_n \otimes P$ over $\Omega_n \otimes B$ and, similarly, x_n, y_n are the torsors over $A \otimes \Omega_n \otimes B$.

In order to get a functor with values in simplicial sets, we apply the simplicial nerve functor $\mathcal{N} : \mathtt{sCat} \to \Delta^{\mathrm{op}}\mathtt{Ens}$ (see 11.3).

10.2.3. **Definition.** Let $B \in \mathtt{dga}(k)$ and let P be a G-torsor over B. The deformation functor $\overline{F}_P : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$ is the composition of $\mathcal{D}_P$ with the simplicial nerve functor

$$\overline{F}_P(A) := \mathcal{N}(\mathcal{D}(P, A)).$$

10.2.4. Let us make some calculation. Suppose that P is a trivial torsor over B.

Fix $A \in \mathtt{dgart}^{\leq 0}(k)$ and calculate the simplicial groupoid $\mathcal{D}(A)$ — we omit P from the notation since P is supposed to be trivial.

The functor # forgetting the differential in dg objects, transforms B-torsors to $B^\#$-torsors.

Lemma. *Let P be a torsor over $B \otimes A$ trivial over B. Then $P^\#$ is a trivial $(B \otimes A)^\#$-torsor.*

Proof. Let $\mathtt{Vectgr}$ be the tensor category of graded vector spaces with the commutativity constraint given by the formula

$$x \otimes y \mapsto (-1)^{\deg x \deg y} y \otimes x.$$

We will use the theory of cotangent complex [Ill]. Similarly to [Ill], ch. II, one defines a cotangent complex $L_{B/A}$ for a commutative A-algebra B in $\mathtt{Vectgr}$.

The cotangent complex commutes with the flat base change (see [Ill], II.2.2.3).

A map of commutative algebras in $\mathtt{Vectgr}$ is *formally smooth* if it satisfies the left lifting property with respect to surjective maps having a nilpotent kernel. A standard result — see e.g. [Ill], III, 3.1.2, for the non-graded case — claims that an A-algebra B is formally smooth if the cotangent complex $L_{B/A}$ is represented by a projective B-module.

Since the algebra R of functions on G is smooth over k, the cotangent complex $L_{R/k}$ is finitely generated projective. Then the base change property and the faithfully flat descent give that $L_{P^\#/B^\#}$ is finitely generated projective $P^\#$-module for any $B^\#$-torsor $P^\#$ under $G = \mathrm{Spec}(R)$. This implies that all graded torsors under G are formally smooth.

This fact implies that the map $P^\# \to P^\# \otimes_{B^\# \otimes A^\#} B^\# \to B^\#$ can be lifted to a map $P^\# \to B^\# \otimes A^\#$ splitting the structure map. This proves triviality of $P^\#$. □

10.2.5. **Corollary.** *For a given $B \in \mathtt{dga}(k)$, the groupoid $\mathcal{C}(A)$ is canonically (in A) equivalent to the following groupoid $\overline{\mathcal{C}}(A)$*

$$\mathrm{Ob}\,\overline{\mathcal{C}}(A) = \mathrm{MC}(\mathfrak{m} \otimes B \otimes \mathfrak{g})$$
$$\mathrm{Hom}_{\overline{\mathcal{C}}(A)}(x, y) = \{g \in \exp((\mathfrak{m} \otimes B)^0 \otimes \mathfrak{g}) | y = g(x)\}.$$

Proof. According to 10.2.4, any $A \otimes B$-torsor trivial over B has form $A \otimes B \otimes R$ as a graded algebra; its differential is defined by its restriction to R which is a derivation $\delta : R \to A \otimes B \otimes R$ trivialized by $A \to k$. This is given by a Maurer-Cartan element of $\mathfrak{m} \otimes B \otimes \mathfrak{g}$. Any automorphism of the graded torsor $(A \otimes B \otimes R)^\#$ is given by a $A \otimes B$-point of G. This one should map to the unit B-point of G under $A \to k$. This gives the second formula of the claim. □

10.2.6. **Proposition.** *For a given commutative k-algebra B the functor*

$$F_{triv} : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

describing deformations of the trivial B-torsor under G is naturally equivalent to the nerve of the Lie algebra $B \otimes \mathfrak{g}$.

Proof. Let B be a commutative k-algebra. According to 10.2.5, one has $\mathrm{Ob}\,\mathcal{D}(A) = \mathrm{MC}(\mathfrak{m}\otimes B\otimes\mathfrak{g})$ is a singleton since $\mathfrak{m}\otimes B\otimes\mathfrak{g}$ is non-positively graded, so $\mathcal{D}(A)$ is actually a simplicial group.

Re-denoting for simplicity $\mathcal{D}(A)$ by $\mathcal{D}$ and $\mathfrak{m}\otimes B\otimes\mathfrak{g}$ by $\mathfrak{g}$ we have

$$\mathcal{D}_n = \{u \in \exp(\Omega_n\otimes\mathfrak{g})^0 | u(0) = 0\} = \mathrm{Stab}_{G_n}(0)$$

in the notation of 8.2.

