INTRODUCTION TO LIE ALGEBRAS.

1. Algebras. Derivations. Definition of Lie algebra

1.1. **Algebras.** Let k be a field. An algebra over k (or k-algebra) is a vector space A endowed with a bilinear operation

$$a, b \in A \mapsto a \cdot b \in A.$$

Recall that bilinearity means that for each $a \in A$ left and right multiplications by a are linear transformations of vector spaces (i.e. preserve sum and multiplication by a scalar).

1.1.1. *Some extra properties.* An algebra A is called associative if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$).

An algebra A is commutative if $a \cdot b = b \cdot a$.

Usually commutative algebras are supposed to be associative as well. Often (but not necessarily in this course) they are also supposed to have a unit 1, that is an element satisfying the condition $1 \cdot a = a \cdot 1 = a$.

1.1.2. *Example.* If V is a vector space, $\mathrm{End}(V)$, the set of (linear) endomorphisms of V is an associative algebra with respect to composition. If $V = k^n$ $\mathrm{End}(V)$ is just the algebra of $n \times n$ matrices over k.

1.1.3. *Example.* The ring of polynomials $k[x]$ over k is a commutative k-algebra. The same for $k[x_1, \ldots, x_n]$, the algebra of polynomials of n variables.

1.1.4. *Example.* If V is a vector space, define an operation by the formula

$$a \cdot b = 0.$$

This is an algebra operation.

1.2. **Subalgebras, ideals, quotient algebras.** A linear map $f : A \to B$ of k-algebras is called *homomorphism* if $f(a \cdot b) = f(a) \cdot f(b)$ for each $a, b \in A$.

The image of a homomorphism is a *subalgebra* (please, give a correct definition). Kernel of f defined as $\{a \in A | f(a) = 0\}$ is *an ideal* in A. Here are the appropriate definitions.

1.2.1. **Definition.** A vector subspace $B \subseteq A$ is called a subalgebra if

$$a, b \in B \Longrightarrow a \cdot b \in B.$$

1.2.2. **Definition.** A vector subspace $I \subseteq A$ is called an ideal if

$$a \in A \& x \in I \Longrightarrow a \cdot x \in I \& x \cdot a \in I.$$

1.2.3. **Lemma.** *Let $f : A \to B$ be a homomorphism of algebras. Then* $\mathrm{Ker}(f)$ *is an ideal in A.*

Proof. Exercise. ☐

An important property of ideals is that one can form a quotient algebra "modulo I". Here is the construction.

Let A be an algebra and I an ideal in A. We define the quotient algebra A/I as follows.

As a set this is the quotient of A modulo the equivalence relation

$$a \sim b \text{ iff } a - b \in I.$$

Thus, this is the set of equivalence classes having form $a + I$, where $a \in A$.

Structure of vector space on A/I is given by the formulas

$$(a + I) + (b + I) = (a + b) + I; \quad \lambda(a + I) = \lambda a + I.$$

Algebra structure on A/I is given by the formula

$$(a + I) \cdot (b + I) = a \cdot b + I.$$

One has a canonical homomorphism

$$\rho : A \to A/I$$

defined by the formula $\rho(a) = a + I$.

As usual, the following theorem (Theorem on homomorphism) is straighforward.

1.2.4. **Theorem.** *Let $f : A \to B$ be a homomorphism of algebras and let I be an ideal in A. Suppose that $I \subseteq$* $\mathrm{Ker}(f)$. *Then there exists a unique homomorphism $\overline{f} : A/I \to B$ such that $f = \overline{f} \circ \rho$ where $\rho : A \to A/I$ is the canonical homomorphism.*

Moreover, $\overline{f}$ is onto iff f is onto; $\overline{f}$ is one-to-one iff $I =$ $\mathrm{Ker}(f)$.

Proof. Exercise. ☐

1.3. **Derivations.** A linear endomorphism $d : A \to A$ is called *derivation* if the following *Leibniz rule* holds.

$$d(a \cdot b) = d(a) \cdot b + a \cdot d(b).$$

The set of all derivations of A is denoted $\mathrm{Der}(A)$. This is clearly a vector subspace of $\mathrm{End}(A)$.

1.3.1. *Composition.* Let $d, d' \in \mathrm{Der}(A)$ let us check that the composition dd' is not a derivation.

$$dd'(a \cdot b) = d(d'(a) \cdot b + a \cdot d'(b)) = d(d'(a) \cdot b) + d(a \cdot d'(b)) = dd'(a) \cdot b + d'(a) \cdot d(b) + d(a) \cdot d'(b) + a \cdot dd'(b) \tag{1}$$

which is not exactly what we need.

1.3.2. *Bracket.* Thus, we suggest another operation. Given $d, d' \in \mathrm{Der}(A)$, define $[d, d'] = dd' - d'd$.

1.3.3. **Theorem.** *If $d, d' \in \mathrm{Der}(A)$ then $[d, d'] \in \mathrm{Der}(A)$.*

Proof. Direct calculation. □

1.3.4. *Properties of this bracket.* 1. $[x, x] = 0$.
2. (Jacobi identity) $[[xy]z] + [[zx]y] + [[yz]x] = 0$
Exercise: check this.

1.4. **Definition of Lie algebra. First examples.** A Lie algebra is an algebra with an operation satisfying the properties 1.3.4.

The operation in a Lie algebra is usually denoted $[,]$ and called (Lie) bracket.

1.4.1. *Anticommutativity.* The first property of a Lie algebra saying $[xx] = 0$ is called anticommutativity. In fact, it implies that $[xy] = -[yx]$ for all x, y.

Proof: $0 = [x + y, x + y] = [xx] + [xy] + [yx] + [yy]$. This implies $[xy] = -[yx]$. The converse is true if $\mathrm{char}\, k \neq 2$. In fact, $[xx] = -[xx]$ implies that $2[xx] = 0$ and, if the charactersitic of k is not 2, this implies $[xx] = 0$.

1.4.2. *Example.* Let $k = \mathbb{R}$, $L = \mathbb{R}$. We are looking for possible Lie brackets on L. Bilinearity and anticommutativity require

$$[a, b] = [a \cdot 1, b \cdot 1] = ab[1, 1] = 0.$$

Thus, there is only one Lie bracket on $L = \mathbb{R}$.

1.4.3. **Definition.** A Lie algebra L having a zero bracket is called *a commutative Lie algebra.*

1.4.4. *Observation.* Fix a field k of characteristic $\neq 2$ and let $L = \langle e_1, \ldots, e_n \rangle$ be n-dimensional vector space over k. In order to define a bilinear operation, it is enough to define it on e_i:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

(this is true for any type of algebra). Elements c_{ij}^k are called *structure constants* of L.

Since we want the bracket to be anti-commutative, one has to have

$$[e_i, e_j] = [e_j, e_i].$$

Bilinearity and this condition imply anti-commutativity of the bracket (check this formally!).

Suppose now we have checked already anticommutativity. To check Jacobi identity let us denote

$$J(x, y, z) = [[xy]z] + [[zx]y] + [[yz]x].$$

One observes that J is trilinear (linear on each one of its three arguments) and antisymmetric (it changes sign if one interchanges any two arguments).

Thus, in order to check $J(x, y, z)$ is identically zero, it is enough to check

$$J(e_i, e_j, e_k) = 0 \text{ for } 1 \leq i < j < k \leq n.$$

1.4.5. *Example.* Suppose $\dim L = 2$. Suppose L is not commutative. Choose a basis $L = \langle e_1, e_2 \rangle$. One has

$$[e_1, e_1] = [e_2, e_2] = 0 \text{ and } [e_1, e_2] = -[e_2, e_1].$$

Let $[e_1, e_2] = y$. Then $y \neq 0$ and any bracket in L is proportional to y (by bilinearity).

Thus, it is convenient to take y as one of generators on L. Choose another one, say x. We have $L = \langle x, y \rangle$ and $[x, y] = \lambda y$. Since L is not commutative, $\lambda \neq 0$. Thus change variables once more setting $x := x/\lambda$.

We finally get

$$L = \langle x, y \rangle \text{ and } [x, y] = y. \tag{2}$$

We have therefore proven that there are only two two-dimensional Lie algebras over k up to isomorphism: a commutative Lie algebra and the one described in (2).

1.4.6. *Example.* The set of $n \times n$ matrices over k is an associative algebra with respect to the matrix multiplication. It becomes a Lie algebra if we define a bracket by the formula

$$[x, y] = xy - yx.$$

This Lie algebra is denoted $\mathfrak{gl}_n(k)$ (sometimes we do not mention the field k). Its dimension is, of course, n^2.

The Lie algebra $\mathfrak{gl}_n$ admits a remarkable Lie subalgebra.

Define $\mathfrak{sl}_n = \{a \in \mathfrak{gl}_n | \operatorname{tr}(a) = 0\}$.

Here $\operatorname{tr}(a) = \sum a_{ii}$ is the trace of a, the sum of the diagonal elements of a.

We claim this is a Lie subalgebra.

1.4.7. *Proof.* Recall that for each pair of matrices a, b one has

$$\mathrm{tr}(ab) = \mathrm{tr}(ba).$$

(Proof is just a direct calculation: both sides are equal to $\sum_{ij} a_{ij}b_{ji}$.)

Then $\mathrm{tr}([a,b]) = \mathrm{tr}(ab) - \mathrm{tr}(ba) = 0$. This proves that $\mathfrak{sl}_n$ is closed under the bracket operation.

1.5. **Classical Lie algebras.** Here is a way to construct many interesting Lie algebras. Let V be a finite dimensional vector space and let $B : V \times V \to k$ be a bilinear form. Define

$$\mathfrak{g}_B = \{f : V \to V | B(f(v), w) + B(v, f(w)) = 0\}$$

for any $v, w \in V$.

1.5.1. **Proposition.** *$\mathfrak{g}_B$ is a Lie subalgebra in $\mathfrak{gl}(V)$.*

Proof. Assume that

$$B(f(v), w) + B(v, f(w)) = 0$$

and

$$B(f(v), w) + B(v, f(w)) = 0.$$

We have

$$B([f,g](v), w) = B(f \circ g(v), w) - B(g \circ f(v), w) = -B(g(v), f(w)) + B(f(v), g(w))$$

and

$$B(v, [f,g](w)) = B(v, f \circ g(w)) - B(v, g \circ f(w)) = -B(f(v), g(w)) + B(g(v), f(w)).$$

Adding two last equalities, we get

$$B([f,g](v), w) + B(v, [f,g](w)) = 0$$

□

Choosing different bilinear forms, one can get a lot of important Lie algebras usually called *classical Lie algebras*.

2. More examples. Ideals. Direct products.

2.1. **More examples.**

2.1.1. Let $k = \mathbb{R}$, $L = \mathbb{R}^3$. Define $[x, y] = x \times y$ — the cross-product. Recall that the latter is defined by the formulas

$$e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2.$$

2.1.2. It is convenient to choose a basis of $\mathfrak{sl}_2$ as follows.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Than the bracket in $\mathfrak{sl}_2$ is given by the formulas

$$[ef] = h,\ [he] = 2e,\ [hf] = -2f.$$

The Lie algebra $\mathfrak{gl}_n$ has many interesting subalgebras. For instance,

2.1.3. $\mathfrak{b}_n = \{a \in \mathfrak{sl}_n | a_{ij} = 0 \text{ for } i > j\}$ — upper-triangular matrices of trace zero.

This algebra has dimension $\frac{n(n+1)}{2} - 1$.

2.1.4. $\mathfrak{n}_n = \{a \in \mathfrak{gl}_n | a_{ij} = 0 \text{ for } i \geq j\}$ — strictly upper-triangular matrices.

This algebra has dimension $\frac{n(n-1)}{2}$.

2.2. **Direct product.** Let L and M be two Lie algebras. Define their direct product $L \times M$ as follows. As a set, this is the Cartesian product of L and M. The operations (multiplication by a scalar, sum and bracket) are defined componentwise. For instance,

$$[(x, y), (x', y')] = ([x, x'], [y, y']).$$

2.2.1. *Example.* If L is commutative of dimension n and L' is commutative of dimension n' then $L \times L'$ is commutative of dimension $n + n'$.

2.2.2. *Example.* The Lie algebra $\mathfrak{gl}_n$ is isomorphic to the direct product $\mathfrak{sl}_n \times k$ (k is the one-dimensional algebra). The map from the direct product to $\mathfrak{gl}_n$ is given by the formula $(a, \lambda) \mapsto a + \lambda I$ where I is the identity matrix.

2.3. **Some calculations.**

2.3.1. *Ideals in $\mathfrak{n}_3$. Quotients.* Choose a basis for $\mathfrak{n}_3$ as follows.

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Multiplication is given by

$$[x, y] = z,\ [x, z] = [y, z] = 0.$$

Let us describe all ideals in $\mathfrak{n}_3$. If I is a non-zero ideal, let $t = ax + by + cz \in I$ be non-zero. Then $[x, t] = bz,\ [y, t] = az,\ [z, t] = 0$. Thus, if $a \neq 0$ or $b \neq 0$ then $z \in I$. If $a = b = 0$ then once more $z \in I$. Therefore, z belongs to any non-zero ideal. Thus, the only one-dimensional ideal is $\langle z\rangle$; any two-dimensional ideal has form $\langle z, ax+by\rangle$. It is easy to see that all these formulas do define ideals.

The quotient-allgebra $\mathfrak{n}_3/\langle z\rangle$ has a basis $\overline{x}$, $\overline{y}$ with the bracket $[\overline{x},\overline{y}]=\overline{z}=0$. Thus, the quotient is a commutative two-dimensional algebra.

2.4. **Adjoint action.** Let L be a Lie algebra, $x\in L$. Define a linear transformation

$$\mathrm{ad}x : L\to L$$

by the formula $\mathrm{ad}_x(y)=[x,y]$.

2.4.1. **Lemma.** ad_x *is a linear transformation.*

In fact, this follows from the linearity of $[,]$ in the first argument.

2.4.2. **Lemma.** ad_x *is a derivation.*

In fact,

$$\mathrm{ad}_x[y,z]=[\mathrm{ad}_x(y),z]+[y,\mathrm{ad}_x(z)]$$

— this follows from the Jacobi identity.

Assembling together ad_x for all $x\in L$ we get therefore a map

$$\mathrm{ad}: L\to \mathrm{Der}(L).$$

2.4.3. **Lemma.** *The map* $\mathrm{ad}: L\to\mathrm{Der}(L)$ *is a homomorphism of Lie algebras.*

One has to check that

$$\mathrm{ad}_{[x,y]}=\mathrm{ad}_x\circ\mathrm{ad}_y-\mathrm{ad}_y\circ\mathrm{ad}_x.$$

This also follows from the Jacobi identity.

2.4.4. **Definition.** Center of a Lie algebra L is defined by the formula

$$Z(L)=\{x\in L|\forall y\in L\quad [x,y]=0\}.$$

By definition of ad, one has $Z(L)=\mathrm{Ker}(\mathrm{ad})$.

For example, $Z(\mathfrak{n}_3)=\langle z\rangle$.

2.5. **Simplicity of $\mathfrak{sl}_2$.**

2.5.1. **Definition.** A Lie algebra L is *simple* if it is not one-dimensional and if it has no non-trivial ideals.

Our aim is to prove the following

2.5.2. **Theorem.** $\mathfrak{sl}_2$ *is simple.*

2.5.3. *Some linear algebra.* Let V be a f.d. vector space and $f \in \mathrm{End}(V)$.

Endomorphism f is called *diagonalizable* if V has a basis of eigenvectors.

If f is diagonalizable then $V = \oplus_{\lambda \in S} V_\lambda$ where $V_\lambda = \{x \in V | f(x) = \lambda x\}$ is the eigenspace corresponding to the eigenvalue λ and S is the set of eigenvalues of f (*spectrum of* f).

2.5.4. **Lemma.** *Let $f \in \mathrm{End}(V)$ be diagonalizable and let W be a f-invariant subspace of V (i.e., $f(W) \subseteq W$). Then*

$$W = \oplus_{\lambda \in S} W_\lambda \text{ where } W_\lambda = W \cap V_\lambda.$$

Proof. We have to prove that if $x \in W$ and if $x = \sum x_\lambda$ with $x_\lambda \in V_\lambda$ then $x_\lambda \in W$.

In fact, $W \ni f^k(x) = \sum f^k(x_\lambda) = \sum \lambda^k x_\lambda$ for each k.

Let $T = \{\lambda \in S | x_\lambda \neq 0\}$ and let $t = |T|$. This is the number of non-zero summands in the decomposition of x. The vectors $x, f(x), \ldots, f^{t-1}(x)$ can be expressed as linear combinations of t linearly independent vectors x_λ. The transition matrix has form

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_t \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{t-1} & \lambda_2^{t-1} & \ldots & \lambda_t^{t-1} \end{pmatrix}.$$

This is Vandermonde matrix. Its determinant is

$$\prod_{i<j} (\lambda_i - \lambda_j) \neq 0.$$

This proves that x_λ can be expressed through $f^k(x)$ and therefore belong to W. □

2.5.5. *Proof of Theorem 2.5.2.* Consider endomorphism ad_h of $\mathfrak{sl}_2$. It is diagonalizable with eigenvalues $-2,\ 0,\ 2$. Any ideal I is invariant with respect to ad_h. Therefore, I should be spanned by a subset of f, h, g. It is easy to check that this is impossible for any nonempty proper subset of generators.

2.6. Problem assignment, 1.

1. Let A be an associative algebra. Define a new operation on A (bracket) by the formula
$$[a, b] = ab - ba.$$
Verify that A is a Lie algebra with respect to this operation.
2. Verify that the set of antisymmetric matrices forms a Lie algebra with respect to the bracket.

3. Derivations.

   (a) Let $A = k[t]$ be the algebra of polynomials. Fix $f \in A$ and define $d : A \to A$ by the formula
$$d(g) = fg'.$$
   Prove d is a derivation.

   (b) The same $A,\ f,\ d : A \to A$ is given by the formula
$$d(g) = fg' + g.$$
   is this a derivation?

   (c) Prove that any derivation of A is of form described in (a). *Hint*: consider the value of d on $1,\ t \in A$.

