THE OUTER BREGMAN PROJECTION METHOD FOR STOCHASTIC FEASIBILITY PROBLEMS IN BANACH SPACES

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We present an iterative method for solving, eventually infinite systems of inequalities $g(\omega, x) \leq 0$, a.e.(Ω), in which the functions $g(\omega, \cdot)$ are defined, convex, lower semicontinuous and essentially uniformly bounded on a Banach space. We show that the proposed method produces sequences which accumulate weakly to solutions of the system, provided that such solutions exist. In the most usual Banach spaces, the size of the constraint violations along the sequences generated by the iterative procedure converges in mean to zero. We prove that this method can be implemented for solving consistent operator equations (like first kind Fredholm or Volterra equations, some potential equations, etc.) when they have solutions in Hilbert, Lebesgue or Sobolev spaces.

1. INTRODUCTION

In this paper we consider the following *stochastic convex feasibility problem*: Given a complete probability space $(\Omega, \mathcal{A}, \mu)$, a uniformly convex, separable and smooth Banach space X and a function $g : \Omega \times X \to \mathbb{R}$ satisfying the following conditions:

$(C1)$ For each $x \in X$, the function $g(\cdot, x)$ is measurable;

$(C2)$ For each $\omega \in \Omega$, the function $g(\omega, \cdot)$ is convex and lower semicontinuous;

$(C3)$ The family of functions $g(\omega, \cdot)$, $\omega \in \Omega$, is essentially uniformly bounded on bounded sets, that is, for any bounded set $E \subseteq X$, there exists a real number $M_E > 0$ such that, for all $x \in E$, $|g(\omega, x)| \leq M_E$, a.e.;

Find a point $x^* \in X$ such that

$$g(\omega, x^*) \leq 0, \text{ a.e.}, \tag{1}$$

provided that such a point exists. One should note that this problem is equivalent to that of finding an almost common point (see [5]) of the sets

$$Q_\omega := \{x \in X : g(\omega, x) \leq 0\}, \ \omega \in \Omega, \tag{2}$$

*The authors gratefully acknowledge the financial support of the Israel Science Foundation. Also, they are grateful to Alfredo N. Iusem and Simeon Reich for comments and suggestions which led to improvements of earlier versions of this paper.

where, according to $(C1)$, $(C2)$ and Theorem 8.2.9 in [3], the point-to-set mapping $\omega \to Q_\omega : \Omega \to X$ is closed, convex valued and measurable.

Our purpose is to show that, in the specified context, the algorithm described below, which we call the *Outer-Bregman-Projection Method (OBPM)*, is well defined and produces bounded sequences whose weak accumulation points are solutions of the given stochastic convex feasibility problem. Moreover, under some additional conditions which, as we will show, are satisfied in the most useful spaces like the Hilbert spaces, the Lebesgue spaces $\mathcal{L}^p$ and the Sobolev spaces $\mathcal{W}^{m,p}$ with $p \in (1,\infty)$, the sequences $\{x^k\}_{k\in\mathbb{N}}$ produced by the OBPM have the property that the size of the constraint violation at each iterative step, $\max\left[g(\omega,x^k),0\right]$, converges in mean to zero.

The OBPM is the following procedure: Fix a number $r \in (1,\infty)$, choose an *initial point* $x^0 \in X$ and, for any nonnegative integer k, choose a measurable selector Γ_k of the point-to-set mapping $\omega \to \partial g(\omega,\cdot)(x^k)$; then, let

$$x^{k+1} = \int_\Omega J_r^*\left(s_k(\omega)\Gamma_k(\omega) + J_r x^k\right) d\mu(\omega), \tag{3}$$

where $J_r : X \to X^*$ and $J_r^* : X^* \to X$ are the duality mappings (see [13]) of weights $\psi_r(t) = rt^{r-1}$ and $\psi_r^{-1}(t) = \left(\frac{t}{r}\right)^{1/(r-1)}$, respectively, and $s_k(\omega)$ is a (necessarily existing) negative solution of the equation

$$\langle \Gamma_k(\omega), J_r^*(s_k(\omega)\Gamma_k(\omega) + J_r x^k) - x^k\rangle = -g(\omega,x^k), \tag{4}$$

when $g(\omega,x^k) > 0$ and $s_k(\omega) = 0$, otherwise.