Now we have to find a natural equivalence from $\Sigma(\mathfrak{g})$ to $\mathcal{N}(\mathcal{D})$. Since $\mathrm{MC}(\mathfrak{g}) = \{0\}$, Corollary 8.2.6 furnishes us a principal fibration with the base $\Sigma(\mathfrak{g})$, the total space $G_\bullet$ with the group $\mathrm{Stab}_{G_\bullet}(0)$ and a canonical pseudo-cross section.

The simplicial set $G_\bullet$ is isomorphic to

$$n \mapsto (\Omega_n\otimes\mathfrak{g})^0$$

which is (a simplicial vector space and) a direct sum of simplicial vector spaces of form $\Omega^p_\bullet$ which are all contractible by [L], p. 44.

Then Theorem 21.13 of [M] provides a canonical homotopy equivalence $\Sigma(\mathfrak{g}) \to \overline{W}(\mathcal{D})$ — see 11.5. According to Lemma 11.5.1, this gives a canonical equivalence in question. $\square$

10.3. **Descent.**

10.3.1. In the previous subsection we defined for each B-torsor P under G, B being a dg commutative algebra, a deformation functor

$$\overline{F}_P : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

and calculated it in terms of the Lie algebra $\mathfrak{g}$ of G in the case when P is trivial and B is a commutative algebra.

In order to proceed further, we need some descent techniques. This will allow us to check that the functor $\overline{F}_P$ is (a sort of) a sheaf in the flat topology.

10.3.2. *"Total space" functors.* Denote by $\mathcal{M}$ the following category. The objects of $\mathcal{M}$ are morphisms $[p]\to[q]$ in Δ. A morphism from $[p]\to[q]$ to $[p']\to[q']$ is a commutative diagram

$$\begin{array}{ccc} [p] & \longrightarrow & [q] \\ {\scriptstyle\alpha}\uparrow & & \downarrow{\scriptstyle\beta} \\ [p'] & \longrightarrow & [q']. \end{array}$$

Let $\mathcal{C}$ be a simplicial category having inverse limits and simplicial *function objects* $\hom(S,X)\in\mathcal{C}$ for $S\in\Delta^{\mathrm{op}}\mathtt{Ens}$ and $X\in\mathcal{C}$. The total space $\mathrm{Tot}(X)$ of a cosimplicial object $X\in\Delta\mathcal{C}$ is defined by the formula

$$\mathrm{Tot}(X) = \lim_{\substack{\longleftarrow}}{}_{\phi\in\mathcal{M}} \hom(\Delta^p, X^q) \tag{23}$$

where $\phi : [p]\to[q]$.

We need three instances of this construction.

Simplicial sets. Put $\mathcal{C} = \Delta^{\mathrm{op}}\mathtt{Ens}$. In this case the definition of Tot is the standard one — see [BK], XI.3.

If $G \in \Delta\mathcal{C}$ is a cosimplicial groupoid, then $\mathrm{Tot}(G)$ is also a groupoid. This is the groupoid of "descent data" for G.

This fact justifies the definition of deformation groupoid in 10.4.3 using homotopy limits.

Simplicial categories. Let $\mathcal{C} = \mathtt{sCat}$ be the category of small simplicial categories. This category admits a structure of simplicial category (see 11.1) with simplicial function objects described as follows. Let $X = \{X_n\} \in \mathtt{sCat}$, $\quad S \in \Delta^{\mathrm{op}}\mathtt{Ens}$. The collection

$$n \mapsto \mathcal{H}om(S, X_n)$$

form a simplicial object in $\mathtt{Cat}$. We define $\mathcal{H}om(S, X)$ to be the simplicial category given by the formulas

$$\mathrm{Ob}\,\mathcal{H}om(S, X) = \mathrm{Ob}\,\mathcal{H}om(S, X_0) \tag{24}$$

$$\mathrm{Hom}_{\mathcal{H}om(S,X)}(x, y)_n = \mathrm{Hom}_{\mathcal{H}om(S,X_n)}(x_n, y_n) \tag{25}$$

where x, y are objects of $\mathcal{H}om(S, X_0)$ and x_n, y_n are their degeneracies in $\mathcal{H}om(S, X_n)$.

Lie algebras. Let $\mathcal{C} = \mathtt{dglie}(k)$ be the category of dg Lie algebras over k with the structure of simplicial category described in 2.4. For $\mathfrak{g} \in \Delta\mathtt{dglie}(k)$ the dg Lie algebra $\mathrm{Tot}(\mathfrak{g})$ coincides with the Thom-Sullivan complex of [HS2], 5.2.

As a complex, $\mathrm{Tot}(\mathfrak{g})$ is canonically quasi-isomorphic to the usual normalization of the cosimplicial complex $\mathfrak{g}$. This construction allows one to define the functor

$$\mathbf{R}\Gamma : \mathtt{dglie}^{qc}(X) \to \mathtt{dglie}(k) \tag{26}$$

from the category of quasi-coherent sheaves of dg Lie algebras on a quasi-compact scheme X over k to the category of dg Lie k-algebras — see details in [HS2], 5.4.

The following theorem relates the first and the third instances of the functor Tot mentioned above to the nerve functor Σ defined in 8.2.