4. Find all ideals and all quotient algebras of the algebra
$$L = \{a \in \mathfrak{gl}_2 | a_{21} = 0\}.$$
   Prove that L is isomorphic to the direct product of k (one-dimensional algebra) and $\mathfrak{b}_2$.

5. (bonus). Let $L = \mathbb{R}^3$ with cross-product as a bracket. Prove that $L_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

   Here $L_{\mathbb{C}}$ denotes the Lie algebra over $\mathbb{C}$ having the base $e_1,\ e_2,\ e_3$ with the bracket given by the formulas
$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2.$$

2.7. **Simplicity of** $(\mathbb{R}^3, \times)$**.** The proof of the simplicity of this Lie algebra is very geometric.

Let I be a non-zero ideal in it and let $0 \neq v \in I$. We can normalize v so that $||v|| = 1$. There exists a pair of vectors v_2, v_3 so that the triple v, v_2, v_3 forms an orthonormal base. Then $v_2 = v \times v_3$ and $v_3 = v \times v_2$ up to sign, therefore, all three vectors belong to I. This proves the assertion.

3. Modules

The notion of module over a Lie algebra is of extreme importance.

3.1. **Two definitions and their equivalence.** Let L be a Lie algebra over a field k.

3.1.1. **Definition.** An L-module is a k-vector space M together with a bilinear map
$$r : L \times M \to M$$
satisfying the following property
$$r([x, y], m) = r(x, r(y, m)) - r(y, r(x, m)).$$

Usually one writes simply xm instead of $r(x,m)$. Then our axiom reads

$$[x,y]m = xym - yxm.$$

To give another definition of L-module recall that for every vector space M the collection of endomorphisms $\mathrm{End}(V)$ admits an associative composition. The operation

$$f,g \in \mathrm{End}(M) \mapsto [f,g] = fg - gf \in \mathrm{End}(M)$$

defines a Lie algebra structure on $\mathrm{End}(M)$. The Lie algebra of endomorphisms so obtained is denoted $\mathfrak{gl}(M)$.

3.1.2. **Definition.** An L-module is a vector space M endowed with a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(M).$$

The proof of the equivalence of the above definitions is fairly standard. Another name for an L-module is *representation of* L. If M is finite dimensional, we are talking about finite dimensional representations.

3.2. **Examples.**

3.2.1. $L = k$. If L is one-dimensional, say, $L = ke$, a module structure

$$\rho : L \to \mathfrak{gl}(M)$$

is given by an endomorphism of M (the image $\rho(e)$).

3.2.2. *L is commutative.* A representation of L is a Lie algebra homomorphism. If $L = \langle e_1, \ldots, e_n \rangle$, a homomorphism $r : L \to \mathfrak{gl}(M)$ is given by the images $r(e_i)$. Since r is a homomorphism, $r(e_i)$ commute. Vice versa, any collection of n commuting endomorphisms of M define on M a structure of L-module.

3.2.3. An $\mathfrak{sl}_2$-module is a vector space M with three endomorphisms E, F, H of M satisfying the conditions

$$EF - FE = H;\ HE - EH = 2E;\ HF - FH = -2F.$$

This means that an $\mathfrak{sl}_2$-module defines a *representation of* $\mathfrak{sl}_2$ *in matrices.* This is the explanation of the term *representation*.

3.2.4. *Natural representation.* By definition, Lie algebra $\mathfrak{gl}_n$ admits an n-dimensional representation. It is given by the identity map

$$\mathrm{id} : \mathfrak{gl}_n \to \mathfrak{gl}(k^n).$$

It is called *the natural representation.* Similarly, if $\mathfrak{g} \subseteq \mathfrak{gl}_n$ is a Lie subalgebra, we have a natural n-dimensional representation of $\mathfrak{g}$.

Examples include $\mathfrak{g} = \mathfrak{sl}_n,\ \mathfrak{b}_n, \mathfrak{n}_n$ and some other algebras.

3.3. **Category of L-modules.** Fix a Lie algebra L.

A linear map $f : M \to N$ is an L-module homomorphism if

$$f(ax) = af(x)$$

for each $a \in L,\ x \in M$. Clearly, composition of homomorphisms is a homomorphism.

3.3.1. **Lemma.** *Let $f : M \to N$ be a bijective homomorphism of L-modules. Then $f^{-1} : N \to M$ is also a homomorphism.*

Proof. Straighforward. □

The notion of submodule and quotient module are defined in a standard way.

3.3.2. *Direct sum.* Given two L-modules M and N, one defines an L-module structure on $M \oplus N$ by the formula

$$a(m, n) = (am, an).$$

3.3.3. *Kernel, image.* Many notions of linear algebra easily generalize to modules.

Given a homomorphism $f : V \to W$ of L-modules, its kernel and image are defined as usual:

$$\mathrm{Ker}(f) = \{v \in V | f(v) = 0\}.$$
$$\mathrm{Im}(f) = f(V) = \{w \in W | \exists v \in V, w = f(v)\}.$$

An important (and easy!) fact is that $\mathrm{Ker}(f)$ is a submodule of V and $\mathrm{Im}(f)$ is a submodule of W.

3.4. **Representation theory.** Representation theory of Lie algebras studies the category of modules over a Lie algebra. Here are the typical questions and the typical notions studied.

3.4.1. *Classification.* Description of all isomorphism classes of L-modules. Sometimes only modules satisfying special properties are considered (e.g., finite dimensional modules).

Today we will see that in the case L is one-dimensional we already know the answer from Linear Algebra.

3.4.2. *Simple modules.* A module is called simple if it does not admit non-trivial submodules. (A synonym: irreducible representation).

3.4.3. *Semisimple modules.* A module is called semisimple if it is isomorphic to a direct sum of simple modules (there are other equivalent definitions). Synonym: a completely reducible representation.

We will study soon the following result.

3.4.4. **Theorem.** *All finite dimensional representations of $\mathfrak{sl}_2$ are completely reducible.*

3.5. **Representations of a one-dimensional Lie algebra.** .

3.5.1. *Isomorphism classes.* We are looking for isomorphism classes of n-dimensional representations. A map $f : M \to N$ is a homomorphism of representations if $f\alpha_M = \alpha_N f$. Since $M = N = k^n$ as vector spaces, we deduce that endomorphisms α_1 and α_2 define isomorphic representations iff there exists an automorphism f such that $\alpha_2 = f\alpha_1 f^{-1}$.

Thus the problem of classifications of n-dimensional representations is equivalent to that of classification of square matrices up to conjugation.

Theory of Jordan normal form answers this question in the case k is algebraically closed.

Let us recall the most important steps in this theory.

3.5.2. *Recollections from Linear Algebra.* Let $f : V \to V$ be an endomorphism of a finite dimensional vector space over an algebraically closed field k. Recall that $\lambda \in k$ is an eigenvalue of f if $f - \lambda I$ is not invertible. The collection of eigenvalues of f is therefore the set of roots of the characteristic polynomial of f defined as

$$P_f(t) = \det(f - tI).$$

In what follows $S(f)$ will denote the set of eigenvalues of f.

Let $\lambda \in S(f)$. A vector $v \in V$ is called an eigenvector corresponding to λ if $f(v) = \lambda v$. Each eigenvalue admits a non-zero eigenvector. Furthermore, $v \in V$ is called generalized eigenvector if there exists $n \in \mathbb{N}$ such that $(f - \lambda I)^n v = 0$.

Fix $\lambda \in S(f)$. Let V^λ denote the set of eigenvectors and V_λ the set of generalized eigenvectors corresponding to λ. These are vector subspaces of V and

$$V_\lambda \supseteq V^\lambda \neq 0.$$

The following are the main results of this study.

- $V = \bigoplus_{\lambda \in S(f)} V_\lambda$.
- $\dim V_\lambda$ equals the multiplicity of λ in the characteristic polynomial of f.
- Each V_λ is isomorphic to a direct sum of Jordan blocks having eigenvalue λ (definition of J. b. see below).

Jordan block having eigenvalue λ is the matrix $A = (a_{ij})$ defined by the formulas

$$a_{ij} = \begin{cases} \lambda & \text{if} \quad i = j, \\ 1 & \text{if} \quad i = j - 1, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

3.5.3. *Simple modules.* Direct consequence of the above: Any simple module has dimension 1; it is defined up to isomorphism by its (only) eigenvalue λ.

This module will be sometimes denoted by k_λ.

3.5.4. *Semisimple modules.* Semisimple module is a direct sum of simple modules. Thus (for k algebraically closed) (V, f) is semisimple iff f is diagonalizable.

3.5.5. *Example of non-semisimple modules.* It is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3.6. **Examples of representations.**

3.6.1. *Adjoint representation.* Let L be any Lie algebra. the map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ defines a repersentation of L called adjoint representation.

Note that submodules of the adjoint representation are the ideals. Therefore, L is simple iff the adjoint representation is irreducible.

3.6.2. *... and its restrictions.* If $M \subseteq L$ is a Lie subalgebra, one can consider L as a M-module restricting the adjoint representation of L on M.

Consider, for example, $L = \mathfrak{sl}_2$ and $M = \langle h \rangle$. Algebra M is one-dimensional and L is an M-module. It is semisimple with eigenvalues $-2,\ 0,\ 2$.

If we take another M, say, $\langle e \rangle$, the picture will change. All eigenvalues are zero and the M-module L is not semisimple.

3.7. **One-dimensional representations.** Let L be a Lie algebra and let $\rho : L \to \mathfrak{gl}(V)$ is a one-dimensional representation. The algebra $\mathfrak{gl}(V)$ is one-dimensional and therefore commutative in this case. Thus,

$$\rho[x, y] = [\rho(x), \rho(y)] = 0.$$

In particular, for $L = \mathfrak{sl}_2$ one gets $\rho = 0$. This proves $\mathfrak{sl}_2$ does not admit non-trivial one-dimensional representations.

3.8. **Problem assignment, 2.**

0. Let L be a Lie algebra and V a vector space. Let $f : L \times V \to V$ be a bilinear map. Define an antisymmetric bilinear operation on $L \oplus V$ by the following properties.
   1. Its restriction to L is the bracket.
   2. Its restriction to V is zero.
   3. Its value on a pair (x, v) with $x \in L,\ v \in V$, is $f(x, v)$. Prove that $L \oplus V$ is a Lie algebra with respect to the operation defined above if and only if f describes on V a structure of L-module.
1. Let $\mathfrak{h} \subseteq \mathfrak{gl}_n$ be the set of diagonal matrices. Check that $\mathfrak{h}$ is a commutative Lie subalgebra. Check that $\mathfrak{gl}_n$ as the $\mathfrak{h}$-module (with respect to adjoint action) is a sum of one-dimensional representations.
2. Prove that the adjoint representation of $\mathfrak{gl}_2$ is isomorphic to a direct sum of a three-dimensional and one-dimensional representations.
3. Prove that the adjoint representation of $\mathfrak{sl}_2$ is not isomorphic to the sum of the natural representation with the trivial representation.

4. Schur Lemma. Representations of $\mathfrak{sl}_2$.

4.1. **Schur's lemma.** Let M, N be two L-modules. The collection of homomorphism of modules is denoted $\mathrm{Hom}_L(M, N)$. It forms a vector space over k.

Thus,

$$\begin{aligned}\mathrm{Hom}_L(M,N) : &= \{\phi \in \mathrm{Hom}_k(M,N) |\ \forall v \in M, \forall x \in L\ \phi(xv) = x\phi(v)\}\\ &= \{\phi \in \mathrm{Hom}_k(M,N) |\ \forall x \in L\ \phi\rho(x) = \rho(x)\phi\}.\end{aligned}$$

4.1.1. **Theorem.** *Suppose the base field k is algebraically closed. If V is a simple finite dimensional module over a Lie algebra L then* $\mathrm{Hom}_L(V, V) = k \cdot \mathrm{id}$.

Proof. Take $\phi \in \mathrm{Hom}_L(V, V)$. For any $c \in k$ the linear operator $(\phi - c \cdot \mathrm{id})$ is a L-homomorphism and so it is either an isomorphism or zero. Let c be an eigenvalue of ϕ; then the operator $(\phi - c \cdot \mathrm{id})$ has a non-zero kernel and so it is not an isomorphism. Hence $\phi - c \cdot \mathrm{id} = 0$ as required. □

4.2. **Application to $\mathfrak{sl}_2(\mathbb{C})$.** Take $k := \mathbb{C}$. Fix the standard basis h, e, f of $\mathfrak{sl}_2$. Recall that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Let $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation. Denote

$$E = \rho(e),\ F = \rho(f), H = \rho(h).$$

Then the above relations imply

$$\begin{aligned} HE - EH &= 2E,\\ HF - FH &= -2F,\\ EF - FE &= H. \end{aligned} \tag{5}$$

Consider the endomorphism

$$Q := H^2 + 2FE + 2EF.$$

This is a linear endomorphism of V. We will check now that Q is an $\mathfrak{sl}_2$-endomorphism. To check this, it is enough to prove

$$QE = EQ,\ QF = FQ,\ QH = HQ.$$

The following easy lemma is useful in calculations.

4.2.1. **Lemma.** *Let $f, g, h \in \mathrm{End}(V)$. Then*

$$[f, gh] = [f, g]h + g[f, h].$$

Here, as usual, the bracket is defined by the formula $[f, g] = fg - gf$.

Proof.

$$[f, g]h + g[f, h] = fgh - gfh + gfh - ghf = fgh - ghf = [f, gh].$$

□

Now one can easily get

4.2.2. **Lemma.** *The operator Q commutes with E, F, H.*

Proof. Recall that all calculations are done in $\mathrm{End}(V)$.

One has

$$[E, Q] = [E, H^2 + 2EF + 2FE] = [E, H]H + H[E, H]+ \\ 2E[E, F] + 2[E, F]E = -2EH - 2HE + 2EH + 2HE = 0.$$

Similarly,

$$[F, Q] = [F, H^2 + 2EF + 2FE] = [F, H]H + H[F, H]+ \\ 2[F, E]F + 2F[F, E] = 2FH + 2HF - 2HF - 2FH = 0$$

and

$$[H, Q] = [H, H^2 + 2EF + 2FE] = 2[H, E]F + 2E[H, F]+ \\ 2[H, F]E + 2F[H, E] = 4EF - 4EF - 4FE + 4FE = 0.$$

□

4.2.3. **Corollary.** *Let V be finite dimensional and simple $\mathfrak{sl}_2$-module. Then $Q = c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$.*

4.3. **Finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.** Our next goal is to describe all finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

As a first step, we will describe a collection of irreducible representations which will turn out to be the collection of all irreducible representations.

We denote by $\mathbb{N}$ the set of non-negative integers.

4.3.1. Recall that $\mathfrak{sl}_2 \subseteq \mathfrak{gl}_2 = \langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$ where E_{ij} denotes the matrix whose only non-zero entry is 1 in the (ij) position.

Note that in this notation $E = E_{12},\ F = E_{21},\ H = E_{11} - E_{22}$.

Consider the polynomial algebra $\mathbb{C}[x, y]$ and define the action of $\mathfrak{gl}_2$ on it by the formulas

$$E_{11}(p) = xp'_x,\quad E_{22}(p) = yp'_y,\quad E_{12}(p) = xp'_y,\quad E_{21}(p) = yp'_x. \tag{6}$$

4.3.2. **Lemma.** *The formulas (6) define a* $\mathfrak{gl}_2$*-module structure on* $\mathbb{C}[x, y]$.

Proof. One can check this claim directly.

Here is another way which allows to avoid most of the calculations. Note that the formulas (6) assign to E_{ij} *derivations* of $\mathbb{C}[x, y]$ (compare to Problem assignment,1, # 1).

Any derivation of $\mathbb{C}[x, y]$ is uniquely defined by its value on the degree one polynomials x and y: if $d(x) = p,\ d(y) = q$ then $d(f) = pf'_x + qf'_y$ (once more, compare to Problem assignment, 1).

Then, in order to prove the formulas (6) are compatible with the brackets it is enough to check them on x and on y. One can see that the formulas (6) restricted on $\langle x, y\rangle$ give just the natural representation of $\mathfrak{gl}_2$. $\square$

The set of homogeneous polynomials of a degree n is, obviously, a $\mathfrak{gl}_2$-submodule and an $\mathfrak{sl}_2$-submodule. Denote this $\mathfrak{sl}_2$-submodule by $V(n)$. Let us show that $V(n)$ is a simple $\mathfrak{sl}_2$-module.

4.3.3. *Module V(n).* Fix n. Consider the following basis of $V(n)$:

$$v_0 := x^n, v_1 := nx^{n-1}y, v_2 := n(n-1)x^{n-2}y^2, \dots,$$
$$v_k := n!/(n-k)!x^{n-k}y^k, \dots, v_n := n!y^n.$$

One has

$$Fv_k = E_{2,1}v_k = v_{k+1},\quad Ev_k = E_{1,2}v_k = k(n+1-k)v_{k-1}, \tag{7}$$
$$Hv_k = (E_{1,1} - E_{2,2})v_k = (n-2k)v_k.$$

We see that H acts diagonally on the basis and all eigenvalues are distinct. By a lemma proven in Lecture 2, any submodule $W \subseteq V(n)$ is spanned by the elements of our basis belonging to W. In particular, any non-zero submodule contains v_k for some k; the relations (7) imply that such a submodule contains all basis elements. Hence $V(n)$ is simple.

4.4. **We have got all of them...** Now we will prove there are no finite dimensional irreducible representations of $\mathfrak{sl}_2$ except for the $V(n)$ described above.

4.4.1. *Definitions.* A vector v of $\mathfrak{sl}_2$-module is called *a weight vector* if $Hv \in \mathbb{C}v$.

A vector v of $\mathfrak{sl}_2$-module is called *of weight* c ($c \in \mathbb{C}$) if $Hv = cv$.

A vector v of $\mathfrak{sl}_2$-module is called *primitive* if $Ev = 0$.

4.4.2. The set of vectors of weight λ in V is denoted V^λ.

Let $v \in V^\lambda$. We claim that $Ev \in V^{\lambda+2}$ and $Fv \in V^{\lambda-2}$. In fact,

$$HEv = EHv + [H,E]v = \lambda Ev + 2Ev = (\lambda+2)Ev$$

and similarly for Fv.