The geometric idea underlying the OBPM consists of obtaining the new iterate x^{k+1} by averaging the Bregman projections of the current iterate x^k with respect to the function $f(x) = \|x\|^r$ onto the closed half space

$$H_k(\omega) = \{y \in X : \langle \Gamma_k(\omega), y - x^k\rangle + g(\omega,x^k) \le 0\}, \tag{5}$$

which contains the closed convex set Q_ω. Recall that, according to [12], the Bregman projection of $x \in X$ with respect to $f(x) = \|x\|^r$ onto a closed, convex, nonempty set $C \subseteq X$ is the (necessarily unique) point

$$\Pi_C^f(x) = \arg\min\left\{D_f(y,x) : y \in C\right\},$$

where $D_f : X \times X \to [0,\infty)$, usually called *Bregman distance,* is given by

$$D_f(y,x) = \|y\|^r - \|x\|^r - \langle f'(x), y - x\rangle, \tag{6}$$

with $f' : X \to X^*$ being the (Gâteaux) derivative of f.

Compared with the methods of solving the stochastic convex feasibility problem (1) which are based on averaging Bregman projections onto the sets Q_ω (see [5], [10], [9], [6], [7], [8] and the references therein) the OBPM presents the advantage that it does not require determining difficult to compute minimizers of nonlinear functions over the sets Q_ω. For the most usual smooth, separable and uniformly convex Banach spaces there are already known formulae for calculating the duality mappings J_r and J_r^* involved in the previous procedure (see [13, pp. 72-73]). Also, a careful analysis of the proof of Theorem 2 in [11], which guarantees the existence of the numbers $s_k(\omega)$ required by the

OBPM, shows that finding the negative solutions $s_k(\omega)$ of (4) amounts to finding zeros of continuous and increasing real functions and this can be done by standard methods (e.g., by Newton's method). Therefore, the implementation of OBPM is quite easy.

The idea of solving convex feasibility problems by using Bregman projections onto half spaces containing the sets Q_ω goes back to [12] and was repeatedly used since then - see, for instance, [15], [17], [18], [19], [20]. Expanding this idea to (possibly infinite) stochastic feasibility problems in infinite dimensional Banach spaces is now possible due to recent results concerning the integrability of measurable families of totally nonexpansive operators (see [9, Chapter 2]) and due to the results established in [11] concerning the total convexity of the powers of the norm in uniformly convex Banach spaces and the relatively simple formulae of computing Bregman projections with respect to them. Guaranteeing that the OBPM works in this more general setting is of interest because of the practically significant applied mathematical problems whose straightforward formulation and natural environment is that of an infinite system of convex inequalities in a uniformly convex Banach space. It was shown in [9, Section 2.4] that finding minima of convex continuous functions, finding Nash equilibria in n-person noncooperative games, solving significant classes of variational inequalities and solving linear operator equations $Tx = b$ in some function spaces is equivalent to solving infinite systems of inequalities (1). Such problems are, usually, unstable in the sense that their "discretization" to finite feasibility problems may lead to "approximate solutions" which are far from the solution set of the problem itself (this phenomenon can be observed, for instance, when one reduce Fredholm integral equations to finite systems of equalities). The use of OBPM for solving the problems in their intrinsic setting may avoid such difficulties.