10.3.3. **Theorem.** *(see* [H1], *Thm. 4.1) Let $\mathfrak{g}$ be a cosimplicial nilpotent dg Lie algebra over k. Suppose that $\mathfrak{g}$ is finitely dimensional in the cosimplicial sense, i.e. that the normalization*

$$N^n(\mathfrak{g}) = \{x \in \mathfrak{g}^n | \sigma^i(x) = 0 \text{ for all } i\}$$

vanishes for sufficiently big n. Then there is a natural homotopy equivalence

$$\Sigma(\mathrm{Tot}(\mathfrak{g})) \to \mathrm{Tot}(\Sigma(\mathfrak{g}))$$

of Kan simplicial sets.

10.3.4. Let $F : X \to \mathcal{C}$ is a functor to a simplicial closed model category $\mathcal{C}$. Recall that the homotopy limit $\mathrm{holim}\, F \in \mathcal{C}$ is defined by the formula

$$\mathrm{holim}\, F = \mathrm{Tot}(\Pi^*(F)).$$

Here the cosimplicial object $\Pi^*(F)$ in $\mathcal{C}$ is defined by the formula

$$\Pi^n(F) = \prod_{(x_0 \to \ldots \to x_n) \in X} F(x_n).$$

10.3.5. Let X be a site, $F : X^{\mathrm{op}} \to \mathcal{C}$ be a presheaf with the values in a simplicial closed model category $\mathcal{C}$ (as examples we will have $\mathcal{C} = \Delta^{\mathrm{op}}\mathtt{Ens}$ and $\mathcal{C} = \mathtt{sCat}$).

For each sieve $\mathcal{U} \to U$ in X we define

$$F(\mathcal{U}) = \operatorname{holim} F \circ s_{\mathcal{U}}$$

where $s_{\mathcal{U}} : \mathcal{U} \to X$ carries any object $f : V \to U$ of $\mathcal{U}$ to $V \in X$.

Definition. $F : X^{\mathrm{op}} \to \mathcal{C}$ is called a sheaf if for each covering sieve $\mathcal{U} \to U$ the natural map

$$F(U) \to F(\mathcal{U})$$

is a weak equivalence.

10.4. **The general case.** Now we are ready to describe deformations of a torsor over an arbitrary scheme — and to present a dg Lie algebra governing these deformations.

First of all, we check in 10.4.1 that the deformation functor $\overline{F}_P$ defined in 10.2.3 is a sheaf in the flat topology.

Then we define in 10.4.3 a deformation functor for a torsor P under G over an arbitrary scheme S. Finally, in 10.4.4 we construct the corresponding dg Lie algebra.

10.4.1. Let $X^{\mathrm{op}} = \mathtt{dga}(k)$. Let $B \in \mathtt{dga}(k)$ and let P be a G-torsor over B. Fix $A \in \mathtt{dgart}^{\leq 0}(k)$.

One has a presheaf $\overline{\mathcal{F}}_P : (X/B)^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathtt{Ens}$ defined by the formula $\overline{\mathcal{F}}_P(f) = \overline{F}_{f^*P}(A)$.

Lemma. *$\overline{\mathcal{F}}_P$ is a sheaf in the flat topology.*

Proof. One can immediately observe that the presheaf

$$\mathcal{D}_P : (X/B)^{\mathrm{op}} \to \mathtt{sCat} \tag{27}$$

is a sheaf.

Applying the simplicial nerve functor to (27) and using Lemma 11.5.2 claiming that holim commutes with the simplicial nerve up to homotopy, we obtain the required result. □

10.4.2. **Corollary.** *Let B be a commutative k-algebra and let P be a G-torsor over B. Denote by $\mathfrak{g}_P$ the corresponding Lie algebra over B. Then the functor $\overline{F}_P : \mathtt{dgart}^{\leq 0}(k) \to \Delta^{\mathrm{op}}\mathtt{Ens}$ is equivalent to the nerve of $\mathfrak{g}_P$.*

Proof. This immediately follows from 10.4.1 together with Theorem 10.3.3. □

10.4.3. Let S be a quasi-compact scheme over k and P be a S-torsor under G.

We define $F_P(A),\ A \in \mathtt{dgart}^{\leq 0}(k)$, by the formula

$$F_P(A) = \operatorname*{holim}_{\substack{i : U \to S \\ U \text{ affine}}} \overline{F}_{i^*P}(A), \tag{28}$$

the inverse homotopy limit being taken over all affine open subschemes of S.

Lemma 10.4.1 implies that the functors F_P and $\overline{F}_P$ are homotopy equivalent for affine S. This implies that F_P is also a sheaf in the flat topology.

In particular, this means that F_P can be calculated using any affine cover of S.

10.4.4. Let S be a quasi-compact scheme over a field k of characteristic zero, G be an affine algebraic group and P be a S-torsor under G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_P$ be the coherent sheaf of Lie algebras on S induced by P.

