

BASIC ALGEBRA - EXERCISE 1

1. Let $A = M_n(k)$ a matrix ring over a field k , $M = k^n$ the vector space of length n columns. Verify that M is a simple A -module.
2. Prove that A considered as a left A -module, is isomorphic to a direct sum of a number of copies of M .
3. Give an example of a \mathbb{Z} -module having no simple submodule.
4. Give an example of a monomorphism of \mathbb{Z} -modules that is not split.
5. Let \mathbb{Q} denote the field of rational numbers, $a, b \in \mathbb{Q}$. Define $\mathbb{H}_{a,b}$ to be the four-dimensional vector space over \mathbb{Q} with the basis $1, i, j, k$ and the multiplication given by the formulas

$$i^2 = a, j^2 = b, ij = k, ji = -k.$$

Write down the rest of the multiplication table. Verify that

- a. $\mathbb{H}_{-1,-1}$ is a division algebra.
 - b. $\mathbb{H}_{2,2}$ is a division algebra.
6. (Weyl algebra) Let $V = F[x]$ ring of polynomials over a field F . Let $E = \text{End}_F(V)$ be the ring of F -linear endomorphisms of V . We define two elements $X, D \in E$ as follows: X is a multiplication by x , so that $X(x^n) = x^{n+1}$. D is the derivative, that is $D(x^n) = nx^{n-1}$. We define A as the span of all monomials $X^n D^m$ where $m, n \geq 0$.

Prove that A is a subring of E . To do so, one has to verify that the product $X^n D^m X^{n'} D^{m'}$ can be expressed as a linear combination of $X^n D^m$.

Hint. Use induction, starting from $DX = XD + 1$.