It is of practical interest to observe that the OBPM described above can be used for solving stochastic convex feasibility problems which are not directly given in the form (1). That is the case of the problem of finding an almost common point of a family of closed convex sets R_ω, $\omega \in \Omega$, where the mapping $\omega \to R_\omega : \Omega \to X$ is measurable. Such problems appear, for instance, from operatorial equations in the form $T(\omega, x) = u(\omega)$ a.e., where $T(\omega, \cdot)$, $\omega \in \Omega$, are continuous linear operators from X to a Banach space Y and the function $u : \Omega \to Y$ as well as the functions $T(\cdot, x)$ whenever $x \in X$, are measurable. Solving this operatorial equation amounts to finding an almost common point of the closed convex sets

$$R_\omega = \{x \in X : T(\omega, x) = u(\omega)\}. \tag{7}$$

In cases like that one can re-write the problem in the form of a system of inequalities (1) with

$$g(\omega, x) := d^2(x, R_\omega), \tag{8}$$

where $d^2(x, A)$ stands for the square of the distance from $x \in X$ to the set $A \subseteq X$. According to [3, Theorem 8.2.11] and [14, Proposition 2.4.1], this function g satisfies the conditions $(C1)$, $(C2)$ and $(C3)$ required above. Moreover, since X is uniformly convex and smooth, the functions $g(\omega, \cdot)$ given by (8) are differentiable and

$$\partial g(\omega, \cdot)(x) = \{J_2(x - P_\omega x)\},$$

where $P_\omega : X \to R_\omega$ is the metric projection operator onto the set R_ω. Thus, in this situation, the OBPM described by (3) reduces to

$$x^{k+1} = \int_\Omega J_r^*\left(s_k(\omega)J_2(x^k - P_\omega x^k) + J_r x^k\right) d\mu(\omega),$$

and it is effectively calculable whenever the metric projections P_ω are so.

A computational feature of the OBPM is its implementability via parallel processing. At each step k of the iterative procedure defined by (3) one can compute in parallel the vector $J_r x^k$ and the subgradients $\Gamma_k(\omega)$ at various points $\omega \in \Omega$ and, then, one can proceed by parallel processing to solve the corresponding equation (4) in order to determine the numbers $s_k(\omega)$.

2. CONVERGENCE ANALYSIS OF THE OBPM

2.1. In this section we consider the stochastic feasibility problem (1) under the conditions $(C1)$, $(C2)$ and $(C3)$ given in Section 1. Our objective is to prove well definedness of the OBPM and to establish useful convergence properties for the sequences it generates. To this end, recall that the *modulus of total convexity of the function f at the point $x \in X$* (see [9, Section 1.2]) is defined by

$$\nu_f(x,t) = \inf\left\{D_f(y,x) : \|y-x\| = t\right\}. \tag{9}$$

We associate with any nonempty set $E \subseteq X$ the function $\nu_f(E,\cdot) : [0,\infty) \to [0,\infty)$ given by

$$\nu_f(E,t) = \inf\left\{\nu_f(x,t) : x \in E\right\}. \tag{10}$$

Clearly, $\nu_f(E,0) = 0$. It was shown in [11] that for the function $f(x) = \|x\|^r$ with $r \in (1,\infty)$, the function $\nu_f(E,\cdot)$ is strictly positive on $(0,\infty)$ when E is bounded. With these facts in mind we state the following result.

Theorem. *For each $r \in (1,\infty)$ the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ generated by the OBPM in a uniformly convex, separable and smooth Banach space X is well defined no matter how the initial point $x^0 \in X$ is chosen. Moreover, the sequence $\left\{x^k\right\}_{k\in\mathbb{N}}$ has the following properties:*