According to Corollary 10.4.2, for each affine open subset $i : U \subseteq S$ the deformation functor $\overline{F}_{i^*P}$ is equivalent to the nerve of the Lie algebra $\mathfrak{g}_P(U)$. Then Theorem 10.3.3 implies that the deformation functor F_P is equivalent to the nerve of the dg Lie algebra $\mathbf{R}\Gamma(S, \mathfrak{g}_P)$ — see (26). This proves the following

Theorem. *The formal stack of deformations of a S-torsor P under G has form*

$$\mathcal{C}(\mathbf{R}\Gamma(S, \mathfrak{g}_P)).$$

10.4.5. A similar result holds in the context of [Ka] where G-local systems are considered.

For this one defines torsors under G over a "dg-ringed space" $(X, \mathcal{O}_X)$ where X is a topological space endowed with a sheaf of commutative dg k-algebras $\mathcal{O}_X$, using the topology generated by surjective open covers of X. Then, given a good topological space X and a torsor P under G over the ringed space (X, k_X) ($\mathcal{O} = k_X$ is the constant sheaf), its deformations over a local dg artinian k-algebra A are defined as $(X, A \otimes k_X)$-torsors under G endowed with a trivialization over (X, k_X). These deformations are governed locally by the sheaf of Lie algebras $\mathfrak{g}_P$ and, therefore, globally, by $\mathbf{R}\Gamma(X, \mathfrak{g}_P)$.

This formula coincides with Kapranov's [Ka], 2.5.1.

11. Appendix: simplicial categories

In this section we present some (more or less) standard results about simplicial categories. It includes the description 11.1.5 of a simplicial CMC structure on the category `sCat` of small simplicial categories. A similar description of CMC structure on `sCat` can be found in [DHK], ch. 48.

The main results of this section are Proposition 11.5.2 and Lemma 11.5.1.

11.1. Weak equivalences and fibrations in `sCat`.

Here we define a closed model category structure on the category `sCat` of simplicial categories.

11.1.1. *(Co)limits.* The category `sCat` admits arbitrary limits and colimits. Inverse limits in `sCat` are induced by inverse limits in `Ens` in the obvious sense.

The existence of inductive limits in `sCat` follows by a general abstract nonsense from the existence of inductive limits in `Ens`. Note that the functor `sCat` $\to$ `Ens` assigning to each simplicial category the set of its objects, commutes with inductive limits. The set of morphisms of an inductive limit is freely generated by the morphisms of all categories involved, modulo an obvious equivalence relation.

Note that the existence of direct limits in `sCat` allows one to mimic the procedure of "adding variables". We will single out the following cases.

Adding an object. Given $\mathcal{C} \in \mathtt{sCat}$, denote $\mathcal{C}\langle * \rangle$ the coproduct of $\mathcal{C}$ with the trivial one-object category $*$.

Adding an ingoing arrow. Given $\mathcal{C} \in \mathtt{sCat},\ x \in \mathrm{Ob}\,\mathcal{C}$, one defines $\mathcal{C}\langle * \to x\rangle$ to be the category having the set of objects $\mathrm{Ob}\,\mathcal{C} \coprod \{*\}$ and the set of morphisms freely generated by $\mathrm{Mor}\,\mathcal{C}$ and by a map $* \to x$.

Adding an outgoing arrow. The category $\mathcal{C}\langle x \to * \rangle$ is defined similarly to the above.

Adding maps between objects Given $\mathcal{C} \in \mathtt{sCat}$, $x, y \in \mathrm{Ob}\,\mathcal{C}$ and a map $\alpha : \mathcal{H}om_{\mathcal{C}}(x, y) \to H$ of simplicial sets, the simplicial category $\mathcal{C}\langle x, y; \alpha\rangle$ has the same objects as $\mathcal{C}$. Its set of morphisms is freely generated by $\mathrm{Mor}\,\mathcal{C}$ and by H, subject to identification $\alpha(f) \sim f,\ f \in \mathcal{H}om_{\mathcal{C}}(x, y)$.

11.1.2. Define the functor

$$\pi_0 : \mathtt{sCat} \to \mathtt{Cat}$$

as follows. For $\mathcal{C} \in \mathtt{sCat}$ the category $\pi_0(\mathcal{C})$ has the same objects as $\mathcal{C}$. For $x, y \in \mathrm{Ob}\,\pi_0(\mathcal{C})$

$$\mathrm{Hom}_{\pi_0(\mathcal{C})}(x, y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(x, y)).$$

11.1.3. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in `sCat` is called a weak equivalence if the following properties are satisfied.

(1) The map $\mathcal{N}(\pi_0(f))$ is a weak equivalence of simplicial sets.

(2) For all $x, x' \in \mathrm{Ob}\,\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is a weak equivalence.

11.1.4. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in `sCat` is called a fibration if it satisfies the following properties

(1) the right lifting property (RLP) with respect to "adding an ingoing or an outgoing arrow"

$$\mathcal{A} \to \mathcal{A}\langle * \to x\rangle,\ \mathcal{A} \to \mathcal{A}\langle x \to * \rangle$$

(see 11.1.1).

(2) For all $x, x' \in \mathrm{Ob}\,\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is a Kan fibration. This is equivalent the the RLP with respect to all maps $\mathcal{A} \to \mathcal{A}\langle x, y; \alpha\rangle$ where α is an acyclic fibration (see 11.1.1).

11.1.5. **Theorem.** *The category* `sCat` *admits a CMC structure with weak equivalences described in 11.1.3 and fibrations as in 11.1.4.*

11.1.6. An explicit description of different classes of morphisms in `sCat` is given below. The proof of Theorem 11.1.5 is standard. It is given in 11.2.

11.1.7. A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathtt{sCat}$ is called an acyclic fibration if it is simultaneously a weak equivalence and a fibration.