4.4.3. Let V be a finite dimensional $\mathfrak{sl}_2$-module. We claim that V has a primitive weight vector.

In fact, $H : V \to V$ is an endomorphism of a finite dimensional vector space. Therefore, H admits an eigenvector $v \in V$. Let $v \in V^\lambda$. Then $E^k v \in V^{\lambda+2k}$. Since V is finite dimensional, this proves that $E^k v = 0$ for k big enough. Thus, if $n = \max\{k | E^k v \neq 0\}$, the element $E^n v$ is a primitive weight vector.

4.4.4. Let V be a finite dimensional $\mathfrak{sl}_2$-module and let v_0 be a primitive vector of weight λ.

Put $v_n = F^n v_0$. One has $Fv_n = v_{n+1}$ and $Hv_n = (\lambda - 2n)v_n$. It turns out there is an very nice formula for Ev_n.

In Lemma 4.4.5 below we will prove the following identity.

$$EF^k = F^k E + kF^{k-1}(H - (k-1)). \tag{8}$$

The formula (8) implies that

$$Ev_n = EF^n v_0 = F^n E v_0 + nF^{n-1}(H - n + 1)v_0 = n(\lambda - n + 1)v_{n-1}.$$

Let us rewrite once more these formulas

$$Fv_n = v_{n+1},\ Hv_n = (\lambda - 2n)v_n,\ Ev_n = n(\lambda - n + 1)v_{n-1}. \tag{9}$$

4.4.5. **Lemma.** *The identity (8) is valid for any $n \geq 1$ for any representation of $\mathfrak{sl}_2$.*

Proof. Induction on k. For $k = 1$ it says that $EF = FE + H$ which is obvious. Suppose it has already been proven for $k = n$ and let $k = n + 1$. We have

$$EF^{n+1} = EF^n F = (F^n E + nF^{n-1}(H-n+1))F = F^n EF + nF^{n-1}(H-n+1)F =$$

$$F^n(FE+H) + nF^{n-1}F(H-n+1) - nF^{n-1}(2F) = F^{n+1}E + (n+1)F^n(H-n).$$

□

4.4.6. We have made a substantial progress. In fact, we already know that any finite dimensional $\mathfrak{sl}_2$ module V contains a primitive weight vector v_0; The collection of $v_n = F^n v_0$ is a submodule. This implies that only finite number of v_i is nonzero.

This has very unexpected consequences. In fact, suppose $n = \max\{i | v_i \neq 0\}$. Then

$$0 = Ev_{n+1} = (n+1)(\lambda - n)v_n$$

and this implies that $\lambda = n$.

We have (easily!) proven the following

4.4.7. **Theorem.** *Let V be a finite dimensional representation and let v_0 be a weight primitive vector of weight λ. Then $\lambda \in \mathbb{N}$. The submodule of V generated by v_0 is $\langle v_0, v_1, \ldots, v_\lambda \rangle$. It has dimension $\lambda + 1$ and its module structure is given by the formulas (9).*

4.4.8. We have proven that any simple finite dimensional $\mathfrak{sl}_2$-module V is isomorphic to $V(n)$ (where $n := \dim V - 1$). In fact, we have already proven that, if a representation V has a primitive vector v_0 of weight n then v_0 generates a submodule spanned by $v_k, k = 0, \ldots, n$, with the action of $\mathfrak{sl}_2$ defined by the formulas

$$Fv_k = v_{k+1},\ Hv_k = (n - 2k)v_k, Ev_k = k(n - k + 1)v_{k-1}.$$

Since the irreducible representation $V(n)$ we constructed above has the primitive weight n vector x^n, it is of the form described above.

4.4.9. **Definition.** A Verma module $M(\lambda)$ of the highest weight λ is an $\mathfrak{sl}_2$-module with a linear basis $v_0, v_1, \ldots$ and the action given by

$$Fv_k = v_{k+1},\ Ev_0 = 0,\ Hv_k = (\lambda - 2k)v_k,\ Ev_{k+1} = (k + 1)(\lambda - k)v_k.$$

An easy exersice is to show that the action is compatible with the relations on E, F, H.

Note that the action of H on a Verma module is semisimple and the action of E is locally nilpotent in the sense of the following definition.

4.4.10. **Definition.** An operator $f : V \to V$ is locally nilpotent if for any $v \in V$ there exists n such that $f^n(v) = 0$.

Of course, a locally nilpotent operator on a finite dimensional vector space is nilpotent. However, E is not nilpotent on the Verma module $M(\lambda)$.

4.4.11. Let N be a $\mathfrak{sl}_2$-module and $v \in N$ be a primitive vector of weight c. Define a linear map $\phi : M(c) \to N$ by setting $\phi(v_k) = F^k v$. By the formulas (8) ϕ is a homomorphism of $\mathfrak{sl}_2$-modules. Since the simple module $V(n)$ is generated by its primitive vector v_0 of weight n, we see that a simple finite dimensional module is a homomorphic image of a certain Verma module.

4.4.12. **Lemma.** *A Verma module $M(\lambda)$ is simple if $\lambda \notin \mathbb{N}$.*

For any $k \in \mathbb{N}$, a Verma module $M(k)$ has a unique non-trivial submodule M' which is isomorphic to $M(-k-2)$. The quotient module M/M' is a simple module of the dimension $k + 1$.

Proof. Let $v_0, v_1, \ldots$ be the basis of $M(\lambda)$ described in Definition 4.4.9. The element H acts diagonally on the basis and all eigenvalues are distinct. By a proposition proved in Lecture 2, any submodule of $M(\lambda)$ is spanned by the elements of our basis belonging to this submodule. If a submodule N contains v_k then it contains $v_{k+1} = Fv_k$ and, if $k \neq \lambda + 1$ it contains also v_{k-1} which is proportional to Ev_k. This implies that $M(\lambda)$ is simple if $\lambda \notin \mathbb{N}$ and that a unique non-trivial submodule M' of $M(\lambda)$ has a basis $v_{\lambda+1}, v_{\lambda+2}, \ldots$ if $\lambda \in \mathbb{N}$.

Fix $\lambda \in \mathbb{N}$. Let $v_0, v_1, \ldots$ be the basis of $M(\lambda)$ described in Definition 4.4.9 and $v'_0, v'_1, \ldots$ be the similar basis for $M(-\lambda - 2)$. It is easy to check that the linear map sending v'_i to $v_{\lambda+1+i}$ is an injective homomorphism. Hence the unique non-trivial submodule M' of $M(\lambda)$ is isomorphic to $M(-\lambda - 2)$. The quotient module M/M' has a basis consisting of the images of vectors $v_0, v_1, \ldots, v_\lambda$. Thus $\dim M/M' = \lambda + 1$. □

4.4.13. In the sequel we need another fact. Let N be a finite dimensional $\mathfrak{sl}_2$-module and $v \in N$ be a primitive non-zero vector of weight c. A homomorphism $\phi : M(c) \to N$ defined by $\phi(v_k) = F^k v$ has non-zero kernel since $\dim N < \infty$. Hence $c \in \mathbb{N}$ and $\ker \phi = M(-c-2)$. Thus ϕ induces an injective homomorphism $V(c) \to N$.

4.5. **Complete reducibility of finite dimensional $\mathfrak{sl}_2$-modules.** In this subsection we prove the following

4.5.1. **Theorem.** *Any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is completely reducible.*

The proof will take a while.

4.5.2. *The Casimir operator.* Let V be a finite dimensional representation. The operator $Q = H^2 + 2EF + 2FE$ is a $\mathfrak{sl}_2$-module endomorphism of V. It is called *Casimir operator*. Recall that $Q : V \to V$ gives rise to a decomposition

$$V = \bigoplus V_\theta \tag{10}$$

where V_θ is the generalized eigenspace corresponding to the eigenvalue θ of Q.

By definition $V_\theta = \{v \in V | \exists n : (Q - \theta \cdot \mathrm{id})^n v = 0\}$. Thus $V_\theta = \cup_n \mathrm{Ker}(Q - \theta \cdot \mathrm{id})^n$. Since Q is $\mathfrak{sl}_2$-invariant, V_θ is an $\mathfrak{sl}_2$-submodule of V. Thus, (10) is a direct decomposition of modules.

Thus, we reduced the claim of our theorem to the case Q has only one eigenvalue θ.

4.5.3. Let now V be any finite dimensional representation and let $P(V) = \{v \in V | Ev = 0\}$ denote the set of primitive elements of V. We claim that $P(V)$ is an H-invariant vector subspace of V.

In fact, if $Ev = 0$ then $EHv = HEv - 2Ev = 0$.

4.5.4. **Proposition.** *The operator H is semisimple on $P(V)$*

Proof. First of all, recall that $(H+2)F = FH$, so $(H-\lambda+2)F = F(H-\lambda)$. Therefore,

$$(H-\lambda)^k v = 0 \Longrightarrow (H-\lambda+2)^k Fv = 0.$$

This implies that the operator F is nilpotent on any finite dimensional $\mathfrak{sl}_2$-module.

Let $v \in P(V)$ and let $k > 0$ be a natural number. Let us check that

$$E^k F^k v = k! H(H-1)\cdots(H-(k-1))v. \tag{11}$$

In fact, the claim is obvious for $k=1$ since v is primitive. Suppose we have already checked it for a given $k \in \mathbb{N}$. Then

$$E^{k+1}F^{k+1}v = E^k(EF^{k+1})v = E^kF^{k+1}Ev + (k+1)E^kF^k(H-k)v = \\ (k+1)!H(H-1)\cdots(H-k)v.$$

Now, since V is finite dimensional, there exists $k \in \mathbb{N}$ big enough, so that $F^k v = 0$ for each $v \in V$.

This implies that the restriction H_P of the operator H on $P(V)$ satisfies the identity

$$H_P(H_P-1)\cdots(H_P-k+1) = 0.$$

This means semisimplicity.

□

4.5.5. Suppose now that $V = V_\theta$. If $v \in P(V)$ is a weight primitive vector of weight n, V admits a simple submodule isomorphic to $V(n)$. By Schur lemma Q acts on $V(n)$ as a multiplication by a number. This number is obviously equal to θ.

On the other hand, one has

4.5.6. **Lemma.** *The Casimir operator Q acts on $V(n)$ as multiplication by $n(n+2)$.*

Proof. The highest weight vector v_0 of $V(n)$ has weight n. Thus, $Qv_0 = H^2(v_0) + 2FE + 2EF = (n^2+2n)v_0$. The rest follows from Schur Lemma. □

As an immediate corollary we deduce that all weight vectors of $P(V)$ have the same weight: this is the natural number n such that $\theta = n^2 + 2n$. This implies that all primitive vectors of V are weight vectors.

4.5.7. *Proof of Theorem 4.5.1.* Choose a basis $\{v^1, \ldots, v^k\}$ of $P(V)$. Each vector v^i is a primitive weight vector. It, therefore, defines a simple submodule $V^i = \langle v^i, Fv^i, \ldots F^n v^i\rangle$. We claim that $V = \oplus V^i$. Here is the proof.

Consider the natural map $f : V(n)^k \to V$ sending the i-th component of $V(n)$ to V^i. The map f is an isomorphism in the weight n part by the choice of v^i. If $\mathrm{Ker}(f) \neq 0$, this is a non-trivial submodule on which Q acts by multiplication by $\theta = n^2 + 2n$. Therefore, its primitive vectors have weight n which contradicts bijectivity of f in weight n.

Similarly, if f is not surjective, the quotient $V/\operatorname{Im}(f)$ would have primitive vector $\bar{v}$ of weight n. Let us show this is impossible. Choose a representative $v \in V$ for $\bar{v}$. Let $v = \sum v_\lambda$ be the (unique) decomposition into sum of generalized eigenvectors for H with eigenvalues λ. The surjective homomorphism $V \to V/\operatorname{Im}(f)$ preserves the generalized weight decomposition, so we can assume from the very beginning that v is annihilated by a power of $H-n$. Then $E(v)$ is annihilated by a power of $H-n-2$ which implies that $Ev = 0$. Therefore, v is primitive and does not belong to $\operatorname{Im}(f)$. Contradiction.

Theorem is proven.

4.5.8. **Corollary.** *A finite dimensional representation V of $\mathfrak{sl}_2$ decomposes as*

$$V = \bigoplus_n V(n)^{d_n}$$

where $d_n = \dim P(V)_n$ is the dimension of the space of primitive vectors of V of weight n.

4.6. **Semisimplicity: generalities.** All modules in this subsection are supposed to be modules over a fixed Lie algebra (or over a fixed associative algebra).

Recall that a module is called simple if it is nonzero and it has no nontrivial submodules.

4.6.1. **Theorem.** *The following conditions on a module V are equivalent.*

- CR1 *V is a sum of its simple submodules.*
- CR2 *V is a sum of a certain family of its simple submodules.*
- CR3 *Any submodule V' of V is a direct summand, that is, there exists a submodule V'' such that $V = V' \oplus V''$.*

A module V saisfying the above properties, is called semisimple or completely reducible.

We will first prove two lemmas.

4.6.2. **Lemma.** *Let V satisfy the property (CR3). Then any submodule of V satisfies the same property.*

Proof. Let $V \supset W \supset W'$. We will prove that W' is a direct summand of W. By the assumption, $V = W' \oplus V''$ for sume V''. Let us define $W'' = W \cap V''$. We will prove that $W = W' \oplus W''$. In fact, $W' \cap W'' \subset W' \cap V'' = 0$. Let us prove that $W = W' + W''$. Let $w \in W$. we can present $w = w' + v''$ where $w' \in W'$ and $v'' \in V''$. Now $v' = w - w' \in W$ so $v'' \in V'' \cap W = W''$. □

4.6.3. **Lemma.** *Let $V \neq 0$ satisfy (CR3). Then V has a simple submodule.*

Proof. Choose $v \in V$, $v \neq 0$. We claim there is a maximal submodule W of V that does not contain v. This directly follows from Zorn lemma. Bt (CR3), one has $V = W \oplus W'$. Let us prove W' is simple. If it has a submodule W_1, one has $W' = W_1 \oplus W_2$. Then $V = W \oplus W_1 \oplus W_2$ and so $v = w + w_1 + w_2$ for some $w \in W$, $w_1 \in W_1$, $w_2 \in W_2$. At least one of w_1, w_2 is nonzero. If $w_2 \neq 0$, $v \in W \oplus W_1$, contradiction. □

Proof of the theorem. The implication (CR2)⇒(CR1) is clear. Let us prove the opposite direction. Let $V = \sum_{i \in I} V_i$, a sum of simple submodules. Define J as a maximal subset of I such that the sum $\sum_{i \in J} V_i$ is direct. Such J exists by Zorn lemma. If $\sum_{i \in J} V_i$ is not the whole V, there exists $j \in I$ such that V_j is not in $\sum_{i \in J} V_i$. The intersection iz a submodule of V_j so it is zero. Contradiction.

(CR1)⇒(CR3). Similar: if $V \supset V'$, let J be maximal among subsets of I such that the sum $\sum_{i \in J} V_j + V'$ is direct. AS above, we deduce that it is the whole V.

(CR3)⇒(CR1). Let V' be the sum of all simple submodules of V. It is nonempty and one has $V = V' \oplus V''$ for some V''. Then V'' has also a simple submodule. Contradiction.

□

Problem assignment, 3

1. Let M, N be two non-isomorphic irreducible representations of a Lie algebra L. Prove that $\mathrm{Hom}_L(M,N)=0$.
2. Let L be a Lie algebra over an algebraically closed field k. Let $M_1,\dots M_n$ be non-isomorphic irreducible L-modules and let
$$M=\bigoplus_{i=1}^{n}M_i^{d_i}.$$
Calculate $\dim \mathrm{Hom}_L(M,M)$.
3. Let $V(n)$ be the standard $n+1$-dimensional representation of $\mathfrak{sl}_2$. Write down the matrices of the action of $E,\ F,H$ in the standard basis $v_0,\dots,v_n$ of $V(n)$.
4. Let V be a finite dimensional $\mathfrak{sl}_2$-module and let V_k be the subspace of weight k vectors. Let $P(V)_k$ be the space of primitive weight k vectors. Prove that for $k\geq 0$
$$\dim P(V)_k=\dim V_k-\dim V_{k+2}.$$
5. Let V be the natural n-dimensional representation of the Lie algebra $\mathfrak{sl}_n$. Consider the map of Lie algebras
$$f:\mathfrak{sl}_2\to\mathfrak{sl}_n$$
sending each matrix $M\in\mathfrak{sl}_2$ to the matrix $f(M)$ defined by the formula
$$f(M)_{ij}=\begin{cases} M_{ij} & \text{if} \qquad i,j\in\{1,2\}\\ 0 & \text{otherwise}\end{cases}$$
This defines an action ρ of $\mathfrak{sl}_2$ on V: $\rho(x)(v)=f(x)v$. For which n the resulting representation is irreducible? Write down the matrices of the operators $E,\ F,H$ acting on V.
6. Define $i:\mathfrak{sl}_2\to\mathfrak{gl}_n$ by the formulas
$$\begin{array}{rcl} i(e) &=& \sum_{i=1}^{n-1}E_{i,i+1}\\ i(h) &=& \sum_{i=1}^{n}a_iE_{i,i}\\ i(f) &=& \sum_{i=1}^{n-1}b_iE_{i+1,i}\end{array}$$
Find a_i and b_i so that i is a Lie algebra homomorphism. Consider $\mathfrak{gl}_n$ as a $\mathfrak{sl}_2$-module with respect to the restriction of the adjoint action along i. Using Problem 2, find the multiplicity of each irreducible module $E(k)$ in $\mathfrak{gl}_n$.

   Note. Whoever prefers working in a more concrete setting is allowed to put $n=3$.
7. We say that a module M satisfies condition (CR4) if for any surjective homomorphism $f:M\to N$ there exists $g:N\to M$ such that $fg=\mathrm{id}_N$.

   Prove that condition (CR4) is equivalent to (CR3).

5. Killing form. Nilpotent Lie algebras

5.1. Killing form.

5.1.1. Let L be a Lie algebra over a field k and let $\rho : L \to \mathfrak{gl}(V)$ be a finite dimensional L-module. Define a map

$$B_\rho : L \times L \to k$$

by the formula

$$B_\rho(x, y) = \operatorname{tr}(\rho(x) \circ \rho(y)),$$

where tr denotes the trace of endomorphism.