(I) If the stochastic feasibility problem (1) *has a solution $z \in X$ such that the function $D_f(z,\cdot)$ is convex on X, then $\{x^k\}_{k\in\mathbb{N}}$ is bounded, has weak accumulation points, any weak accumulation point x^* of it is a solution of the stochastic convex feasibility problem* (1)*, the following limit exists and we have*

$$\lim_{k\to\infty}\left\|x^{k+1} - x^k\right\| = 0. \tag{11}$$

(II) If, in addition to the requirements of (I), the function $f(x) = \|x\|^r$ has the property that for each nonempty bounded set $E \subseteq X$ there exists a function $\eta_E : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ such that, for any $a \in [0,+\infty)$, the function $\eta_E(a,\cdot)$ is continuous, convex, strictly increasing and

$$\nu_f(E,t) \geq \eta_E(a,t), \tag{12}$$

whenever $t \in [0, a]$, *then, for any OBPM generated sequence* $\{x^k\}_{k\in\mathbb{N}}$, *the size of constraint violations converges in mean to zero, i.e.,*

$$\lim_{k\to\infty} \int_\Omega \max\left[0, g(\omega, x^k)\right] d\mu(\omega) = 0. \tag{13}$$

The proof of this result consists of a sequence of lemmas which is given below.

2.2. We start our proof by establishing the following technical facts.

Lemma. *For any* $x \in X$ *the following statements hold:*
(i) *The point-to-set mapping* $G : \Omega \to X^*$ *given by*

$$G(\omega) = \partial g(\omega, \cdot)(x) \tag{14}$$

is nonempty, convex, closed valued, measurable, and it has a measurable selector;
(ii) *If* $\omega \in \Omega$ *and* $\gamma \in \partial g(\omega, \cdot)(x)$, *then the set* Q_ω, *given by* (2), *is contained in the closed half space*

$$N(\omega, \gamma, x) = \{y \in X : \langle \gamma, y - x\rangle + g(\omega, x) \le 0\}. \tag{15}$$

(iii) *If* $\Gamma : \Omega \to X^*$ *is a measurable selector of the mapping* G *given by* (14), *then the point-to-set mapping* $H^\Gamma : \Omega \times X \to X$ *defined by*

$$H^\Gamma(\omega, x) = N(\omega, \Gamma(\omega), x), \tag{16}$$

has the property that $H^\Gamma(\cdot, x)$ *is measurable.*

Proof. (i) Note that, for each $\omega \in \Omega$, the set $G(\omega)$ is convex and weakly* closed in X^* (cf. Proposition 1.11 in [23]). Consequently, $G(\omega)$ is closed in X^*. Let $\{y^k\}_{k\in\mathbb{N}}$ be a dense countable subset of X (such a subset exists because X is separable). For each $k \in \mathbb{N}$, define the function $F_k : \Omega \times X^* \to \mathbb{R}$ by

$$F_k(\omega, \zeta) := \langle \zeta, y^k - x\rangle - g(\omega, y^k) + g(\omega, x)$$

and put

$$G_k(\omega) := [F_k(\omega, \cdot)]^{-1}(-\infty, 0].$$

It is easy to verify that

$$G(\omega) = \bigcap_{k=1}^{\infty} G_k(\omega). \tag{17}$$

Since X is reflexive and separable, Theorem 1.14 in [1] applies and it shows that X^* is separable. Therefore, one can apply Theorem 8.2.9 in [3] to the Carathéodory mapping F_k and deduce that the point-to-set mappings $\omega \to G_k(\omega)$ are measurable for all $k \in \mathbb{N}$. Consequently, Theorem 8.2.4 in [3] combined with (17) show that the point-to-set mapping G defined by (14) is measurable, too. According to [3, Theorem 8.1.3], the measurable mapping G has a measurable selector.