Lemma. *A map $f : \mathcal{C} \to \mathcal{D}$ is an acyclic fibration iff the following conditions are satisfied.*

(1) the map $\mathrm{Ob}\, f : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ is surjective. In other words, f satisfies the RLP with respect to "adding an object map" $\mathcal{A} \to \mathcal{A}\langle * \rangle$.

(2) For all $x, x' \in \mathrm{Ob}\,\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is an acyclic Kan fibration.

Proof. If f satisfies (1), (2), it is clearly an acyclic fibration. In the other direction, suppose f is an acyclic fibration. Then the property (2) is clear. We have only to check that $\mathrm{Ob}\, f$ is surjective. Since f satisfies the RLP with respect to ingoing and outgoing arrows, $\mathcal{D}$ is a disjoint union of the full subcategories, defined by the image of $\mathrm{Ob}\, f$ and by its complement. Since $\pi_0(f)$ is a weak equivalence, it induces a bijection of the connected components of $\mathcal{C}$ and $\mathcal{D}$ and this proves the claim. □

11.1.8. A map $f : \mathcal{C} \to \mathcal{D}$ will be called *a standard cofibration* if there is a collection of maps $f_i : \mathcal{C}_i \to \mathcal{C}_{i+1},\ i \in \mathbb{N}$ such that $\mathcal{C} = \mathcal{C}_0,\ \mathcal{D} = \varinjlim \mathcal{C}_i$, and each f_i is a coproduct of maps of one of the following two types:

(1) Adding an object $\mathcal{C}_i \to \mathcal{C}_i\langle * \rangle$;

(2) Adding maps between objects $\mathcal{C}_i \to \mathcal{C}_i\langle x, y; \alpha \rangle$ with α injective.

By 11.1.7, standard cofibrations satisfy LLP with respect to all acyclic fibrations.

11.1.9. A map $f : \mathcal{C} \to \mathcal{D}$ will be called *a standard acyclic cofibration* if there is a collection of maps $f_i : \mathcal{C}_i \to \mathcal{C}_{i+1},\ i \in \mathbb{N}$ such that $\mathcal{C} = \mathcal{C}_0,\ \mathcal{D} = \varinjlim \mathcal{C}_i$, and each f_i is a coproduct of maps of one of the following three types:

(1+) Adding an ingoing arrow $\mathcal{C}_i \to \mathcal{C}_i\langle * \to x \rangle$;

(1–) Adding an outgoing arrow $\mathcal{C}_i \to \mathcal{C}_i\langle x \to * \rangle$;

(2) Adding maps between objects $\mathcal{C}_i \to \mathcal{C}_i\langle x, y; \alpha \rangle$ with α acyclic cofibration.

By 11.1.4, standard acyclic cofibrations satisfy LLP with respect to all fibrations.

11.1.10. The following description of cofibrations and of acyclic cofibrations results from the proof of Theorem 11.1.5.

Corollary. *1. Any cofibration in* $\mathtt{sCat}$ *is a retract of a standard cofibration.*

2. Any acyclic cofibration in $\mathtt{sCat}$ *is a retract of a standard acyclic cofibration.*

11.2. **Proof of Theorem 11.1.5.** The axioms (CM 1), (CM 2), (CM 3), (CM 4)(ii) are immediately verified.

(CM 5)(ii) Let $f : X \to Y$ be a map in $\mathtt{sCat}$. Adding objects to X, we can ensure that the map $f : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ is surjective. Then, adding maps between objects, we can decompose f into a standard cofibration followed by an acyclic fibration.

This implies, in particular, that any cofibration is a retract of a standard cofibration.

To check the axiom (CM 5)(i) we need the following

11.2.1. **Lemma.** *Standard acyclic cofibrations are acyclic cofibrations.*

Proof. It is enough to prove that a map $\mathcal{C} \to \mathcal{D}$ is a weak equivalence when $\mathcal{D}$ is obtained from $\mathcal{C}$ by one of the following ways.

(1) adding a number of ingoing arrows;

(2) adding a number of outgoing arrows;

(3) adding (simultaneously) maps between objects x_i and y_i along acyclic cofibrations $\alpha_i : \mathcal{H}om(x_i, y_i) \to H_i$.

In the first two cases the map $\mathcal{C} \to \mathcal{D}$ is easily split by an acyclic fibration.

The shortest way to get the result in the case (3) is to use Proposition 7.2 of [DK] which claims the existence of CMC structure on the category of simplicial categories having a fixed set of objects.

□

(CM 5)(i) Let $f : X \to Y$ be a map in $\texttt{sCat}$. Adding ingoing and outgoing arrows to X, we can ensure that the image of $\mathrm{Ob}(X)$ under f consists of a number of connected components of $\mathrm{Ob}(Y)$. From now on we can suppose, without loss of generality, that f is surjective on objects. Then to costruct a decomposition $f = p \circ i$ it is enough to check that p satisfies condition (2) of 11.1.4 to ensure p is a fibration.

Now applying step by step the procedure of adding maps between objects $\mathcal{C} \to \mathcal{C}\langle x, y; \alpha\rangle$ along acyclic cofibrations α, we can construct a decomposition $f = p \circ i$ with p fibration and i a standard acyclic cofibration. According to Lemma 11.2.1, i is an acyclic cofibration.