5.1.2. **Lemma.** *The map B_ρ is bilinear and symmetric.*

Proof. Bilinearity is obvious. Symmetricity follows from the well-known property of trace we have already used:

$$\operatorname{tr}(fg) = \operatorname{tr}(gf).$$

□

The map B_ρ satisfies another property called *invariance*.

5.1.3. **Definition.** Let V, W, K be three L-modules. A map

$$f : V \times W \to K$$

is called L-invariant if for all $x \in L,\ v \in V,\ w \in W$ one has

$$xf(v, w) = f(xv, w) + f(v, xw).$$

5.1.4. **Lemma.** *Let $\rho : L \to \mathfrak{gl}(V)$ be a finite dimensional representation. The bilinear form $B_\rho : V \times V \to k$ is invariant (k is the trivial representation). This means that $B_\rho([x, y], z) + B_\rho(y, [x, z]) = 0$.*

Proof. Let $X = \rho(x)$, and similarly Y and Z. One has

$$B_\rho([x, y], z) = \operatorname{tr}([X, Y]Z) = \operatorname{tr}(XYZ - YXZ) = \operatorname{tr}(XYZ) - \operatorname{tr}(YXZ).$$

Similarly,

$$B_\rho(y, [x, z]) = \operatorname{tr}(Y[X, Z]) = \operatorname{tr}(YXZ) - \operatorname{tr}(YZX).$$

Finally, $\operatorname{tr}(X(YZ)) = \operatorname{tr}((YZ)X)$ and the lemma is proven. □

5.1.5. **Definition.** Killing form on a Lie algebra L is the bilinear form

$$B : L \times L \to k$$

defined by the adjoint representation.

5.1.6. **Example.** Let L be commutative. Then the Killing form on L is zero.

5.1.7. **Example.** Let $L = \mathfrak{sl}_2$. Then the Killing form is non-degenerate, i.e. for any nonzero $x \in L$ there exists $y \in L$ such that $B(x, y) \neq 0$. (see Problem assignment, 1).

5.1.8. Let $B : V \times V \to k$ be a symmetric bilinear form. The kernel of B is defined by the formula

$$\mathrm{Ker}(B) = \{x \in V | \forall y \in V \quad B(x, y) = 0\}.$$

The form is *non-degenerate* if its kernel is zero. In this case the linear transformation

$$B' : V \to V^*$$

from V to the dual vector space V^* given by the formula

$$B'(x)(y) = B(x, y),$$

is injective. If $\dim V < \infty$ this implies that B' is an isomorphism.

5.1.9. **Proposition.** *Let $\rho : L \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then the kernel of B_ρ is an ideal in L.*

Proof. Suppose $x \in \mathrm{Ker} B_\rho$. This means that $\mathrm{tr}(\rho(x)\rho(y)) = 0$ for all y. Then for all $z \in L$

$$B_\rho([z, x], y) = -B_\rho(x, [z, y]) = 0$$

by the invariantness of B_ρ. □

Today we will study a large class of algebras having vanishing Killing form.

5.2. **Nilpotent Lie algebras.** Let $V, W \subseteq L$ be two vector subspaces of a Lie algebra L. We define $[V, W]$ as the vector subspace of L spanned by the brackets $[v, w],\ v \in V,\ w \in W$.

Jacobi identity implies that if V, W are ideals in L then $[V, W]$ is also an ideal in L. Define a sequence of ideals $C^k(L)$ by the formulas

$$C^1(L) = L; \quad C^{n+1}(L) = [L, C^n(L)].$$

5.2.1. **Lemma.** *One has $[C^r(L), C^s(L)] \subseteq C^{r+s}(L)$.*

Proof. Induction in r. □

5.2.2. **Example.** Recall that

$$\mathfrak{n}_n = \{A = (a_{ij}) \in \mathfrak{gl}_n | a_{ij} = 0 \text{ for } j < i + 1\}.$$

If $L = \mathfrak{n}_n$ then

$$C^k(L) = \{A = (a_{ij}) \in \mathfrak{gl}_n | a_{ij} = 0 \text{ for } j < i + k\}.$$

Check this!

5.2.3. **Definition.** A Lie algebra L is called nilpotent if $C^n(L) = 0$ for $n \in \mathbb{N}$ big enough.

Thus, commutative Lie algebras as well as the algebras $\mathfrak{n}_n$ are nilpotent.

5.3. **Engel theorem.**

5.3.1. **Lemma.** *Let $L, R \in \operatorname{End}(V)$ be two commuting nilpotent operators. Then $L+R$ is also nilpotent.*

Proof. Since L and R commute, one has a usual Newton binomial formula for $(L+R)^n$. This implies that if $L^n = R^n = 0$ then $(L+R)^{2n} = 0$. □

5.3.2. **Theorem.** *Let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra, $\dim(V) < \infty$. Suppose that all $x \in L$ considered as the operators on V, are nilpotent. Then there exists a non-zero vector $v \in V$ such that $xv = 0$ for all $x \in L$.*

Proof.
Step 1. Let us check that for each $x \in L$ the endomorphism ad_x of L is nilpotent. In fact, let $L_x : \operatorname{End}(V) \to \operatorname{End}(V)$ be the left multiplication by x and R_x be the right multiplication. Then $\operatorname{ad}_x = L_x - R_x$. The operators L_x and R_x commute. Both of them are nilpotent since x is a nilpotent endomorphism of V. Therefore, ad_x is nilpotent by 5.3.1.

Step 2. By induction in $\dim L$ we assume the theorem has been already proven for Lie algebras K of dimension $\dim K < \dim L$.

Step 3. Consider a maximal Lie subalgebra K of L strictly contained in L. We will check now that K is a codimension one ideal of L.

Let us prove first K is an ideal in L. Let

$$I = \{x \in L | \forall y \in K \quad [x, y] \in K\}.$$

This is a Lie subalgebra of L (Jacobi identity). Obviously $I \supseteq K$. We claim $I \neq K$. By maximality of K this will imply that $I = L$ which precisely means that K is an ideal in L.

In fact, consider the (restriction of the) adjoint action of K on L. K is a K-submodule of L. Consider the action of K on L/K. By the induction hypothesis (here we are using ad_x is nilpotent!), there exists a non-zero element $\overline{x} \in L/K$ invariant with respect to K. This means $[a, \overline{x}] = 0$ for all $a \in K$ or, in other words, choosing a representative $x \in L$ of $\overline{x}$, we get $[a, x] \in K$ for all $a \in K$. Thus, $x \in I \setminus K$ and we are done.

Now we know that K is an ideal. Let us check $\dim\ L/K = 1$. In fact, If $x \in L \setminus K$, the vector space $K + kx \subseteq L$ is a subalgebra of L. Since K was chosen to be maximal, $K + kx = L$.

Step 4. Put

$$W = \{v \in V | Kv = 0\}.$$

Check that W is an L-submodule of V. If $x \in K,\ y \in L$ and $w \in W$, one has

$$x(yw) = y(xw) + [x, y]w = 0.$$

By the induction hypothesis, $W \neq 0$. Choose $x \in L \setminus K$. This is a nilpotent endomorphism of W. Therefore, there exists $0 \neq w \in W$ such that $xw = 0$. Clearly, w annihilates the whole of L.

Theorem is proven. □

5.3.3. **Corollary.** *Let $\rho : L \to \mathfrak{gl}(V)$ be a representation. Suppose that for each $x \in L$ the operator $\rho(x)$ is nilpotent. Then one can choose a basis $v_1, \dots, v_n$ of V so that*

$$\rho(L) \subseteq \mathfrak{n}_n \subseteq \mathfrak{gl}_n = \mathfrak{gl}(V).$$

Note that the choice of a basis allows one to identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n$.

Proof. Induction on $n = \dim V$.

One can substitute L by $\rho(L) \subseteq \mathfrak{gl}(V)$. By Theorem 5.3.2 there exists a nonzero element $v_1 \in V$ satisfying

$$xv_1 = 0 \text{ for all } x \in L.$$

Now consider L-module $W = V/\langle v_1 \rangle$ and apply the inductive hypothesis. □

5.3.4. **Corollary.** *(Engel theorem) A Lie algebra L is nilpotent iff the endomorphism* ad_x *is nilpotent for all $x \in L$.*

Proof. Suppose L is nilpotent, $x \in L$. By definition $ad_x(C^k(L)) \subseteq C^{k+1}(L)$. This implies that ad_x is nilpotent.

In the other direction, suppose ad_x is nilpotent for all $x \in L$.

By 5.3.3 there exists a basis $y_1, \dots, y_n$ of L such that

$$\mathrm{ad}_x(y_i) \in \langle y_1, \dots, y_{i-1} \rangle \text{ for all } x \in L, i.$$

This implies by induction that

$$C^i(L) \subseteq \langle y_1, \dots, y_{i-n+1} \rangle.$$

Therefore, $C^{n+1}(L) = 0$. □

Problem assignment, 4

1. Calculate the Killing form for $\mathfrak{sl}_2$ in the standard basis.
2. Let $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V(n))$ be the irreducible representation of $\mathfrak{sl}_2$ of highest weight $n > 0$. Prove that B_ρ is non-degenerate.
   Hint. Check that $B_\rho(h, h) \neq 0$.
3. Prove that the Killing form of a nilpotent Lie algebra vanishes.
4. Prove that subalgebra and quotient algebra of a nilpotent Lie algebra is nilpotent.
   Let K be a nilpotent ideal of L and let L/K be nilpotent. Does this imply that L is nilpotent?
5. Prove that a nilpotent three-dimensional Lie algebra L is either abelian or isomorphic to $\mathfrak{n}_3$.

6. Solvable Lie algebras

Recall that if V, W are vector subspaces of a Lie algebra L then $[V, W]$ denotes the vector subspace of L generated by all elements $[v, w]$ where $v \in V,\ w \in W$.

Recall also that if I, J are ideals in L then $[I, J]$ is also an ideal of L (Jacobi identity).

6.1. **Definition and first properties.** Define a sequence of ideals of L (the derived series) by

$$D^0(L) := L,\ D^1(L) := [L, L],\ D^{i+1}(L) := [D^i(L), D^i(L)].$$

6.1.1. **Definition.** A Lie algebra L is called *solvable* if $D^n(L) = 0$ for some n.

6.1.2. *Examples.* Recall that the sequence of ideals $C^n(L)$ was defined by the formulas

$$C^1(L) = L; \quad C^{n+1}(L) = [L, C^n(L)].$$

A Lie algebra is nilpotent if $C^n(L) = 0$ for some n.

Therefore, any nilpotent Lie algebra is solvable since $D^n(L) \subset C^{n+1}(L)$.

Define $\mathfrak{b}_n(k) \subset \mathfrak{gl}_n(k)$ as the Lie subalgebra consisting of upper triangular matrices. *Please take care: this contradicts our notation at the beginning of the course where* $\mathfrak{b}$ *denoted traceless upper triangular matrices. Sorry.*

The algebra $\mathfrak{b}_n(k)$ is solvable since $D^1(\mathfrak{b}) = \mathfrak{n}$ and so $D^i(\mathfrak{b}) \subset C^i(\mathfrak{n})$. However $\mathfrak{b}$ is not nilpotent since $C^2(\mathfrak{b}) = D^1(\mathfrak{b}) = \mathfrak{n}$ and $C^3(\mathfrak{b}) = C^2(\mathfrak{b}) = \ldots$. (Check this!)

6.1.3. **Proposition.** *The following conditions are equivalent*

(i) $D^n(L) = 0$.

(ii) *There exists a chain of ideals*

$$L = I_0 \supset I_1 \supset \ldots \supset I_n = 0$$

such that I_k/I_{k+1} is a commutative Lie algebra that is $[I_k, I_k] \subset I_{k+1}$ for all k.

Proof. (i)$\Longrightarrow$ (ii) since the chain $I_k := D^k(L)$ satisfies the condition of (ii). Moreover, this is the minimal chain which satisfies the condition: if $I_0, \ldots, I_n$ is such that $[I_k, I_k] \subset I_{k+1}$ for all k, then $I_k \supset D^k(L)$. Hence (ii)$\Longrightarrow$ (i). $\square$

6.2. **Lie theorem.**

6.2.1. **Theorem.** *Assume that the base field k is algebraically closed and has characteristic zero. Let L be a solvable finite dimensional Lie algebra and let $\rho : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of L. Then one can choose a basis of V so that the image $\rho(L)$ is a subalgebra of $\mathfrak{b}_n(k)$, $n := \dim V$.*

The formulation is slightly similar to that of Engel theorem. There is a number of differences.

- The assumptions in Engel theorem are about operators $\rho(x)$ whereas in Lie theorem the assumptions are about the Lie algebra L.

  For instance, if L is nilpotent, Engel theorem does not imply that $\rho(L) \subset \mathfrak{n}_n$.
- On the other hand, Lie theorem has assumptions on the field (algebraically closed of characteristic zero).

The proof is however similar to the proof of Engel theorem.

First of all, note that the theorem is an easy consequence (by induction on the dimension of V) of the following assertion.

6.2.2. **Theorem.** *Let L and ρ be as above. Suppose that* $\dim\ V > 0$. *Then all endomorphisms $\rho(g), g \in L$ have a common eigenvector.*

In fact, suppose 6.2.2 have been proven. Then Theorem 6.2 immediately follows by induction on $\dim\ V$. In fact, by 6.2.2 there exists a common eigenvector $v_1 \in V$ for all operators $\rho(x),\ x \in L$. This means that v_1 generates a one-dimensional L-submodule of V. By the induction hypothesis, the quotient module V/kv_1 admits a basis $\overline{v_2}, \ldots, \overline{v_n}$ satisfying the requirements of the theorem. Then choose representatives $v_i \in V$ of the classes $\overline{v_i}$ and the basis $v_1, v_2, \ldots, v_n$ is the one we were looking for.

From now on we will concentrate on proving Theorem 6.2.2. The proof will go by induction on $\dim\ L$.

Here are the steps of the proof.

- Find in L an ideal I of codimension 1.
- By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have a common eigenvector v_0. Let W be the space of all common eigenvectors of $\rho(g), g \in I$ with the same eigenvalues. We prove that W is L-submodule.
- Choose $x \in L$ such that $L = I + kx$. The element $\rho(x)$ has an eigenvector in W. It will be automatically a common eigenvector to all $\rho(g), g \in L$.

6.2.3. *Proof of 6.2.2.* We are proving the theorem by induction on dim L.

Basis of the induction.

For dim $L = 1$ one has $L = kx$. Since the field k is algebraically closed and $\dim V < \infty$, $\rho(x)$ has an eigenvector v. Obviously v is an eigenvector of all endomorphisms $\rho(g), g \in L$.

The step of the induction

1. First of all, we will prove there exists an ideal I in L having codimesion one.

Since L is solvable, $[L, L] \neq L$.

Observe that any subspace of L containing the commutant $[L, L]$ is an ideal of L.

Hence L has an ideal I of codimension 1: $L = I \oplus kx$.

2. By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have a common eigenvector $v_0 \neq 0$. For any $g \in I$ denote by $\chi(g)$ the scalar satisfying $\rho(g)v_0 = \chi(g)v_0$. This uniquely defines a map $\chi : L \to k$ satisfying two properties

- χ is linear;
- $\chi([x, y]) = 0$ for all $x, y \in I$.

(Explanation: χ is the homomorphism $I \to \mathfrak{gl}_1$ corresponding to the one-dimensional representation generated by v_0).

3. The space $W := \{v \in V|\ \rho(g)v = \chi(g)v,\ \forall g \in I\}$ contains v_0. Any vector of W is a common eigenvector for all endomorphisms $\rho(g), g \in I$. To finish the proof, it is enough to verify that $\rho(x)W \subset W$ since this inclusion implies the existence of an eigenvector of $\rho(x)$ in W.

4. Hence we need to show that $\rho(x)W \subset W$ or, equivalently, that $\rho(g)\rho(x)v = \chi(g)\rho(x)v$ for all $v \in W, g \in I$.

One has

$$\rho(g)\rho(x)v = \rho(x)\rho(g)v + \rho([x, g])v = \chi(g)\rho(x)v + \chi([g, x])v$$

since $[g, x] \in I$. Therefore, our aim is to prove that $\chi([x, g]) = 0$ for each $g \in I$.

5. Fix $v \in W \setminus \{0\}$ and for each $k \in \mathbb{N}$ denote by V_k the span of $v = \rho(x)^0 v, \rho(x)v, \ldots, \rho(x)^k v$.

Let us show that for any $g \in I, k \in \mathbb{N}$ $\rho(g)V_k \subset V_k$ and, moreover, $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1} (set $V_{-1} = 0$). We show this by induction on k. For $k = 0$ the assertion follows from the definition of W. For the induction step, observe that

$$\rho(g)\rho(x)^{k+1}v = \rho(x)\rho(g)\rho(x)^k v + \rho([g, x])\rho(x)^k v.$$

Since I is an ideal, $[g, x] \in I$. Then, by the induction hypothesis, $\rho([g, x])\rho(x)^k v \in V_k$ and $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1}. Therefore $\rho(g)\rho(x)^{k+1}v = \chi(g)\rho(x)^{k+1}v$ modulo V_k as required.

Let $k \in \mathbb{N}$ be maximal such that the vectors $v, \rho(x)v, \ldots, \rho(x)^k v$ are linearly independent. Then $V_m = V_k$ for all $m \geq k$. As we showed, V_k is $\rho(I)$-stable and relative to the basis $v, \rho(x)v, \ldots, \rho(x)^k v$ each matrix $\rho(g)$ $(g \in I)$ is upper triangular with the diagonal entries equal to $\chi(g)$. Obviously $\rho(x)V_k = V_k$ and so for any $g \in I$ $\rho([x,g]) = \rho(x)\rho(g) - \rho(g)\rho(x)$ is a traceless endomorphism. In the other hand, the trace of $\rho([x,g])$ is $(k+1)\chi([x,g])$. Since $\mathrm{char}(f) = 0$, this implies $\chi([x,g]) = 0$ as required.

Thus, W is $\rho(x)$-invariant, therefore, W contains a $\rho(x)$-invariant vector which is invariant with respect to the whole of L.

The theorem is proven.