(ii) For any $y \in Q_\omega$, we have $g(\omega, y) \leq 0$ and, therefore,

$$-g(\omega, x) \geq g(\omega, y) - g(\omega, x) \geq \langle \gamma, y - x \rangle .$$

(iii) Consider the function $\Phi_x : \Omega \times X \to \mathbb{R}$ given by

$$\Phi_x(\omega, y) = \langle \Gamma(\omega), y - x \rangle + g(\omega, x).$$

For any $y \in X$, the function $\omega \to \langle \Gamma(\omega), y - x \rangle$ is measurable and, therefore, so is the function $\Phi_x(\cdot, y)$. For each $\omega \in \Omega$, the function $\Phi_x(\omega, \cdot)$ is continuous because $g(\omega, \cdot)$ is continuous (it is convex and lower semicontinuous on X). Hence, Φ_x is a Carathéodory function to which Theorem 8.2.9 in [3] applies and it shows that the point-to-set mapping $\omega \to [\Phi_x(\omega, \cdot)]^{-1}((-\infty, 0])$ is measurable. Since we have

$$H^\Gamma(\omega, x) = [\Phi_x(\omega, \cdot)]^{-1}((-\infty, 0]),$$

for any $\omega \in \Omega$, it results that the point-to-set mapping $H^\Gamma(\cdot, x)$ is measurable. □

2.3. A consequence of Lemma 2.2 is that the OBPM generated sequences are well defined as shown by the following result.

Lemma. *For any initial point $x^0 \in X$ and for each integer $k \geq 0$, there exists a measurable selector Γ_k of the point-to-set mapping $\omega \to \partial g(\omega, \cdot)(x^k)$ and the vector x^{k+1} given by* (3) *is well defined.*

Proof. We proceed by induction upon k. Let $k = 0$ and observe that, according to Lemma 2.2(i), there exists a measurable selector Γ_k of the point-to-set mapping $\omega \to \partial g(\omega, \cdot)(x^k)$. According to [11, Theorem 2], when $g(\omega, x^k) > 0$, the Bregman projection of x^k with respect to the function $f(x) = \|x\|^r$ onto the half space $H_k(\omega) := H^{\Gamma_k}(\omega, x^k)$, given by (5), exists and it is exactly

$$\Pi^f_{H_k(\omega)}(x^k) = J_r^* \left[s_k(\omega)\Gamma_k(\omega) + J_r x^k \right] . \tag{18}$$

Applying Proposition 2.1.5 in [9] one deduces that the family of operators $x \to \Pi^f_{H_k(\omega)}(x) : X \to X$, $\omega \in \Omega$, is totally nonexpansive with respect to the function f. This family of operators is also measurable because of [3, Theorem 8.2.11]. Consequently, one can apply Corollary 2.2.5 in [9] and it shows that the function $\omega \to \Pi^f_{H_k(\omega)}(x^k)$ is integrable, that is, the integral in (3) exists and x^{k+1} is well defined.

Now, assume that for some $k > 0$ the vector x^k is well defined. Then, repeating the reasoning above with x^k instead of x^0, we deduce that the measurable selector Γ_k exists and the vector x^{k+1} is well defined. □

2.4. Our further argumentation is based on several properties of the Bregman distance which are summarized in the next lemma. Note that, according to (6), for any $x \in X$, the function $D_f(\cdot, x)$ is continuous and convex. Continuity of the function $D_f(y, \cdot)$ for any $y \in X$ is established below. Convexity of this function occurs only in special circumstances as we will show later. It was noted in Subsection 2.1 that the function $f(x) = \|x\|^r$ with $r > 1$ has the property that the function $\nu_f(E, \cdot)$ is positive on $(0, \infty)$, whenever $E \subset X$ is nonempty and bounded. Strict monotonicity of this function, an essential feature in our analysis of OBPM, is established here.

Lemma. (i) *For any* $y \in X$ *the function* $D_f(y,\cdot)$ *defined by* (6) *is continuous on* X *and we have*

$$D_f(y,x) = \|y\|^r + (r-1)\|x\|^r - \langle J_r(x), y\rangle ; \tag{19}$$

(ii) *If the set* $E \subset X$ *is nonempty and* $c \in [1,\infty)$, *then* $\nu_f(E,ct) \geq c\nu_f(E,t)$, *for all* $t \in [0,\infty)$;

(iii) *If the set* $E \subset X$ *is nonempty and bounded, then the function* $\nu_f(E,\cdot)$ *is strictly increasing on* $[0,\infty)$.