Now, applying the proof of (CM 5)(i) to any acyclic cofibration f, we deduce that f is a retract of a standard acyclic cofibration.

(CM 4)(i) By definition, any standard acyclic cofibration satisfies LLP with respect to all fibrations. Any acyclic fibration is a retract of a standard acyclic fibration, and therefore satisfies as well LLP with respect to all fibrations.

Theorem is proven.

11.3. **Simplicial nerve.**

11.3.1. In what follows we identify $\texttt{Cat}$ with the full subcategory of $\Delta^{\mathrm{op}}\texttt{Ens}$. Then every simplicial category (and even every $\mathcal{C} \in \Delta^{\mathrm{op}}\texttt{Cat}$) can be seen as a bisimplicial set; its diagonal will be called *the nerve* of $\mathcal{C}$ and will be denoted $\mathcal{N}(\mathcal{C})$. If $\mathcal{C}$ is a "usual" category, $\mathcal{N}(\mathcal{C})$ is its "usual" nerve.

The functor $\mathcal{N} : \texttt{sCat} \to \Delta^{\mathrm{op}}\texttt{Ens}$ admits a left adjoint functor

$$\texttt{SC} : \Delta^{\mathrm{op}}\texttt{Ens} \to \texttt{sCat}$$

defined by the properties

$\operatorname{Ob} \mathtt{SC}(\Delta^n) = [n] = \{0, \dots, n\}$;
$\operatorname{Mor} \mathtt{SC}(\Delta^n)$ is freely generated by $a_i \in \mathcal{H}om_n(i-1, i),\ i = 1, \dots, n$;
$\mathtt{SC}$ commutes with arbitrary colimits.

11.3.2. **Proposition.** *The nerve functor $\mathcal{N} : \mathtt{sCat} \to \Delta^{\mathrm{op}}\mathtt{Ens}$ preserves weak equivalences, fibrations and cofibrations.*

Proof. 1. To check that $\mathcal{N}$ preserves the fibrations, it is enough to prove that the adjoint functor $\mathtt{SC}$ preserves acyclic cofibrations. For this we have to check that $\mathtt{SC}$ transforms any map $\Lambda^n_i \to \Delta^n$ to an acyclic fibration. This is an easy exercise (one should consider the cases $n = 1$ and $n > 1$ separately). Note that the same reasoning (even easier!) proves that $\mathcal{N}$ preserves acyclic fibrations — this is because $\mathtt{SC}$ preserves cofibrations.

2. It is clear that $\mathcal{N}(f)$ is a weak equivalence provided f is a weak equivalence *bijective on objects*. To prove the general claim, we present f as a composition of an acyclic fibration with an acyclic cofibration and therefore reduce the problem to the case f is an acyclic cofibration. Using 11.1.10, we can suppose that f is of one of the types (1+), (1–), (2) of 11.1.9. The type (2) does not change the set of objects, so we have nothing to prove. The maps of types (1+), (1–) split, and the splitting map is an acyclic fibration. This proves the claim.

3. The claim about cofibrations is obvious. $\square$

11.3.3. **Lemma.** *Let $f : X \to Y$ in $\mathtt{sCat}$ induce a fibration $\mathcal{N}(f) : \mathcal{N}(X) \to \mathcal{N}(Y)$. Then the functor $f_0 : X_0 \to Y_0$ induces a fibration of the corresponding (usual) nerves*

$$N(f_0) : N(X_0) \to N(Y_0).$$

Proof. One has a natural map $i : N(X_0) \to \mathcal{N}(X)$. A map $[0] \to [n]$ in $\Delta^{\mathrm{op}}\mathtt{Ens}$ sending 0 to i for $i = 0, \dots, n$, defines a map $v_i : \mathcal{N}_n(X) \to N_n(X_0)$ splitting i_n. The maps v_i do not form a map of simplicial sets; however, for any $i \neq j$ one has the following identity

$$d_j \circ v_i = v'_k \circ d_j \tag{29}$$

where $k = i$ for $j > i$ and $i - 1$ otherwise, and $v'_k : \mathcal{N}_{n-1}(X) \to N_{n-1}(X_0)$ is defined as v_i above.

Fix $i = 0, \dots, n$ and let $x_j,\quad j \neq i$ be a collection of compatible $(n-1)$-simplices in $N(X_0)$. Let $y \in N_n(Y_0)$ be such that $f(x_j) = d_j y$. Our aim is to find $x \in N_n(X_0)$ such that $y = f(x)$ and $x_j = d_j x$.

Since $\mathcal{N}(f)$ is a fibration, there exists $x \in \mathcal{N}_n(X)$ such that $i(y) = f(z)$ and $i(x_j) = d_j(z)$. Put $x = v_i(z)$. Then $f(x) = y$ since v_i splits i. Also $d_j(x) = d_j(v_i(z)) = x_j$ by (29). $\square$

11.4. **Simplicial structure on $\mathtt{sCat}$.**

11.4.1. Let S be a simplicial set, $\mathcal{C}$ be a category and let $N\mathcal{C}$ be the nerve of $\mathcal{C}$. The simplicial set $\mathcal{H}om(S, N\mathcal{C})$ is the nerve of a category which will be denoted by $\mathcal{C}^S$.