6.3. **Remarks.**

6.3.1. The Lie theorem immediately implies that if V is a finite dimensional irreducible representation of a solvable Lie algebra then it is one-dimensional.

6.3.2. Even though Lie theorem looks very similar to Engel theorem, they are very different. Lie theorem provides an information about the structure of representations of a solvable Lie algebra. Engel theorem says nothing about representations of a nilpotent Lie algebra; it is applicable only if the image of the Lie algebra consists of nilpotent endomorphisms. For example, if L is one-dimensional (therefore nilpotent), it has representations corresponding to any (not necessarily nilpotent) endomorphisms. However, existence of Jordan normal form shows that any (single) representation can be written by as an upper-triangular matrix, provided the base field is algebraically closed.

6.3.3. **Corollary.** *Let k be algebraically closed field of characteristic zero. Let L be a solvable Lie algebra. Then there exists a sequence of ideals*

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = L$$

where $\dim\ I_k = k$.

Proof. Apply Lie theorem to the adjoint representation. □

6.3.4. **Corollary.** *Let k, L be as above. Then $[L,L]$ is nilpotent Lie algebra.*

Proof. Let $x \in [L,L]$. The element x acts trivially on any one-dimensional representation of L, in particular, on any factor I_k/I_{k-1} of the sequence of ideals guaranteed by 6.3.3. This implies that ad_x is nilpotent. Then by Engel theorem $[L,L]$ is nilpotent. □

6.3.5. **Corollary.** *Let k, L be as above. Let $K : L \times L \to k$ be the Killing form. Then $K(x,y) = 0$ if $x \in L, \quad y \in [L,L]$.*

Proof. According to 6.3.3, L admits a base in which ad_x has an apper triangular form. Then ad_y for $y \in [L,L]$ has in this base a strictly upper triangular form. This implies the claim. □

It turn out the converse is also true. It is called Cartan criterion (of solvability). We will use it without proof (the proof is tricky).

6.3.6. **Theorem.** *Let k be a field of characteristic zero and let L be a Lie subalgebra of $\mathfrak{gl}(V)$. The following conditions are equivalent.*

- *The Lie algebra L is solvable.*
- *For each $x \in L$ and $y \in DL = [L, L]$ one has $tr(x \circ y) = 0$.*

6.3.7. **Corollary.** *Let L be a finite dimensional Lie algebra. Then L is solvable iff $K(x, y) = 0$ for any $x \in L$, $y \in [L, L]$.*

Proof. Corollary 6.3.5 gives one direction of the equivalence. In the opposite direction, we apply Cartan criterion to the algebra $\mathrm{ad}(L) \in \mathfrak{gl}(L)$. We deduce that $\mathrm{ad}(L)$ is solvable. The kernel of ad is the center of L which is obviously a solvable ideal. This implies that L is solvable, by Exercise 3 of the Problem assignment 5. □

Problem assignment, 5

1. Prove that any subalgebra of a solvable Lie algebra is solvable.
2. Prove that any quotient algebra of a solvable Lie algebra is solvable.
3. Let L be a Lie algebra and I be an ideal in L. Prove that if L and L/I are solvable then L is solvable.
4. Let L be a Lie algebra. Prove that L contains a (unique) maximal solvable ideal.
5. Let $L = V \oplus k \cdot e$ with the bracket defined by the formulas $[v, v'] = 0$ $(v, v' \in V)$, $[e, v] = A(v)$ where $A : V \to V$ is a fixed operator.
   - a Verify that L is a Lie algebra.
   - b Verify that L is nilpotent iff A is a nilpotent operator.
   - c Verify that the Killing form vanishes iff $\mathrm{tr}(A^2) = 0$.
   - d Give example of a non-nilpotent operator A such that $\mathrm{tr}(A^2) = 0$.

7. Semisimplicity

7.1. **Radical.** Recall (see Problem 4 of the last problem assignment) that any finite dimensional Lie algebra has a unique maximal solvable ideal.

This ideal $R \subseteq L$ is called *the radical of L*.

7.2. Semisimple Lie algebras.

7.2.1. **Definition.** A Lie algebra L is called semisimple if its radical is trivial.

It is clear that any simple algebra is semisimple.

Here is an equivalent definition.

7.2.2. **Definition.** A Lie algebra L is semisimple if it has no nontrivial abelian ideals.

In fact, any abelian ideal is solvable, so if L has nontrivial abelian ideals, it is not semisimple. In the other direction, if L is not semisimple, i.e. if its radical

R is non-zero, consider the sequence of derived ideals $D^k(R)$; the last non-zero ideal $D^n(R)$) will be abelian.

There exist very nice criteria of semisimplicity.

The first one uses the Killing form

$$K(x,y) = \mathrm{tr}((\mathrm{ad}_x \mathrm{ad}_y).$$

The proof of the following lemma is based on Cartan criterion.

7.2.3. **Lemma.** *Let L be a finite dimensional Lie algebra over a field of characteristic zero and let I be the kernel of the Killing form K:*

$$I = \{x \in L | \forall y \in L \ K(x,y) = 0\}. \tag{12}$$

The I is a solvable ideal in L.

Proof. I is an ideal since K is invariant. In fact, if $x \in I$ and $z \in L$ then

$$K([x,z],y) = K(x,[z,y]) = 0 \text{ for all } y \in L.$$

Solvability of I immediately follows from Cartan criterion. □

7.2.4. **Theorem.** *A Lie algebra L is semisimple iff its Killing form is non-degenerate.*

Proof. If L is semisimple, the kernel of the Killing form should be trivial or the whole L. In the second case L should be solvable, which contradicts semisimplicity of L. Thus, the Killing form on a semisimple Lie algebra is non-degenerate.

In the other direction, suppose that L is not semisimple. Then L admits a nontrivial abelian ideal I. Then for $x \in I,\ y \in L$ one has

- $\mathrm{ad}_x \mathrm{ad}_y(L) \subset I$;
- $\mathrm{ad}_x \mathrm{ad}_y(I) = 0$.

This implies that $\mathrm{ad}_x \mathrm{ad}_y$ is nilpotent, therefore, $K(x,y) = 0$. □

The above criterion is very powerful.

7.2.5. **Corollary.** *Let L be a semisimple Lie algebra and let I be an ideal in L. Put $J = I^\perp$ (the orthogonal complement to I with respect to K). Then J is an ideal in L and $L = I \times J$.*

Proof. J is ideal by invariance of K: $y \in J$ iff $\forall x \in I\ K(x,y) = 0$. Then for $z \in L$ and for all $x \in I$ one has

$$K([z,y],x) = -K([y,z],x) = -K(y,[z,x]) = 0.$$

The intersenction $I \cap J$ is solvable by Cartan criterion applied to the restriction of the adjoint representation of L. Therefore, $I \cap J = 0$ and this implies $L = I \oplus J$ as vector spaces. The rest is obvious. □

7.2.6. **Lemma.** *The direct product $L \times M$ is semisimple iff both L and M are semisimple.*

Proof. Killing form on $L \times M$ is the orthogonal sum of the Killing forms on L and on M. Orthogonal sum is non-degenerate iff the summands are non-degenerate. □

Taking into account everything said above we deduce

7.2.7. **Theorem.** *A Lie algebra L is semisimple iff it is a direct product of simple algebras.*

7.3. **The algebra $\mathfrak{sl}_n$.** We will prove here that $\mathfrak{sl}_n$ is a simple Lie algebra. The idea is similar to the case $n = 2$.

It is more convenient to work with the algebra $L = \mathfrak{gl}_n$. Our aim is the following

7.3.1. **Theorem.** *The only ideals of $\mathfrak{gl}_n$ are $\mathfrak{sl}_n$ and kI where I is the unit matrix.*

In particular, since $\mathfrak{gl}_n = \mathfrak{sl}_n \times kI$, this implies that $\mathfrak{sl}_n$ is a simple algebra.

Let us choose a convenient basis.

Let e_{ij} be the matrix having 1 at (i, j)-th place and 0 elsewhere.

Denote $H = \langle e_{11}, \ldots, e_{nn}\rangle$. This is an n-dimensional commutative Lie subalgebra. We can easily describe L as a H-module with respect to the adjoint action. The result is given by the formula

$$L = H \oplus \bigoplus_{i \neq j} \langle e_{ij}\rangle. \tag{13}$$

Where H is a sum of trivial H-modules and each $\langle e_{ij}\rangle$ is a one-dimensional H-module defined by the character $\chi_{ij} : H \to k$ given by the formula

$$\chi_{ij}(h) = h_{ii} - h_{jj}.$$

Let I be an ideal of L.

We are in the situation of Theorem 4.6.1. L is a completely reducible representation of H, I is a subrepresentation. Therefore, I is also completely reducible. Every simple H-submodule of L is either $\langle e_{ij}\rangle$ for $i \neq j$ or a one-dimensional subspace of H.

Suppose there exists $i \neq j$ so that $e_{ij} \in I$. Then $[e_{ji}, e_{ij}] = e_{jj} - e_{ii} \in I$. Then $[e_{jj} - e_{ii}, e_{kl}]$ which is e_{kl} up to a non-zero constant whenever $k \neq l$ and $\{k, l\} \cap \{i, j\} \neq \emptyset$.

Then it is easy to see that $e_{ij} \in I$ for all $i \neq j$ and finally $I \supseteq \mathfrak{sl}_n$.

Suppose now that none of the e_{ij}, $i \neq j$, belongs to I. Then $I \subseteq H$. Since $e_{ij} \in I$, for each $h \in I$ one should have

$$[h, e_{ij}] = (h_{ii} - h_{jj})e_{ij} = 0$$

which implies that $I \subseteq kI$.

Theorem is proven.

8. Jordan decomposition: theme with variations

8.1. Recall that $f \in \operatorname{End}(V)$ is semisimple if f is diagonalizable (over the algebraic closure of the base field). Equivalently, this means that V admits a basis of eigenvectors. Equivalently, this means that the minimal polynomial for f has distinct roots. This formulation is convenient to prove that if W is an invariant subspace of V and f is semisimple, then $f|_W$ is semisimple as well.

8.1.1. **Proposition.** *Let $x \in \operatorname{End}(V)$.*

1. *There exist unique elements $s, n \in \operatorname{End}(V)$ such that $x = s + n$, s is semisimple, n is nilpotent and $[s, n] = 0$.*
2. *There exist polynomials $p, q \in k[t]$ with no constant term, such that $s = p(x)$, $n = q(x)$. Thus, s and n commute with every endomorphism commuting with x.*
3. *If $A \subseteq B \subseteq V$ and $x(B) \subseteq A$ then $s(B) \subseteq A$ and $n(B) \subseteq A$.*

Proof. Existence in 1. follows from the standard linear algebra theorem (Jordan decomposition). The claim 2 seems to be ugly. It will, however, help us to prove the rest of the claims.

Let a_i, $i = 1, \ldots, k$ be the eigenvalues of x with multiplicities m_i. Then $V = \sum V_i$ and the characteristic polynomial of $x|_{V_i}$ is $(t - a_i)^{m_i}$.

We claim there is a polynomial $p \in k[t]$ such that
$p \equiv a_i \mod (t - a_i)^{m_i}$ and $p \equiv 0 \mod t$ (the proof see below; this claim is called Chinese remander theorem).

Then for each i the restriction of the operator $p(x)$ on V_i is $p(x) = a_i + q_i(x)(x - a_i)^{m_i} = a_i$. This proves that $p(x) = s$. If we put $q = t - p$, we get $q(x) = n$.

Thus, we have proven claim 2 for the specific decomposition $x = s + n$. Let us now prove uniqueness of the decomposition. Let $x = s + n = s' + n'$. Since s' and n' commute with x and $s = p(x)$, $n = q(x)$, s' commutes with s and n' commutes with n. Then one has

$$s - s' = n' - n.$$

The sum of two commuting nilpotent elements is nilpotent. The sum of two commuting semisimple elements is semisimple. A nilpotent semisimple element is zero. This together gives the uniqueness claim.

Finally, since p, q have no constant term, the claim 3 follows. □

Here is the famous Chinese Remainder theorem.

8.1.2. **Lemma.** *Let $f_1, \ldots, f_k$ be pairwise coprime polynomials in $k[t]$. Let $a_1, \ldots a_k \in k[t]$. Then there exists a polynomial $p \in k[t]$ satisfying the equality*

$$p \equiv a_i \mod (f_i).$$

Proof. We proceed by induction in k. The case $k = 1$ is obvious, let us settle the case $k = 2$. Since f_1, f_2 are coprime, there exist g_1, g_2 such that $f_1g_1 + f_2g_2 = 1$. Then it is easy to see that $p = a_1f_2g_2 + a_2f_1g_1$ satisfies the requirements.

The general case is proven by induction. Let $k > 2$. By the above, there exists q such that $q = a_i(\mod f_i)$ for $i = 1, 2$. We now replace f_1, f_2 with one polynomial f_1f_2 and we are looking for p such that $p = q(\mod f_1f_2)$, $p = a_i(\mod f_i)$ for $i > 2$. By the induction assumption this problem has a solution. □

8.1.3. **Lemma.** *Let $x \in \mathrm{End}(V)$, $x = s + n$. Then $\mathrm{ad}_x = \mathrm{ad}_s + \mathrm{ad}_n$ is the Jordan-Chevalley decomposition of ad_x.*

Proof. ad_s is semisimple and ad_n is nilpotent (see Lemma in Engel theorem). The rest is obvious. □

8.2. Let L be a semisimple Lie algebra. Each element $x \in L$ defines an endomorphism $\mathrm{ad}_x \in End(L)$ which has a unique semisimple and nilpotent part

$$\mathrm{ad}_x = s + n.$$

We will see later that the elements s and n can be also expressed (in a unique way) as

$$s = \mathrm{ad}_{x_s}; \quad n = \mathrm{ad}_{x_n}.$$

The presentation $x = x_s + x_n$ is called *the abstract Jordan decomposition.*

The existence of such decomposition in a semisimple Lie algebra is a first step in the classification of semisimple Lie algebras.

8.2.1. **Lemma.** *Let V be a finite dimensional algebra and $D = \mathrm{Der}(A)$. If $x = s + n \in D$ then $s \in D$, $n \in D$.*

Proof. Let $V = \oplus_a V_a$ be the decomposition of V into generalized eigenspaces with respect to the eigenvalues of x. We claim that $V_a \cdot V_b \subseteq V_{a+b}$. In fact, if $v \in V_a, \quad w \in V_b$, so that

$$(x - a)^i v = 0, \quad (x - b)^j w = 0,$$

then

$$(x - a - b)^{i+j}(vw) = \sum_k \binom{i+j}{k} (x - a)^{i+j-k}(v)(x - b)^k(w) = 0.$$

Here is the explanation of the last formula. We prove it by induction in $n = i + j$. For $n = 1$ it says

$$(x - a - b)(vw) = (x - a)(v)w + v(x - b)(w)$$

that immediately follows from Leibniz rule. To prove the formula for $n = k + 1$, we apply the operator $(x-a-b)^k$ to $(x-a-b)(vw) = (x-a)(v)w+v(x-b)(w)$ and use the standard binomial identities.

The endomorphism s has value a on V_a. Therefore, Leibniz identity is obvious for s.

Finally, since $x, s \in D$, $n = x - s$ is in D as well. □

8.2.2. **Proposition.** *Let L be semisimple. Then the map*

$$ad : L \to \mathrm{Der}(L)$$

is a Lie algebra isomorphism.

Proof. L is semisimple, therefore, has no center. Thus, $\mathrm{ad} : L \to D = \mathrm{Der}(L)$ is injective. Identify L with $\mathrm{ad}(L)$. We claim L is an ideal in D. In fact, if $x \in L$ and $d \in D$ then $[d, \mathrm{ad}_x] = \mathrm{ad}_{d(x)}$.

Let us check that the Killing form of D restricted to L, gives the Killing form of L. Choose a base in L and complete it to a base in D. Then one sees that for $x, y \in L$ one has

$$\mathrm{tr}_D(\mathrm{ad}_x \mathrm{ad}_y) = \mathrm{tr}_L(\mathrm{ad}_x \mathrm{ad}_y)$$

since $\mathrm{ad}_y(D) \subseteq L$, so $\mathrm{ad}_x \mathrm{ad}_y(D) \subseteq L$, and the trace depends on diagonal elements only.

Now, use that the Killing form of L is non-degenerate. This means that $L^\perp \cap L = 0$ which implies $D = L \oplus L^\perp$. Since the Killing form is invariant and L is an ideal, we deduce that $L^\perp$ is also an ideal. Therefore, $[L, L^\perp] = 0$ that is $D = L \times L^\perp$.

Finally, if $d \in L^\perp$ and $x \in L$ then $[d, \mathrm{ad}_x] = \mathrm{ad}_{d(x)}$ which implies that $d(x) = 0$. Thus, $d = 0$ and we are done. □

8.3. We are now ready to deduce the main result.

8.3.1. **Proposition.** *Let L be a semisimple Lie algebra, $x \in L$. Then there exist unique elements $x_s,\ x_n \in L$ such that*

- *$x = x_s + x_n$, and the three elements commute with each other.*
- *ad_{x_s} is semisimple and ad_{x_n} is nilpotent.*

Proof. ad_x is a derivation, therefore its semisimple and nilpotent parts are derivations by Lemma 8.2.1. Then by Proposition 8.2.2 the semisimple and nilpotent parts of ad_x come also from L. □

Problem assignment, 6

1. The algebra $\mathfrak{sp}_4$ consists of 4×4-matrices of form
$$\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$
where M, N, P, Q are two-by-two matrices satisfying the conditions
$$N, P \text{ are symmetric; } Q = -N^t.$$
Prove that $\mathfrak{sp}_4$ is simple.
Hint. The reasoning similar to that of Theorem 7.3.1, where H is once more the subalgebra of diagonal matrices.
2. Compute the basis in $\mathfrak{sl}_2$ dual to the standard basis e, f, h with respect to the Killing form.
3. Let $L = L_1 \times L_2$ is a product of semisimple Lie algebras. Let $x \in L$ be presented $x = x^1 + x^2$ with $x^i \in L_i$. Prove that $x_s = x_s^1 + x_s^2$.
4. Calculate the Killing form for the two-dimensional non-abelian Lie algebra.