Proof. (i) In order to prove continuity of $D_f(y,\cdot)$, let $\{u^k\}_{k\in\mathbb{N}}$ be a convergent sequence in X whose limit is u. Then, we have

$$\begin{aligned} 0 \leq |D_f(y,u^k) - D_f(y,u)| &\leq |f(u^k) - f(u)| + |\langle f'(u^k) - f'(u), y\rangle| \\ &\quad + |\langle f'(u^k), u^k\rangle - \langle f'(u), u\rangle|, \end{aligned}$$

where, obviously, the first and the second term converge to zero as $k \to \infty$ because f is continuous and f' is norm-to-weak* continuous (see [23, Proposition 2.8]), respectively. Taking into account that

$$\begin{aligned} 0 \leq |\langle f'(u^k), u^k\rangle - \langle f'(u), u\rangle| &\leq |\langle f'(u^k), u^k - u\rangle| + |\langle f'(u^k) - f'(u), u\rangle| \\ &\leq \|f'(u^k)\|_* \|u^k - u\| + |\langle f'(u^k) - f'(u), u\rangle| \end{aligned}$$

and using the boundedness of the weakly* convergent sequence $\{f'(u^k)\}_{k\in\mathbb{N}}$, we obtain that the sum on the right hand-side converges to zero as $k \to \infty$. Therefore,

$$\lim_{k\to+\infty} |D_f(y,u^k) - D_f(y,u)| = 0,$$

and the continuity of $D_f(y,\cdot)$ is proved.

Now, for proving (19), we apply Asplund's Theorem (see, for instance, [13, p. 18]) which implies that, in the current setting, $f' = J_r$. Hence, we have

$$\begin{aligned} D_f(y,x) &= \|y\|^r - \|x\|^r - \langle J_r(x), y - x\rangle \\ &= \|y\|^r - \|x\|^r + \langle J_r(x), x\rangle - \langle J_r(x), y\rangle \\ &= \|y\|^r + (r-1)\|x\|^r - \langle J_r(x), y\rangle, \end{aligned}$$

because

$$\langle J_r(x), x\rangle = \psi_r(\|x\|)\,\|x\| = r\,\|x\|^r .$$

(ii) results from [9, Proposition 1.2.2] which shows that, for any $x \in X$, we have $\nu_f(x,ct) \geq c\nu_f(x,t)$, whenever $c \geq 1$.

(iii) If $0 < t_1 < t_2 < \infty$, then, applying (ii), we obtain

$$\nu_f(E,t_2) = \nu_f(E, \frac{t_2}{t_1}t_1) \geq \frac{t_2}{t_1}\nu_f(E,t_1).$$

Since, as noted in Subsection 2.1, $\nu_f(E,t_1) > 0$ because E is nonempty and bounded, we deduce

$$\nu_f(E,t_2) \geq \frac{t_2}{t_1}\nu_f(E,t_1) > \nu_f(E,t_1)$$

and the proof is complete. □

2.5. Lemma 2.3 guarantees that, no matter how the initial point x^0 is chosen in X, the OBPM generated sequence $\{x^k\}_{k\in\mathbb{N}}$ with the initial point x^0 is well defined. From now on we assume that for some solution z of (1) the function $D_f(z,\cdot)$ is convex. The next result shows that, under these circumstances, any sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by the OBPM has some of the properties listed at point (I) of Theorem 2.1.