11.4.2. Let now S be a simplicial set and $X \in \mathtt{sCat}$. The categories X^S_n form a simplicial object in $\mathtt{Cat}$. However, they have different sets of objects. We define a simplicial category X^S by the formulas

$$\mathrm{Ob}\ X^S = \mathrm{Ob}\ X^S_0 \tag{30}$$

$$\mathrm{Hom}_{X^S}(x,y)_n = \mathrm{Hom}_{X^S_n}(x_n,y_n) \tag{31}$$

where x,y are objects of X^S_0 and x_n, y_n are their degeneracies in X^S_n.

11.4.3. The following proposition means that the formulas (30)–(31) define on $\mathtt{sCat}$ a right framing in sense of [DHK], ch. 53.

Proposition. *1. For any $X \in \mathtt{sCat}$ the natural map $X \to X^{\Delta^n}$ is a weak equivalence.*

2. Let $X \in \mathtt{sCat}$ be fibrant. Then for each $n \in \mathbb{N}$ the natural map

$$X^{\Delta^n} \to X^{\partial\Delta^n}$$

is a fibration.

The first claim of the proposition is standard; the second claim is a special case of the following

11.4.4. **Lemma.** *Let $\alpha : S \to T$ be a cofibration in $\Delta^{\mathrm{op}}\mathtt{Ens}$ and let $\pi : X \to Y$ be a fibration in $\mathtt{sCat}$. Then the induced map*

$$\rho : X^T \to X^S \times_{Y^S} Y^T \tag{32}$$

is a fibration in $\mathtt{sCat}$.

Proof. We have to check that ρ satisfies the properties (1), (2) of 11.1.4.

1. *adding an ingoing or an outgoing arrow.* Given an object $x \in X^T$ and a 0-arrow $f : \rho(x) \to y$ in $X^S \times_{Y^S} Y^T$, we have to lift f to a 0-arrow $\widetilde{f} : x \to \widetilde{y}$.

The object $x \in X^T$ is given by a map $x : T \to N(X_0)$; a 0-arrow in X^T is given by a map $T \times \Delta^1 \to N(X_0)$. The pair x, f defined above gives rise to a commutative square

$$\begin{array}{ccc} T \sqcup^S (S \times \Delta^1) & \longrightarrow & NX_0 \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle Nf_0} \\ T \times \Delta^1 & \longrightarrow & NY_0 \end{array}$$

The map β induced by $\alpha : S \to T$, is acyclic cofibration. The map Nf_0 is a fibration by Lemma 11.3.3. This proves the assertion.

2. Let us check property (2) of 11.1.4. It is convenient to define for $X \in \mathtt{sCat}$ and $S \in \Delta^{\mathrm{op}}\mathtt{Ens}$ a category $X(S)$ with the objects $\mathrm{Ob}\, X(S) = \mathrm{Ob}\, X_0$ and morphisms given by

$$\mathrm{Hom}_{X(S)}(x,y) = \mathrm{Hom}(S, \mathcal{H}om_X(x,y)).$$

A straightforward check shows that if $U \to V$ is an acyclic cofibration in $\Delta^{\mathrm{op}}\mathtt{Ens}$ and $X \to Y$ is a fibration in $\mathtt{sCat}$ then the induced map

$$NX(V) \to NX(U) \times_{NY(U)} NX(V)$$

is an acyclic fibration. This easily implies property (2) of 11.1.4. $\square$

11.5. **Functor $\overline{W}$.** The following construction is a minor generalization of the functor $\overline{W}$, see [M], chapter IV.

Let $\mathcal{C} \in \mathtt{sCat}$. Define a simplicial set $\overline{W}(\mathcal{C})$ as follows.

The n-simplices of $\overline{W}(\mathcal{C})$ are the sequences

$$(g_1|\ldots|g_n),\ g_i \in \hom_{\mathcal{C}_{n-i}}(v_{i-1}, v_i) \text{ where } v_0, \ldots, v_n \in \mathrm{Ob}\,\mathcal{C}.$$

The faces and the degeneracies are given by the following formulas

$$\begin{aligned}
&d_i(g_1|\ldots|g_n) = (d_{i-1}g_1|\ldots|d_1 g_{i-1}|g_{i+1}d_0 g_i|\ldots|g_n) \text{ for } i \neq 0, n\\
&d_0(g_1|\ldots|g_n) = (g_2|\ldots|g_n)\\
&d_n(g_1|\ldots|g_n) = (d_{n-1}g_1|\ldots|d_1 g_n)\\
&s_0(g_1|\ldots|g_n) = (\mathrm{id}\,|g_1|\ldots|g_n)\\
&s_i(g_1|\ldots|g_n) = (s_{i-1}g_1|\ldots|s_0 g_i|\,\mathrm{id}\,|g_{i+1}|\ldots|g_n) \text{ for } i > 0.
\end{aligned}$$

11.5.1. One has the following canonical maps connecting $\mathcal{N}(\mathcal{C})$ and $\overline{W}(\mathcal{C})$

$$\pi : \mathcal{N}(\mathcal{C}) \rightleftarrows \overline{W}(\mathcal{C}) : \rho$$

given by the formulas

$$\begin{aligned}
\pi[f_1|\ldots|f_n] &= (d_0 f_1|d_0^2 f_2|\ldots|d_0^n f_n)\\
\rho(g_1|\ldots|g_n) &= [s_0 g_1|s_0^2 g_2|\ldots|s_0^n g_n].
\end{aligned}$$

Lemma. *Suppose that $\mathcal{C}$ is a simplicial groupoid. Then the maps π, ρ are homotopy equivalences.*

Proof. Note that for each two objects x, y of $\mathcal{C}$ the simplicial set

$$n \mapsto \mathrm{Hom}_{\mathcal{C}_n}(x, y) \tag{33}$$

is fibrant. In fact, it is empty when $x \not\sim y$ and is isomorphic to a simplicial group otherwise.