9. Complete reducibility

9.1. The aim of this section is proving the following important theorem.

9.1.1. **Theorem.** *Let L be a semisimple Lie algebra. Then any finite dimensional representation of L is completely reducible.*

Note that the theorem has already been proven for $L = \mathfrak{sl}_2$. We have seen that the complete reducibility fails for non-finitely dimensional modules: nonsimple Verma modules are indecomposable into a sum of irreducible modules. Therefore, this theorem is as general as possible. We will see later that the converse of the theorem is also correct: if L is a finite dimensional Lie algebra such that all finite dimensional L-modules are completely reduclible, then L is semisimple. The proof will result from a sequence of steps. One of the steps is a long digression about tensor algebra of L-modules.

9.2. **Step 1: Killing form.** Let L be semisimple and $\rho : L \to \mathfrak{gl}(V)$ be injective. Then the Killing form B_ρ given by the formula $B_\rho(x, y) = \mathrm{tr}_V(\rho(x) \circ \rho(y))$, is nondegenerate.

In effect, the kernel $I = \mathrm{Ker}(B_\rho)$ is an ideal of L. It is solvable by the Cartan criterion. Therefore, $I = 0$.

9.3. **Step 2: Casimir.** Let B be a non-degenerate invariant symmetric bilinear form on L (here L is not required to be semisimple!). Let $x_1, \ldots, x_n$ be a basis in L and let $y_1, \ldots, y_n$ be the dual basis: $B(x_i, y_j) = \delta_{ij}$. We claim that the endomorphism
$$Q = \sum \rho(x_i) \circ \rho(y_i) : V \to V$$

does not depend on the choice of the basis $x_1, \dots, x_n$; that it commutes with any L-homomorphism of representations and with the action of L. The proof is via a direct calculation. In what follows we omit ρ from the formulas.

Let $x_i' = \sum a_{ij} x_j$. Then $y_i = \sum a_{ji} y_j'$ and

$$\sum x_i y_i = \sum a_{ji} x_i y_j' = \sum x_j' y_j'.$$

This means that the resulting operator does not depend on the choice of a basis in L. The homomorphism Q commutes with any L-homomorphism since this is true for any expression of this kind. Let us check that it commutes with the action of L. We have to prove that for any $z \in L$ one has

$$\sum z x_i y_i = \sum x_i y_i z : V \to V$$

or, in other words, that one has the following identity

$$[z, \sum x_i y_i] = 0$$

in the algebra of endomorphisms of any L-module V. Note that

$$[z, \sum x_i y_i] = \sum [z, x_i] y_i + \sum x_i [z, y_i].$$

Since the form is invariant,

$$B([z, x_i], y_j) + B(x_i, [z, y_j]) = 0$$

which implies that

$$[z, x_i] = \sum B([z, x_i], y_j) x_j = -\sum B(x_i, [z, y_j]) x_j$$

and similarly

$$[z, y_i] = \sum B([z, y_i], x_j) y_j.$$

Therefore,

$$\sum [z, x_i] y_i + \sum x_i [z, y_i] = -\sum B(x_i, [z, y_j]) x_j y_i + \sum B([z, y_i], x_j) x_i y_j = 0.$$

We will present later on a more “scientific” explanation of this fact.

9.4. **Step 3: calculation.** Let $B = B_\rho$ as in Step 1 and let Q be the corresponding Casimir endomorphism of V. Then $\mathrm{tr}_V(Q) = \dim L$.

9.5. **Step 4: The case V is simple.** Note that if V is simple (we still assume that ρ is injective), $Q : V \to V$ is a non-zero isomorphism.

9.6. **Step 5: a special case.** Assume that $V \subset W$ is a pair of representations with $\dim(W/V) = 1$. We claim that in this case the embedding $V \to W$ splits so that W is isomorphic to a direct sum $V \oplus \mathbf{1}$ where $\mathbf{1}$ is the trivial (one-dimensional) representation. This is the most important part of the proof.

First of all, any one-dimensional representation is defined by a character $\rho : L \to F$ whose kernel contains $[L, L]$. Since L is semisimple, $L = [L, L]$, so such character is trivial. Therefore, W/V is a trivial representation $\mathbf{1}$.

We will prove by induction in $\dim(V)$ that the embedding $V \subset W$ splits. The splitting of the embedding is equivalent to the existence of $w \in W$ such that $xw = 0$ for all $x \in L$ and $W = V + Fw$.

First of all, we will show that one can assume V to be irreducible.

In fact, if V' is a nontrivial submodule of V, we can factor both V and W by V' and we get an embedding of representations $\bar{V} = V/V' \to \bar{W} = W/V'$ of codimension one, having smaller dimension of the submodule. By the iductive assumption, there exists $\bar{w} \in \bar{W}$ such that $x\bar{w} = 0$ for all $x \in L$ and $\bar{W} = \bar{V} + F\bar{w}$.

Now define W' as the preimage of $F\bar{w}$ under the projection $W \to \bar{W}$. We have the embedding $V' \to W'$ of codimension 1, so, one more by the inductive hypothesis, we can find $w \in W'$ with the required properties. Thus, V can be assumed to be irreducible.

9.6.1. Let us now show that we can also assume that the map $\rho : L \to \mathfrak{gl}(V)$ is injective. Let $I = \ker(\rho)$. This means that any $x \in I$ annihilates V. Since any $x \in L$ carries W to V, $[I, I]$ annihilates W. Since L is semisimple, I is also semisimple, so $[I, I] = I$, therefore, both V and W are annihilated by I. Thus, they are both modules over L/I which is also semisimple. This reduces everyting to the case ρ is injective.

9.6.2. All reductions made, we have the following. L is a semisimple Lie algebra, V an irreducible representation such that $\rho : L \to \mathfrak{gl}(V)$ is injective, and $V \subset W$ so that $W/V = \mathbf{1}$. Look now at the Casimir $Q_\rho : W \to W$. Its restriction to V is a multiplication by a nonzero constant, and its action on W/V is zero. Therefore, $Q_\rho : W \to W$ has a one-dimensional kernel K. This is a one dimensional L-submodule, so any nonzero $w \in K$ satisfies the required conditions.

9.7. **The general case.** We wish to prove that any embedding $V \subset W$ of finite dimensional L-modules splits, that is that there exists a submodule U of W such that $W = V \oplus U$. We know that this implies complete reducibility. Equivalently, we have to prove that there exists a homomorphism $\pi : W \to V$ of L-modules such that $\pi|_V = \mathrm{id}_V$.

We will deduce this from the special case studied above. To proceed, we need some general construction.

9.7.1. *Hom.* Let V and W be two prepresentations of L. We will now define a structure of L-module on the vector space $\mathrm{Hom}(V, W)$.

Here is the formula. For $f : V \to W$ and $x \in L$, we define $xf : V \to W$ by the formula

$$(xf)(v) = xf(v) - f(xv).$$

The map xf so defined is obviously linear. Therefore, the only thing to verify is the identity

$$[x, y]f = x(yf) - y(x(f).$$

This is an easy exercise.

Here are a couple of special cases.

First, assume $V = \mathbf{1} = F1$ is the trivial representation. Then $\mathrm{Hom}(\mathbf{1}, W) = W$, the isomorphism carrying $f : \mathbf{1} \to W$ to $f(1)$.

It is easy to see that the structure of L-module on $\mathrm{Fun}(\mathbf{1}, W)$ is compatible with this isomorphism.

Another special case is the L-module structure on the dual space $V^* = \mathrm{Hom}(V, \mathbf{1})$. The L-action is given by the formula

$$xf(v) = -f(xv).$$

Note the following obvious

9.7.2. **Lemma.** *An element $f \in \mathrm{Hom}(V, W)$ is L-invariant, that is, $xf = 0$ for all $x \in L$ iff $f : V \to W$ is a homomorphism of L-modules.*

9.7.3. *End of the proof.* We are now ready to prove that for any L-submodule V of W there exists a splitting $s : W \to V$ that is a homomorphism of L-modules satisfying the condition $s|_V = \mathrm{id}_V$.

We proceed as follows. Let $\mathrm{Hom}(W, V)$ be the L-module defined above and let $H \subset \mathrm{Hom}(W, V)$ be the collection of linear transformations $\phi : W \to V$ such that the restriction $\phi|_V$ is a multiplication by a scalar. Define the map $p : H \to F$ as the one carrying ϕ to the scalar $c \in F$ such that $\phi|_V = c\mathrm{id}_V$. One can easily see that p is a homomorphism of L-modules. The kernel of p is the collection of maps $\phi : W \to V$ such that $\phi|_V = 0$. It identifies with $\mathrm{Hom}(W/V, V)$. Thus, the vector space $\mathrm{Hom}(W/V, V)$ is a subspace of H having codimension one. It is an L-submodule as it is defined as the kernel of a homomorphism of L-modules. According to the special case of the theorem proven above, there exists $s \in H$ such that $xs = 0$ for all $x \in L$ and $H = \mathrm{Ker}(p) + Fs$. We have $p(s) \neq 0$, so, replacing s with $p(s)^{-1}s$, we will assume $p(s) = 1$. This means that $s|_V = \mathrm{id}_V$. One the other hand, $s : W \to V$ is an L-homomorphism by Lemma 9.7.2, since $xs = 0$ for all $x \in L$.

9.8. **Digression: Tensor product of L-modules.** In the proof of complete reducibility of finite dimensional modules over a semisimple Lie algebra, we used an operation Hom assigning an L-module $\operatorname{Hom}(V, W)$ to a pair of modules V, W. In this subsection we will discuss another operation closely related to Hom.

9.8.1. *Tensor product of vector spaces.* Fix a field k.

9.8.2. **Definition.** Let V, W, X be three vector spaces over F. An F-bilinear map $f : V \times W \to X$ is a map satisfying the following properties.

1. $f(v + v', w) = f(v, w) + f(v', w)$.
2. $f(av, w) = af(v, w)$ for any $a \in F$.
3. $f(v, w + w') = f(v, w) + f(v, w')$.
4. $f(v, aw) = af(v, w)$ for any $a \in F$.

Of course, we have already seen this definition in a special case $V = W$ and $X = F$ — this was the definition of a bilinear form on V.

The set of bilinear maps $V \times W \to X$ is a vector space: the sum of two bilinear maps is bilinear and a bilinear map multiplied by a constant is bilinear. We denote $\mathtt{Bil}(V, W; X)$ this vector space.

9.8.3. The assignment $(V, W, X) \mapsto \mathtt{Bil}(V, W; X)$ is a functor (see below) in three arguments, covariant in X and contravariant in V and in W. Here is the precise statement.

Given $a : V' \to V$, $b : W' \to W$ and $c : X \to X'$ linear maps, a map

$$\mathtt{Bil}(V, W; X) \to \mathtt{Bil}(V', W', X')$$

is defined as the one carrying $f : V \times W \to X$ to the composition

$$V' \times W' \xrightarrow{a \times b} V \times W \xrightarrow{f} X \xrightarrow{c} X'.$$

Of course, one has to verify that the above composition remains bilinear.

9.8.4. It turns out, for given V and W, there exists a universal bilinear map $u : V \times W \to U$ in the following sense.

As we said above, any linear map $\phi : U \to X$ defines, by composition, a bilinear map $\phi \circ u : V \times W \to X$.

Definition. A bilinear map $u : V \times W \to U$ is called universal if for any vector space X the map

$$\operatorname{Hom}(U, X) \to \mathtt{Bil}(V, W; X)$$

is a bijection (an isomorphism of vector spaces).

The definition above says nothing about existence or uniqueness of the universal bilinear map. We will prove existence later. We will start explaining in what sense it is unique.

9.8.5. **Lemma.** *Let $u : V \times W \to U$ and $u' : V \times W \to U'$ be both universal. Then there exists a unique isomorphism $\theta : U \to U'$ such that $u' = \theta \circ u$.*

Proof. Since u is universal, there exists a unique homomorphism $\theta : U \to U'$ such that $u' = \theta \circ u$. Similarly, since u' is universal, there exists a unique homomorphism $\theta' : U' \to U$ such that $u = \theta' \circ u'$. We claim that θ and θ' are inverse to each other. In fact, $\theta' \circ \theta : U \to U$ satisfies the property

$$u = (\theta' \circ \theta) \circ u$$

and id_U should be the only map $U \to U$ satisfying this property (once more, because of universality of u). The proof of $\theta \circ \theta' = \mathrm{id}_{U'}$ goes along the same lines. $\square$

9.8.6. *Existence.* We will now prove existence of a universal bilinear map. Let X be a basis of V and Y a basis of W. This means that any $v \in V$ can be uniquely presented as a lilnear combination of elements of X, and any element $w \in W$ has a unique presentation of elements of Y. A bilinear map $f : V \times W \to Z$ is uniquely defined by its values on $X \times Y$, $f(x, y) \in Z$. This is the reasoning we know from the theory of bilinear forms.

This leads us to the following construction of a universal bilinear map. Set U to be the vector space with the basis $X \times Y$. We will denote the pair $(x, y) \in X \times Y$ considered as an element of the basis of U, as $x \otimes y$. (At the moment, this is just a notation!)

The map $u : V \times W \to U$ is the one carrying the pair $(x, y) \in V \times W$ to the basis vector $x \otimes y$ of U.

The above description is not easy to understand. To understand it better, let us add that, for $v = \sum c_i x_i$, $w = \sum d_j y_j$, $c_i, d_j \in k$, $x_i \in X$, $y_j \in Y$, one has

$$u(v, w) = \sum_{i,j} c_i d_j x_i \otimes y_j.$$

This easily follows from bilinearity of U and from the condition $u(x_i, y_j) = x_i \otimes y_j$.

9.8.7. We define the tensor product of V and W as "the" universal bilinear map $u : V \times W \to U$. We denote $U = V \otimes W$. This is a vector space, together with a bilinear map $u : V \times W \to U \otimes W$ defined uniquely up to a unique isomorphism.

We also denote $u(v, w)$ as $v \otimes w \in V \otimes W$. This extends the notation $x \otimes y$ we introduced in the construction of $V \otimes W$.

9.8.8. **Corollary.** $\dim(V \otimes W) = \dim(V) \dim(W)$.

$\square$

A bilinear map $f : V \times W \to X$ can be otherwise defined as a linear map $\tilde{f} : V \to \mathrm{Hom}(W, X)$ from V to the vector space $\mathrm{Hom}(W, X)$ of linear maps from

W to X. Since one has a bijection $\mathtt{Bil}(V, W; X) = \mathrm{Hom}(V \otimes W, X)$, we get a functorial isomorphism

$$\mathrm{Hom}(V \otimes W, X) \tilde{\to} \mathrm{Hom}(V, \mathrm{Hom}(W, X)). \tag{14}$$

The above formula connects two functors: tensor product and Hom. In the language of category theory, this means that the functors $\otimes$ and Hom are *adjoint*.

9.8.9. There is another conection between tensor product and the functor Hom. We know that, if V and W have (finite) dimensions m and n respectively, than both $V \otimes W$ and $\mathrm{Hom}(V, W)$ have dimension mn. Here is a more precise connection between the two notions.

For any pair of vector spaces V and W we define a linear map

$$\theta : V^* \otimes W \to \mathrm{Hom}(V, W)$$

as follows. We start with a bilinear map

$$\Theta : V^* \times W \to \mathrm{Hom}(V, W)$$

by the formula

$$\Theta(f, w)(v) = f(v) \cdot w.$$

Linearity in $f \in V^*$ and in $w \in W$ is obvious. Therefore, by universality of tensor product, we have a linear map θ. We have

Proposition. *Assume that V is finite dimensional. Then θ is an isomorphism.*

Proof. Choose a basis $v_1, \ldots, v_n$ of V. Let $f_1, \ldots, f_n$ be the dual basis for V^*. Recall that this means that $f_j(v_j) = \delta^i_j$, the Kronecker's delta. If $\{w_\alpha\}$ is a basis for W (finite or infinite), the pairs (f_i, w_α) form a basis for $V^* \otimes W$. The map θ carries such pair to the map $\phi_{i,\alpha} : V \to W$ carrying v_i to w_α and v_j for $j \neq i$ to zero. Such $\phi_{i,\alpha}$ form obviously a basis for $\mathrm{Hom}(V, W)$. □

Using the direct sum decomposition of the tensor product presented below one can prove that θ is also an isomorphism if W is finite dimensional. It is not an isomorphism in general!

9.8.10. *Direct sum.* There is a sort of distributiva law for tensor products.

Given V_1, V_2 and W, we will construct an isomorphism

$$V_1 \otimes W \oplus V_2 \otimes W \to (V_1 \oplus V_2) \otimes W.$$

In order to construct the map $f : X_1 \oplus X_2 \to Y$, it is enough to construct $f_i : X_i \to Y$, $i = 1, 2$: then $f(x_1 + x_2) = f_1(x_1) + f_2(x_2)$. Thus, to construct the required map, it is enough to present $V_i \otimes W \to (V_1 \oplus V_2) \otimes W$. The rest is clear. To prove that the constructed map is an isomorphism, we choose bases and make an easy verification.

9.8.11. *Tensor product of representations.* Let V and W be L-modules. We will now present a natural L-module structure on $V \otimes W$. For any $x \in L$ we have to present an endomorphism of $V \otimes W$. Here we use functorial properties of a tensor product. Let us fix a notation. For $f : V \to V'$, we denote by $f \otimes 1 : V \otimes W \to V' \otimes W$ the map defined by the formula $(f \otimes 1)(v \otimes w) = f(v) \otimes w$. Similarly, for $g : W \to W'$, we define $1 \otimes g : V \otimes W \to V \otimes W'$ by the formula $(1 \otimes g)(v \otimes w) = v \otimes g(w)$. We can now define the endomorphism of $V \otimes W$ given by $x \in L$ as $x \otimes 1 + 1 \otimes x$. That is, we have $x(v \otimes w) = xv \otimes w + v \otimes xw$. It is an easy exercise to verify that this formula determines an action of L on $V \otimes W$.

9.8.12. *Some identities.* Let V, W, X be three L-modules. We can now verify that the standard isomorphisms of vector spaces

$$\mathrm{Hom}(V, \mathrm{Hom}(W, X)) \to \mathrm{Hom}(V \otimes W, X),$$

$$\theta : V^* \otimes W \to \mathrm{Hom}(V, W)$$

are isomorphisms of L-modules.