Lemma. *Any sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by the OBPM satisfies*

$$D_f(z,x^{k+1}) + D_f(x^{k+1},x^k) \le D_f(z,x^k). \tag{20}$$

Also, $\{x^k\}_{k\in\mathbb{N}}$ is bounded, has weak accumulations points, the following limits exist and

$$\lim_{k\to\infty} \left\|x^{k+1}-x^k\right\| = \lim_{k\to\infty} D_f(x^{k+1},x^k) = \lim_{k\to\infty} \int_\Omega D_f(\Pi^f_{H_k(\omega)}(x^k),x^k) = 0. \tag{21}$$

Proof. Let $z\in X$ be a solution of the stochastic convex feasibility problem such that $D_f(z,\cdot)$ is convex. It was noted in the proof of Lemma 2.3 that the Bregman projections $\Pi^f_{H_k(\omega)}(x^k)$ exist and are given by (18). According to [9, Proposition 2.1.5] we also have that, for each $k\in\mathbb{N}$,

$$D_f(z,\Pi^f_{H_k(\omega)}(x^k)) + D_f(\Pi^f_{H_k(\omega)}(x^k),x^k) \le D_f(z,x^k), \tag{22}$$

for all $\omega\in\Omega$. The functions $D_f(\cdot,x^k)$ and $D_f(z,\cdot)$ are convex and continuous (cf. Lemma 2.4(i)). Therefore, by integrating the inequality (22) with respect to $\omega\in\Omega$ and applying Jensen's inequality we obtain (20). From (20) we see that the nonnegative sequence $\{D_f(z,x^k)\}_{k\in\mathbb{N}}$ is nonincreasing and, thus, convergent. Now, using Jensen's inequality again in (22) we deduce that

$$0 \le \int_\Omega D_f(\Pi^f_{H_k(\omega)}(x^k),x^k)d\mu(\omega) \le D_f(z,x^k) - D_f(z,x^{k+1}),$$

where, according to (20), the right hand side converges to zero as $k\to\infty$. Consequently,

$$\lim_{k\to\infty} \int_\Omega D_f(\Pi^f_{H_k(\omega)}(x^k),x^k)d\mu(\omega) = 0.$$

Since

$$0 \le D_f(x^{k+1},x^k) \le \int_\Omega D_f(\Pi^f_{H_k(\omega)}(x^k),x^k)d\mu(\omega),$$

we also have that

$$\lim_{k\to\infty} D_f(x^{k+1},x^k) = 0. \tag{23}$$

The monotonicity of the sequence $\{D_f(z,x^k)\}_{k\in\mathbb{N}}$ which results from (20) shows that the entire sequence $\{x^k\}_{k\in\mathbb{N}}$ is contained in the set

$$R^f_0(z) := \{y\in X : D_f(z,y) \le D_f(z,x^0)\}. \tag{24}$$

According to Corollary 1(i) in [11], this set is bounded. Hence, the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded and, therefore, it has weak accumulation points (since the space X is reflexive). Let E be the bounded set of all terms of the sequence $\{x^k\}_{k\in\mathbb{N}}$. From (10) and (9) we get that

$$0 \leq \nu_f(E, \|x^{k+1} - x^k\|) \leq \nu_f(x^k, \|x^{k+1} - x^k\|) \leq D_f(x^{k+1}, x^k),$$

for all $k \in \mathbb{N}$. This and (23) implies that

$$\lim_{k\to\infty} \nu_f(E, \|x^{k+1} - x^k\|) = 0.$$

According to Lemma 2.4(iii), this can not happen unless $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$. □

2.6. Now we are in position to complete the proof of point (I) of the theorem.

Lemma. *If x^* is a weak accumulation point of a OBPM generated sequence $\{x^k\}_{k\in\mathbb{N}}$, then x^* is a solution of the stochastic convex feasibility problem* (1).