We will directly check that if $\mathcal{C} \in \mathtt{sGrp}$ the map π satisfies the RLP with respect to the maps $\partial\Delta^n \to \Delta^n$.

A map $\Delta^n \to \overline{W}(\mathcal{C})$ is given by a collection

$$g = (g_1|\ldots|g_n),\ g_i \in \mathcal{H}om_{n-i}(v_{i-1}, v_i).$$

A map $\partial\Delta^n \to \mathcal{N}(\mathcal{C})$ is given by a compatible collection

$$x^i = [x_1^i|\ldots|x_{n-1}^i] \text{ with } \deg x_j^i = n-1,$$

the compatibility conditions being the conditions (A), (B) below.

(A) The condition $d_i(x^k) = d_{k-1}(x^i)$ for $i < k$ amounts to the system

$$\begin{aligned}
&d_i(x_j^k) = d_{k-1}(x_j^i) \text{ for } j \leq i-1 \text{ or } j \geq k+1\\
&d_i(x_{i+1}^k \circ x_i^k) = d_{k-1}(x_i^i) \text{ for } i \leq k-2\\
&d_i(x_{j+1}^k) = d_{k-1}(x_j^i) \text{ for } i < j \leq k-2\\
&d_i(x_k^k) = d_{k-1}(x_k^i \circ x_{k-1}^i) \text{ for } i \leq k-2\\
&d_{k-1}(x_k^k \circ x_{k-1}^k) = d_{k-1}(x_k^{k-1} \circ x_{k-1}^{k-1})
\end{aligned}$$

(B) The compatibilities of x^i with g say that $\pi(x^i) = d_i(g)$ which gives

$$d_0^j x_j^i = \begin{cases} d_{i-j} g_j \text{ for } j \le i-1 \\ g_{i+1} \circ d_0 g_i \text{ for } j = i \\ g_{j+1} \text{ for } j > i. \end{cases}$$

(C) Now, we have to construct a map $\Delta^n \to \mathcal{N}(\mathcal{C})$ i.e. a collection $[f_1|\ldots|f_n]$ satisfying the conditions $d_i f = x^i$, $\pi(f) = g$ which can be rewritten as a system

(1) $d_0^j(f_j) = g_j$

(2) $d_i(f_j) = \begin{cases} x_j^i, & i \ge j+1 \\ x_{j-1}^i, & i \le j-2 \end{cases}$

(3) $d_{j-1}(f_j) \circ d_{j-1}(f_{j-1}) = x_{j-1}^{j-1}$

One is looking for f_j by induction. For $j = 1$ we have prescribed values for $d_i(f_1)$, $i \ne 1$.

Thus, checking their compatibility and using the fibrantness of (33), we deduce that the system admits a solution f_1.

Suppose that f_i have already been found for $i < j$ so that the equations above are satisfied. One checks first of all that for $j > 1$ the equation (C.1) follows from (C.2). Afterwards, one can find the value for $d_{j-1}(f_j)$ since $\mathcal{C}_{n-1}$ is a groupoid. Then we have the prescribed values for $d_i(f_j)$ for all $i \ne j$ and we have only to check using the compatibility conditions (A), (B) that these prescriptions given by (C.2), (C.3), are compatible. ☐

11.5.2. Let I be a category and let $\mathcal{C} : I \to \mathtt{sCat}$ be a functor.

Proposition. *Suppose that $\mathcal{C}(i)$ is fibrant for each $i \in I$ (for instance, $\mathcal{C}(i) \in \mathtt{sGrp}$). Then there is a natural homotopy equivalence*

$$\mathcal{N}(\operatorname{holim} \mathcal{C}) \sim \operatorname{holim}(\mathcal{N} \circ \mathcal{C}). \tag{34}$$

Proof. It is very convenient here to use the approach of [DHK], Ch. XIV. Namely, since $\mathcal{C}(i)$ are fibrant in $\mathtt{sCat}$, $\operatorname{holim}\mathcal{C}$ represents the right derived functor $\mathbf{R}\lim : \mathrm{Ho}(\mathtt{sCat}^I) \to \mathrm{Ho}(\mathtt{sCat})$ (Note that our definition of homotopy limit coincides with that of [DHK] by Proposition 11.4.3.) On the other hand, this functor can be calculated using "virtually fibrant" resolutions of $\mathcal{C}$ — see [DHK], 58.5.

The functor $\mathcal{N}$ is "right Quillen" — it admits a left adjoint functor and preserves fibrations and weak equivalences. Then by [DHK], 12.6, it sends virtually fibrant diagrams in $\mathtt{sCat}$ to virtually fibrant diagrams of simplicial sets. Since $\mathcal{N}$ commutes with inverse limits, this proves the assertion. ☐

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