10. Levi theorem. Reductive Lie algebras

10.1. **Levi theorem.** Recall that a Lie algebra L is semisimple if it has no solvable ideals. Recall as well that any Lie algebra has a maximal solvable ideal R called *the radical* (as the sum of two solvable ideals is a solvable ideal) and that therefore the quotient L/R is semisimple (prove this!) We are going to prove now that L contains a subalgebra isomorphic to L/R. More precisely, one has the following theorem.

10.1.1. **Theorem.** *(Levi) Let $f : L \to S$ be a surjective Lie algebra homomorphism with S semisimple. Then there exists a Lie algebra homomorphism $g : S \to L$ splitting f, that is, satisfying $f \circ g = \mathrm{id}_S$.*

10.1.2. *Semidirect product.* Let $I = \mathrm{Ker}(f)$. One obviously has $L = I \oplus g(S)$ as vector spaces. Note that I is an ideal and $g(S)$ is a Lie subalgebra in L. Since I is an ideal, $[g(x), y] \in I$ for all $x \in S, y \in I$, so the formula

$$x(y) = [g(x), y]$$

defines on I a structure of S-module. In the opposite direction, any Lie algebra homomorphism $a : S \to \mathrm{Der}(I)$ allows one to define a Lie algebra structure on $L := S \oplus I$ by the formula

$$[(x, y), (x', y')] = ([x, x'], [y, y'] + a(x)(y') - a(x')(y)).$$

The above construction is called *the semidirect product* of Lie algebras S and I along a homomorphism a. In the special case $a = 0$ this is just a direct product of Lie algebras. Thus, Levi theorem can be reformulated as follows.

10.1.3. **Corollary.** *Any finite dimensional Lie algebra is a semidirect product of its radical with a semisimple Lie algebra.*

An image $g(S)$ of the quotient of L by the radical is called a Levi factor. Note that the Levi factor is not unique (it is in fact unique up to conjugation) as there is a freedom of choice of teh section g.

10.1.4. *Proof of Levi theorem.* Note that there are similar theorems in the theories of associative algebra (Wedderburn-Malcev theorem).

Step 1. Reduce the problem to the case when I is a simple L-module. This is done by induction in $\dim(I)$. If I is not simple, it contains an ideal J of a smaller nonzero dimension. By induction, the projection $L/J \to S$ splits, so that there exists $\bar{g} : S \to L/J$ with $L/J = \bar{g}(S) \oplus I/J$. Let L' be the preimage of $\bar{g}(S)$ in L. This is a subalgebra of L having a projection $L' \to S$ with the kernel J. Once more, by induction, there is a splitting of this projection.

Step 2. Let R be the radical of L. The image of R, $f(R)$, is a solvable ideal in S, so $f(R) = 0$ so $R \subset I$. Since I is simple, either $R = 0$ or $R = I$. If $R = 0$, L is semisimple and any ideal I in it is a direct factor (using Killing form). Thus, we can assume $I = R$ and in this case I is commutative as otherwise $[I, I]$ is a nontrivial submodule in I.

Step 3. Consider the case when L acts trivially on I. Then I lies in the center of L; therefore it is the center (as otherwise $L/I = S$ would have a center). Then L is a module over $L/I = S$ which is semisimple. Therefore, by complete reducibility we have $L = I \oplus K$ as S-modules, therefore, $K = L/I$ is isomorphic to S. This case is settled.

Step 4. We now assume that I is a simple nontrivial L-module and $[I, I] = 0$. This is the main part of the proof. We have to find a Lie subalgebra K of L isomorphic to S such that $L = I \oplus K$.

The idea of the construction of K is the following. We will present an L-module W and and element $w \in W$ such that the linear map

$$a : L \to W,\ a(x) = xw$$

satisfies the following properties.

- $a|_I$ is injective.
- $a(L) = a(I)$.

Let us show how everything will follow from the properties of W and w. We define $K = \mathrm{Ker}(a)$, that is $K = \{x \in L | xw = 0\}$. This means that K is the stabilizer of w, in particular, it is a Lie subalgebra.

The homomorphism $a : L \to W$ produces a map of vector spaces $p : L \to I$ carrying $x \in L$ to the unique $y \in I$ such that $a(x) = a(y)$. This means that $p|_I = \mathrm{id}_I$, $\mathrm{Ker}(p) = K$. This gives a decomposition $L = I \oplus K$ as vector spaces.

Let us now construct W and w as required. We put $W = \mathrm{Hom}(L, L)$ with the L-module structure defined by the standard formula

$$(xf)(y) = [x, f(y)] - f([x, y]).$$

Define the following subspaces of W.

$$P = \{ad_a,\ a \in I\}.$$

$$Q = \{\phi \in W | \phi(L) \subset I \& \phi(I) = 0\}.$$

$$R = \{\phi \in W | \phi(L) \subset I \& \phi|_I = \lambda \cdot \mathrm{id}_I, \lambda \in F\}.$$

These are L-submodules of W. In fact, if $x \in L$ then $x\mathrm{ad}_a = ad_{[x,a]}$ for $a \in I$. This proves that P is an L-submodule. The vector subspace Q indetifies with $\mathrm{Hom}(L/I, I)$ which is obviously an L-submodule of W. An easy calculation shows that R is an L-submodule of W. One has a short exact sequence of L-modules

$$0 \to Q \to R \to \mathbf{1} \to 0$$

where, as usual, $\mathbf{1}$ is the trivial representation of L. Furthemore, $P \subset Q$ and one deduces the following short exact sequence

$$0 \to Q/P \to R/P \to \mathbf{1} \to 0.$$

Note that I acts trivially on the quotients, so this short exact sequence can be seen as a sequence of $L/I = S$-modules. Since S is semisimple, the sequence splits, so that there is an element $\bar{w} \in R/P$ such that $x\bar{w} = 0$ for all $x \in L$ and $R/P = Q/P \oplus F\bar{w}$. We choose $w \in R$ as a preimage of $\bar{w}$. The element w so chosen is a linear map $w : L \to I$ whose restriction to I is a multiplication by a nonzero constant and such that for any $x \in L$ one has $xw = \mathrm{ad}_a$ for some $a \in I$. Multiplying if necessary w by a nonzero constant, we can assume that $w|_I = \mathrm{id}_I$.

Let us verify that $w \in W$ satisfies the required properties.

- If $a \in I$ and $aw = 0$, we have $[\mathrm{ad}_a, w] = 0$ that is $\mathrm{ad}_a \circ w(x) = w \circ \mathrm{ad}_a(x)$ that is $w([a, x]) = [a, w(x)] = 0$, the latter as $w(x) \in I$ and $[I, I] = 0$. Finally, $w([a, x]) = [a, x]$ so $[a, x] = 0$ for all $a \in I$ and $x \in L$ which is impossible as the action of L on I is nontrivial.
- It remains to verify that for any $x \in L$ there exists $a \in I$ such that $x \cdot w = \mathrm{ad}_a(w)$. This is just invariance of $\bar{w}$.

The theorem is proven.

10.2. **Reductive Lie algebras.** A finite dimensional Lie algebra L is called reductive if it is completely reducible when considered as an L-module with respect to the adjoint action. A semisimple Lie algebra is reductive as its all finite dimensional representations are completely reducible. A commutative Lie algebra is reductive as it is a direct sum of trivial representations.

10.2.1. **Lemma.** *A product of two reductive Lie algebras is reductive.*

10.2.2. **Theorem.** *Any reductive Lie algebra is a direct product of semisimple and commutative algebras.*

Proof. The submodules of L are just the ideals. If I and J are two ideals such that $I \cap J = 0$ then obviously $[I, J] = 0$. Thus, L decomposes into a finite product of simple ideals. Any simple ideal is either a simple Lie algebra or one-dimensional. This implies the result. □

10.3. **Problem assignment, 7.**

1. Prove that the radical of a reductive Lie algebra L coincides with its center.
2. Prove that the Levi factor of a reductive Lie algebra L is (isomorphic to) $[L, L]$.
3. Prove that if any finite dimensional representation of L is completely reducible then L is semisimple.

11. Study of semisimple Lie algebras

We assume here that the base field is algebraically closed.

Let L be a semisimple Lie algebra.

11.1. **Maximal toral subalgebras and roots.** If L does not contain ad-semisimple elements, all its elements are ad-nilpotent by Jordan-Chevalley theorem and L is nilpotent by Engel theorem.

Since this is not the case, L contains semisimple elements.

A Lie subalgebra of L is called *toral* if it consists of semisimple elements. The above reasoning implies that any semisimple Lie algebra contains nontrivial toral subalgebras.

One has

11.1.1. **Lemma.** *(For k algebraically closed) Any toral subalgebra is commutative.*

Proof. Let T be a toral subalgebra and $x \in T$. Observe that $(\mathrm{ad}x)T \subset T$ and denote the restriction of $\mathrm{ad}x$ to T by $\mathrm{ad}_T x$. We need to show that $\mathrm{ad}_T x = 0$. Obviously $\mathrm{ad}_T x$ is semisimple.

Let $y_0 \in T$ be an eigenvector of $\mathrm{ad}_T x$ that is $(\mathrm{ad}x)y_0 = cy_0$ for some $c \in k$. We have to prove $c = 0$.

Since $\mathrm{ad}_T y_0$ is semisimple, T admits a basis $y_0, y_1, \ldots, y_r$ of eigenvectors of $\mathrm{ad}_T y$. One has $(\mathrm{ad}y_0)T \subset \{y_1, \ldots, y_r\}$ because $(\mathrm{ad}y_0)y_0 = 0$. Now the equality $(\mathrm{ad}x)y_0 = -(\mathrm{ad}y_0)x$ gives $c = 0$. □

11.1.2. **Example.** In the case $k = \mathbb{R}$, a linear operator $\psi \in \mathrm{End}(V)$ is called semisimple if it complexification $\psi_{\mathbb{C}} \in \mathrm{End}(V \otimes_{\mathbb{R}} \mathbb{C})$ is diagonalizable.

Let $L = \texttt{Vect}$ be a three dimensional Lie algebra over $\mathbb{R}$ with respect to vector product: $[v, w] := v \times w$. This algebra is simple (see Lecture 2) since an ideal containing a given non-zero element v contains also all vectors which are orthogonal to v and so coincides with `Vect`. Any element $x \in L$ is ad-semisimple since all eigenvalues of ad$x$ are distinct $(0, i, -i)$. However L is not commutative. Thus, the condition $k = \overline{k}$ is very important.

11.1.3. Fix a toral subalgebra H in L. Since H consists of commuting semisimple elements, L has a basis for which all matrices ad$x : x \in H$ are diagonal. Let v be a non-zero common eigenvector of the linear operators ad$x : x \in H$. An element μ of the dual space H^* is called *the weight of* v if $(\text{ad}x)(v) = \mu(x)v$ for all $x \in H$. One has

$$L = L_0 \oplus \bigoplus_{\mu \in \Delta} L_\mu$$

where

$$\Delta := \{\mu \in H^* \setminus \{0\}|\ L_\mu \neq 0\}, L_\mu := \{x \in L|\ (\text{ad}h)(x) = \mu(h)x,\ \forall h \in H\}.$$

The elements of Δ are called *the roots* of L.

Denote by K the Killing form of L.

11.1.4. **Lemma.** *i)* $[L_\mu, L_\nu] \subset L_{\mu+\nu}$,
ii) for $\alpha \in \Delta$ *all elements of* L_α *are nilpotent,*
iii) if $\alpha + \beta \neq 0$ *then* $\forall x \in L_\alpha, y \in L_\beta$ *one has* $K(x, y) = 0$,
iv) the restriction of K *to* L_0 *is non-degenerate.*

Proof. (i) follows from the Jacobi identity. (ii) follows from (i) and the fact that the set of weights of L with respect to H is a finite set (it is equal to $\Delta \cup \{0\}$). (iii) follows from ad-invariance of K. Finally, (iv) is an immediate consequence of non-degeneracy of K and (iii). □

11.1.5. **Corollary.** *For any* $\alpha \in \Delta$ *one has* $\dim L_\alpha = \dim L_{-\alpha}$.

Proof. Combining the non-degeneracy of K and (iii), we conclude that for any non-zero $x \in L_\alpha$ there exists $y \in L_{-\alpha}$ such that $K(x, y) \neq 0$. Therefore the formula $x \mapsto f_x :\ f_x(y) := K(x, y),\ \forall y \in L_{-\alpha}$ defines an embedding $L_\alpha \to L^*_{-\alpha}$. In particular, $\dim L_\alpha \leq \dim L_{-\alpha}$. Applying the last inequality for $\alpha' := -\alpha$, one concludes $\dim L_\alpha = \dim L_{-\alpha}$. □

Since any toral subalgebra is commutative, $L_0 \supseteq H$.

11.1.6. **Proposition.** *Let* H *be a maximal toral subalgebra. Then* $L_0 = H$. *That is, a maximal toral subalgebra coincides with its centralizer.*

Proof. Note that by definition $L_0 = \{x \in L | [h, x] = 0\ \forall\ h \in H\}$ is the centralizer of H.

Step 1. Let $x = s + n$ be the Jordan-Chevalley decomposition of an element $x \in L_0$. Then s and n are in L_0.

In effect, $L_0 = \{x \in L | \mathrm{ad}_x(H) = 0\}$. Thus, by the property of Jordan decomposition, see 8.1.1(3), both ad_s and ad_n satisfy the same property.

Step 2. If $x \in L_0$ is semisimple then $x \in H$. This follows from maximality of H: x commutes with H, therefore $H \oplus k \cdot x$ consists of semisimple elements.

Step 3. The restriction of K to H is nondegenerate. Let $h \in H$ and let $K(h, H) = 0$. We have to prove that $h = 0$. We will first check that $K(h, L_0) = 0$ and then we deduce $h = 0$ from the nondegeneracy of $K|_{L_0}$. For a general $x \in L_0$ one has $x = s + n$ where $s \in H$ by Step 2. Obviously, $\mathrm{tr}(\mathrm{ad}_h \cdot \mathrm{ad}_n) = 0$ since h and n commute and n is nilpotent. Thus, $K(h, x) = K(h, s) + K(h, n) = 0$ for all x.

Step 4. L_0 is nilpotent. By Engel theorem it suffices to check that ad_x is nilpotent for all $x \in L_0$. This is true for ad_s since $s \in H$ so $\mathrm{ad}_s = 0$ and this is true for ad_n. Therefore, this is true for ad_x.

Step 5. $H \cap [L_0, L_0] = 0$. In effect, $K(H, [L_0, L_0]) = 0$ since K is invariant. If $h \in H \cap [L_0, L_0]$, $K(h, H)$ would vanish, therefore, h would vanish since $K|_H$ is nondegenerate.

Step 6. L_0 is commutative. Otherwise $[L_0, L_0] \neq 0$ and then by Engel theorem 5.3.3 $[L_0, L_0] \cap Z(L_0) \neq 0$. Let $x \in [L_0, L_0] \cap Z(L_0)$ and let $x = n + s$ be the Jordan decomposition. One has $n \neq 0$ since otherwise x would be semisimple, which is impossible by Steps 2 and 5. The nilpotent element n belongs to L_0 and therefore to the center of L_0 by the properties of the Jordan decomposition.

Then $K(n, x) = \mathrm{tr}(ad_n \cdot \mathrm{ad}_x) = 0$ for all $x \in L_0$ which contradicts to the nondegeneracy of $K|_{L_0}$.

Step 7. Finally, assume $L_0 \neq H$. Then there exists a nonzero nilpotent element $x \in L_0$ by Steps 1,2. Then $K(x, y) = \mathrm{tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y) = 0$ for all $y \in L_0$ since ad_x is nilpotent and commutes with ad_y. This contradicts nondegeneracy of $K|_{L_0}$.

□

11.2. **Root space decomposition.** Let L be a semisimple complex Lie algebra. Recall that the Killing form

$$K(x, y) := \mathrm{tr}(\mathrm{ad}x \cdot \mathrm{ad}y)$$

is a non-degenerate invariant bilinear form on Ł.

11.2.1. Recall that a subalgebra $H \subset L$ is called toral if it consists of semisimple elements:

$$\forall x \in H \quad \mathrm{ad}x : L \to L \text{ is a semsimple linear operator.}$$

We have shown that any toral subalgebra is commutative and that a maximal toral subalgebra coincides with its centralizer. Moreover the restriction of the Killing form K to a maximal toral subalgebra is non-degenerate.

11.2.2. Fix a maximal toral subalgebra H in L and denote by Δ the set of roots:

$$L = H \oplus \bigoplus_{\alpha\in\Delta} L_\alpha.$$

Recall that the restriction of K to H is nondegenerate. This means that for each $\alpha \in H^*$ there exists a unique element $t_\alpha \in H$ such that $K(t_\alpha, h) = \alpha(h)$ for $h \in H$.

The set of roots Δ satisfies the following properties.

11.2.3. **Proposition.** 1. *The set $\Delta \subset H^*$ spans H^*.*
2. *$\alpha \in \Delta$ iff $-\alpha \in \Delta$.*
3. *If $\alpha \in \Delta$ then $[L_\alpha, L_{-\alpha}]$ is one-dimensional spanned by t_α.*
4. *$\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Delta$.*
5. *Let $\alpha \in \Delta$, $0 \neq x \in L_\alpha$. There exists an element $y \in L_{-\alpha}$ such the triple $(x, y, h_\alpha = [x, y])$ generate a subalgebra of L isomorphic to $\mathfrak{sl}_2$ and $h_\alpha = \frac{2t_\alpha}{K(t_\alpha,t_\alpha)}$.*

Proof. 1. If Δ does not span H^*, there exists a non-zero element $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. This implies that h commutes with the elements of L_α for all $\alpha \in \Delta$. Then h is central. Since the center of L is trivial, this leads to a contradiction.

2. Since $K(L_\alpha, L_\beta) = 0$ for $\beta \neq -\alpha$ and since K is nondegenerate, Δ is symmetric with respect to $\alpha \mapsto -\alpha$.

3. Let $x \in L_\alpha,\ y \in L_{-\alpha}$ and let $h \in H$. One has

$$K(h, [x, y]) = K([h, x], y) = \alpha(h)K(x, y) = K(t_\alpha, h)K(x, y) = K(K(x, y)t_\alpha, h).$$

This implies that H is orthogonal to $[x, y] - K(x, y)t_\alpha$ which in turn yields

$$[x, y] = K(x, y)t_\alpha.$$

4. Assume $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) = 0$. Choose $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $K(x, y) = 1$ so that $[x, y] = t_\alpha$ and $[t_\alpha, x] = [t_\alpha, y] = 0$. One can consider $\mathrm{Span}\{x, y, t_\alpha\}$ as a solvable subalgebra of $\mathfrak{gl}(L)$; thus, its commutator containing t_α is nilpotent; since it is in H, it is as well semisimple, that is $\mathrm{ad}_{t_\alpha} = 0$ or t_α is in the center of L.

5. If $x \in L_\alpha$ and $y \in L_{-\alpha}$ so that $K(x, y) = c$, we have

$$[x, y] = ct_\alpha,\ [ct_\alpha, x] = c\alpha(t_\alpha)x,\ [ct_\alpha, y] = -c\alpha(t_\alpha)y.$$

Thus, if we set $c = \frac{2}{K(t_\alpha,t_\alpha)}$, we get the required $\mathfrak{sl}_2$-triple. □

11.2.4. Let S_α be the Lie subalgebra of L spanned by an element $x \in L_\alpha$, $y \in L_{-\alpha}$ such that $[x, y] = h_\alpha := \frac{2t_\alpha}{K(t_\alpha,t_\alpha)}$, and h_α. Note that we cannot, at the moment, claim that S_α so defined is unique.

The map $S_\alpha \to L$ is a map of Lie algebras and S_α acts on L via the adjoint action.

We know that L decomposes, as S_α-module, into a direct sum of irreducible modules whose structure we fortunately know.

Consider the vector subspace M of L of the form

$$M = H \oplus \bigoplus_{c\in\mathbb{C}^*} L_{c\alpha}.$$

This is a S_α-submodule since $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$. Since $h_\alpha(c\alpha) = 2c$ and the weights of M are integral (M is finite dimensional), $c \in \frac{1}{2}\mathbb{Z}$. Note that $H = \mathbb{C} \cdot h_\alpha \oplus H'$ where $H' = \{h \in H | \alpha(h) = 0\}$, and each element of H' generates a trivial S_α-submodule. Since S_α has weights $0, \pm 2$, and H' has codimension 1 in H, the sum $S_\alpha \oplus H'$ contains all irreducible S_α-modules of even weight (those having a zero weight space). In particular, we deduce that if $\alpha \in \Delta$ then $2\alpha \notin \Delta$. But then $\alpha/2 \notin \Delta$ as well. This implies that 1 is not a weight of M. Therefore, $M = H' \oplus S_\alpha$.

This immediately implies the following property of the set of roots Δ and corresponding Lie algebra.

Proposition. 1. *Let α and $c\alpha$ belong to a root system Δ. Then $c = \pm 1$.*
2. *For each $\alpha \in \Delta$ one has* $\dim\ L_\alpha = 1$.

As a corollary, we get, for each $\alpha \in \Delta$, a unique Lie subalgebra S_α isomorphic to $\mathfrak{sl}_2$, spanned by $L_\alpha, L_{-\alpha}$ and $h_\alpha \in H$.

11.3. **L as S_α-module.** A lot of information can be deduced from considering L as S_α- module. Let $\beta \neq \pm\alpha$ be an element of Δ. Define

$$M = \oplus_{k\in\mathbb{Z}} L_{\beta-k\alpha}.$$

Obviously M is an S_α-submodule of L. Its weights with respect to h_α are

$$(\beta - k\alpha)(h_\alpha) = \beta(h_\alpha) - 2k.$$

This implies that M is a simple S_α-module. This implies the following result.

11.3.1. **Proposition.** 1. *For any $\alpha, \beta \in \Delta$ one has $\beta(h_\alpha) \in \mathbb{Z}$.*
2. *The set of $k \in \mathbb{Z}$ for which $\beta - k\alpha \in \Delta$ is a segment $[-r, s]$ and $\beta(h_\alpha) = s - r$.*

11.4. **Δ is a root system.** We will now see that the set $\Delta \in H^*$ satisfies very special symmetricity properties. For any $\alpha \in \Delta$ we define a linear operator s_α in H^* by the formula

$$s_\alpha(\gamma) = \gamma - \gamma(h_\alpha)\alpha. \tag{15}$$

We immediately see that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha^2 = \mathrm{id}$. Thus, s_α is a reflection. We have

11.4.1. **Lemma.** *For any $\alpha \in \Delta$ one has $s_\alpha(\Delta) = \Delta$.*

Proof. $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha$. The number $\beta(h_\alpha)$ is equal to $s - r$ that always belongs to the segment $[-r, s]$, see 11.3.1(2). □

It is a good time to define the abstract notion of a root system.

11.4.2. **Definition.** A finite subset R of an Euclidean space V (that is, a real vector space with an inner product $\langle\ ,\ \rangle$) is called a root system if

1. R spans V and does not contain 0.
2. If α and $c\alpha$ belong to R then $c = \pm 1$.
3. For $\alpha, \beta \in R$ one has $\langle \alpha, \beta \rangle \in \mathbb{Z}$.
4. For any $\alpha \in R$ one has $s_\alpha(R) = R$. Here s_α is the orthogonal reflection of V carrying α to $-\alpha$.

11.4.3. Let us write down the explicit formula for s_α. The conditions are that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(v) = v$ whenever $\langle v, \alpha \rangle = 0$. This yields the formula

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

11.4.4. In order to see that Δ is a root system, we have to embed it into an Euclidean space so that the reflections s_α that we defined earlier by the formula $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha$ become the orthogonal reflections.

It can be easily done. Let us first define the real vector space V. This can be nothing but the real span of Δ, $V = \mathrm{Span}_{\mathbb{R}}(\Delta)$. Killing form on L restructs to a nondegenerate form on H. This induces an isomorphism $\kappa : H \to H^*$ which in turn gives a symmetric bilinear for on H^*. We will verify that its restriction to V is positively definite.

Look at the real vector space $H_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}(h_\alpha)$ spanned by all h_α, $\alpha \in \Delta$. It is easy to calculate the restriction of the Killing form on it. Given $h = \sum a_\alpha \alpha$, $a_\alpha \in \mathbb{R}$, one has $K(h, h) = \mathrm{tr}(ad_h \circ \mathrm{ad}_h) = \sum_{\alpha \in \Delta} \alpha(h)^2$. This is a nonnegative number as all $\alpha(h)$ are real. Thus, K induces a positively definite form on $H_{\mathbb{R}}$. The real vector space $H_{\mathbb{R}}$ spans H as h_α span H. On the other hand, if we choose an orthonormal basis in $H_{\mathbb{R}}$, it will become orthonormal in H, thereforem linearly independent over $\mathbb{C}$. This implies that $\dim_{\mathbb{R}} H_{\mathbb{R}} = \dim H$.

We are more interested in another real space, $V = \mathrm{Span}_{\mathbb{R}}(\Delta)$. Fortunately, the map $\kappa : H \to H^*$ carries $H_{\mathbb{R}}$ to V. Thus, the symmetric form on V induced by the Killing form is positive definite. This also proves that V spans H^* and $\dim_{\mathbb{R}}(V) = \dim(H)$.

It remains to verify that the formula

$$s_\alpha(v) = v - v(h_\alpha)\alpha$$

is an orthogonal reflection, with respect to the inner product. This follows from the formula

$$\kappa(h_\alpha) = \frac{2}{\langle \alpha, \alpha \rangle}\alpha.$$

11.5. Note the following fact without proof.

Theorem. *All maximal toral subalgebras are conjugate: if H, H' are maximal toral subalgebras of L then there exists an automorphism $\psi : L \to L$ such that $\psi(H) = H'$.*

This fact implies in particular that the root system defined by a semisimple Lie algebra is independent of the choice of a maximal toral subalgebra.

It turns out that, in the opposite direction, the root system Δ completely determines L (see the book of Serre for a proof).

To provide some detail, let $\Delta \subset V$ be a root system in a real vector space V. We denote H to the the complexification of the dual vector space V^*. The semisimple Lie algebra determined by the root system Δ will be, as a vector space, the direct sum

$$L = H \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

where L_α are one-dimensional vector spaces with a fixed generator x_α. The bracket should satisfy the properties

- $x_\alpha, x_{-\alpha}] = h_\alpha \in H$.
- $[x_\alpha, x_\beta] = 0$ if $\alpha + \beta \notin \Delta$.
- $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ for certain constants $c_{\alpha,\beta}$.

The remaining problem of finding the constants $c_{\alpha,\beta}$ is not very easy. Still, there is a unique algebra, up to isomorphism, with a given root system Δ.

Thus, classification of (complex) semisimple Lie algebras reduces to the classification of the root systems.

11.6. There is a full classification of the root systems. Irreducible root systems consist of four infinite series A_n, B_n, C_n, D_n ($n = \dim H$ is called the rank of a root system) and 5 exceptional series E_6, E_7, E_8, F_4, G_2.

We will list the rank one and rank two root systems

11.6.1. *Rank one, A_1.* This root systems consists of two vectors, α and $-\alpha$, in $\mathbb{R}^1$. It corresponds to the Lie algebra $\mathfrak{sl}_2$.

11.6.2. *Direct product.* Let Δ be a root system in V and Δ' a root system in R'. We define a root system $\Delta \times \Delta'$ in $V \oplus V'$ as the set of pairs (α, β) with $\alpha \in \Delta$ and $\beta \in \Delta'$.

Direct product of root systems corresponds to a direct product of semisimple Lie algebras. An algebra is simple if the corresponding root system is irreducible that is cannot be presented as a product of root systems.

11.6.3. *Rank two, $A_1 \times A_1$.* This one corresponds to the Lie algebra $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

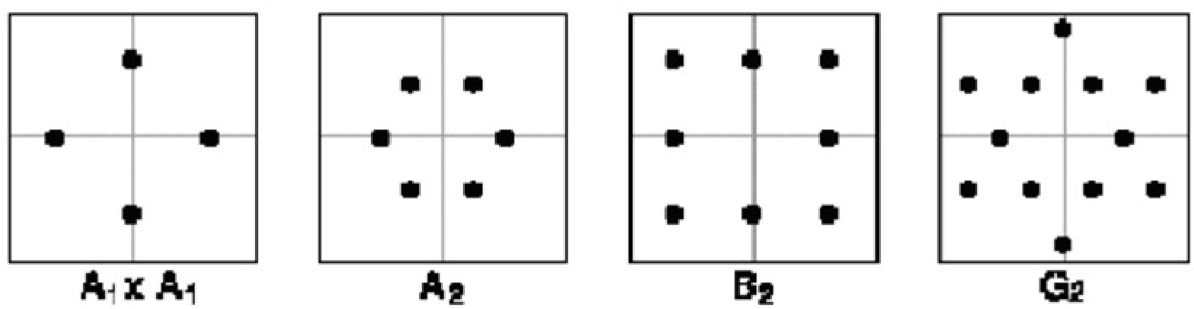
11.6.4. *Rank two, A_2.* This root systems consists of 6 vectors that form a right hexagon, see the picture on the next page.

11.6.5. *Rank two, B_2.* This root system has 8 elements, the vectors with coordinates $(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)$, see the picture on the next page.

11.6.6. *Rank two, G_2.* See the picture on the next page.

11.6.7. *Weyl group.* Recall that any root $\alpha \in \Delta$ gives rise to a reflection s_α so that $s_\alpha(\Delta) = \Delta$.

Let W be the subgroup of automorphisms of H^* generated by s_α. This is a finite group (should be proven) acting on Δ. It is called the Weyl group of the root system Δ (and of the corresponding semisimple Lie algebra L).


A1 x A1
A2
B2
G2

11.7. **Example: $\mathfrak{sl}(n)$.** To describe a root system of $\mathfrak{sl}(n)$, it is convenient to start from $\mathfrak{gl}(n)$. The latter is reductive: $\mathfrak{gl}(n) = \mathfrak{sl}(n) \times \mathbb{C}z$ and we can define a maximal toral subalgebra in the same manner; it is easy to check that such a subalgebra is of the form $\mathfrak{h}' = \mathfrak{h} \times \mathbb{C}z$ where $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{sl}(n)$.

The natural choice for $\mathfrak{h}'$ is the set of diagonal matrices

$$\mathfrak{h}' := \{\sum a_i E_{i,i},\ a_i \in \mathbb{C}\}.$$

The natural choice for $\mathfrak{h}$ is the set of traceless diagonal matrices

$$\mathfrak{h} := \{\sum a_i E_{i,i}|\ \sum a_i = 0\}.$$

Let $\{\epsilon_i\}_{i=1}^n$ be a basis of $(\mathfrak{h}')^*$ which is dual to the basis $E_{i,i}$. Since $\mathfrak{h}$ is a subspace of $\mathfrak{h}'$, the dual space $\mathfrak{h}^*$ may be naturally viewed as a factor space

$$\mathfrak{h}^* = span\{\epsilon_i\}_{i=1}^n / \sum_{i=1}^n \epsilon_i.$$

In this notation, one has

$$\Delta := \{\epsilon_i - \epsilon_j\}_{i \neq j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $(\epsilon_i - \epsilon_j)$ is

$$E_{i,j},\ h_{\epsilon_i - \epsilon_j} := E_{i,i} - E_{j,j},\ E_{j,i}.$$

The $(\epsilon_i - \epsilon_j)$-strings take form

$$\epsilon_k - \epsilon_i; \epsilon_k - \epsilon_j.$$

Therefore the integers $\beta(h_\alpha)$ are

$$(\epsilon_k - \epsilon_i)(h_{\epsilon_i - \epsilon_j}) = -1$$
$$(\epsilon_i - \epsilon_j)(h_{\epsilon_i - \epsilon_j}) = 2$$
$$(\epsilon_k - \epsilon_j)(h_{\epsilon_i - \epsilon_j}) = 1$$

where $k \neq i, j$ and zero for all remaining cases.

11.8. **Example: $\mathfrak{sp}(n)$ $(n = 2l)$.** This is a Lie subalgebra of $\mathfrak{gl}(n)$ which consists of all matrices T satisfying $TA + AT^t = 0$ where

$$A = \begin{pmatrix} 0 & | & I_l \\ -- & - & -- \\ -I_l & | & 0 \end{pmatrix}$$

and I_l stands for the identity $l \times l$ matrix.

The matrices in $\mathfrak{sp}(n)$ are of the form

$$T_{x,y,z} := \begin{pmatrix} x & | & y \\ -- & - & -- \\ z & | & -x^t \end{pmatrix}$$

where x, y, x are $l \times l$ matrices and y, z are symmetric: $y^t = y, z^t = z$. We have a natural embedding $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ $(x \mapsto T_{x,0,0})$.

Let $\mathfrak{h}$ be the set of diagonal matrices

$$\mathfrak{h} := \{\sum_{i=1}^{l} a_i(E_{i,i} - E_{l+i,l+i})\}$$

(it corresponds to $\mathfrak{h}'$ in the previous example). Retain notation for the dual basis.

Obviously this is a commutative Lie subalgebra. To check that $\mathfrak{h}$ is a maximal toral subalgebra, let us show that it consists of ad-semisimple elements and coincides with the own centralizer (so it is a maximal commutative subalgebra).

Indeed, if $x = E_{i,j}, y = z = 0$ then $T_{x,0,0}$ has weight $\epsilon_i - \epsilon_j$ (in this example i, j are assumed to be distinct integeres from 1 to l).

If $x = 0, y = E_{i,i}$ then $T_{0,y,0} = E_{i,l+i}$ and has weight $2\epsilon_i$.

Similarly, if $x = 0, z = E_{i,i}$ then $T_{0,0,z} = E_{l+i,i}$ and has weight $-2\epsilon_i$.

If $x = 0, y = E_{i,j} + E_{j,i}$ then $T_{0,y,0}$ has weight $\epsilon_i + \epsilon_j$.

If $x = 0, z = E_{i,j} + E_{j,i}$ then $T_{0,0,z}$ has weight $-(\epsilon_i + \epsilon_j)$.

Thus $\mathfrak{h}$ is a maximal toral subalgebra and

$$\Delta := \{\epsilon_i - \epsilon_j; \pm(\epsilon_i + \epsilon_j); \pm 2\epsilon_i\}.$$

We have the following $\mathfrak{sl}(2)$-triples. The $\mathfrak{sl}(2)$ triple corresponing to $(\epsilon_i - \epsilon_j)$ comes from $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ and takes form

$$E_{i,j} - E_{l+j,l+i},\ h_{\epsilon_i - \epsilon_j} := (E_{i,i} - E_{j,j}) - (E_{l+i,l+i} - E_{l+j,l+j}),\ E_{j,i} - E_{l+i,l+j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $2\epsilon_i$ is

$$E_{i,l+i}, h_{\epsilon_i + \epsilon_j} := E_{i,i} - E_{l+i,l+i}, E_{l+i,i}$$

Finally, the $\mathfrak{sl}(2)$ triple corresponing to $(\epsilon_i + \epsilon_j)$ is

$$E_{i,l+j} + E_{j,l+i},\ h_{\epsilon_i + \epsilon_j} := (E_{i,i} + E_{j,j}) - (E_{l+i,l+i} + E_{l+j,l+j}),\ E_{l+j,i} + E_{l+i,j}.$$

Examples of strings:

$$2\epsilon_2 - \text{string}: \quad \epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2;$$

$$(\epsilon_1 - \epsilon_2) - \text{string}: \quad 2\epsilon_2, \epsilon_1 + \epsilon_2, 2\epsilon_1;$$

$$(\epsilon_1 + \epsilon_2) - \text{string}: \quad -2\epsilon_2, \epsilon_1 - \epsilon_2, 2\epsilon_1.$$

The numbers $< \alpha, \beta >= \alpha(h_\beta)$:

$$< \epsilon_1 - \epsilon_2, 2\epsilon_2 >= -1, \quad < \epsilon_1 + \epsilon_2, 2\epsilon_2 >= 1$$

$$< 2\epsilon_2, \epsilon_1 - \epsilon_2 >= -2, \quad < 2\epsilon_2, \epsilon_1 + \epsilon_2 >= 2$$