Proof. Denote by

$$\theta_k(\omega) = \|\Pi^f_{H_k(\omega)}(x^k) - x^k\|. \tag{25}$$

Let E be the set of all terms of the sequence $\{x^k\}_{k\in\mathbb{N}}$. This set is bounded (cf. Lemma 2.5). Note that $\omega \to \nu_f(E, \theta_k(\omega))$ is measurable because it is the composition of the monotone function $\nu_f(E, \cdot)$ (see Lemma 2.4(iii)) with the function θ_k which, according to [3, Theorem 8.2.11], is measurable. From (9) and (10) we have

$$0 \leq \int_\Omega \nu_f(E, \theta_k(\omega)) d\mu(\omega) \leq \int_\Omega D_f(\Pi^f_{H_k(\omega)}(x^k), x^k) d\mu(\omega),$$

for all $k \in \mathbb{N}$. This, together with Lemma 2.5, implies that

$$\lim_{k\to\infty} \int_\Omega \nu_f(E, \theta_k(\omega)) d\mu(\omega) = 0. \tag{26}$$

From (4) and (18) we have that

$$\begin{aligned} g(\omega, x^k) &\leq \left|\langle \Gamma_k(\omega), J_r^*(s_k(\omega)\Gamma_k(\omega) + J_r x^k) - x^k \rangle\right| \\ &\leq \|\Gamma_k(\omega)\|_* \left\|J_r^*(s_k(\omega)\Gamma_k(\omega) + J_r x^k) - x^k\right\| \\ &= \|\Gamma_k(\omega)\|_* \left\|\Pi^f_{H_k(\omega)}(x^k) - x^k\right\| = \|\Gamma_k(\omega)\|_* \theta_k(\omega). \end{aligned} \tag{27}$$

Let

$$E' = \bigcup_{k=0}^{\infty} \{y \in X : \|y - x^k\| \leq 2\}.$$

Since E is bounded the set E' is bounded too and, obviously, $E \subseteq E'$. For each $k \in \mathbb{N}$ and $\omega \in \Omega$ for which $\Gamma_k(\omega) \neq 0$ denote

$$x^k_\omega = x^k + \frac{1}{\|\Gamma_k(\omega)\|_*} J_2^*(\Gamma_k(\omega)).$$

Clearly, $x_\omega^k \in E'$ because $\|J_2^*(\Gamma_k(\omega))\| = 2\|\Gamma_k(\omega)\|_*$. Let $\overline{M} = M_{E'}$ be a positive real number such that

$$|g(\omega, x)| \leq \overline{M}, \text{ a.e.}$$

for each $x \in E'$. Such a number exists because of condition $(C3)$. Then, for each $k \in \mathbb{N}$ and for almost any $\omega \in \Omega$, we have

$$\begin{aligned} 2\overline{M} &\geq \left|g(\omega, x_\omega^k)\right| + \left|g(\omega, x^k)\right| \\ &\geq g(\omega, x_\omega^k) - g(\omega, x^k) \geq \left\langle \Gamma_k(\omega), x_\omega^k - x^k \right\rangle \\ &= \frac{1}{\|\Gamma_k(\omega)\|_*} \left\langle \Gamma_k(\omega), J_2^*(\Gamma_k(\omega)) \right\rangle = 2\|\Gamma_k(\omega)\|_*, \end{aligned}$$

provided that $\Gamma_k(\omega) \neq 0$. This shows that, for each $k \in \mathbb{N}$, we have $\|\Gamma_k(\omega)\|_* \leq \overline{M}$ for almost all $\omega \in \Omega$. Combining this and (27) we deduce that

$$g(\omega, x^k) \leq \overline{M}\theta_k(\omega), \text{ a.e.}, \tag{28}$$

for all $k \in \mathbb{N}$.

Let $\{x^{i_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{x^k\}_{k\in\mathbb{N}}$ whose weak limit is x^*. Note that, according to (26), the sequence of functions $\{\nu_f(E, \theta_{i_k}(\omega))\}_{k\in\mathbb{N}}$ converges in mean to zero. Therefore, $\{\nu_f(E, \theta_{i_k}(\omega))\}_{k\in\mathbb{N}}$ is fundamental in measure (cf. [21, Theorem 38.7]). Hence, there exists a set $\Omega_0 \subseteq \Omega$ and a subsequence $\{\nu_f(E, \theta_{j_k}(\omega))\}_{k\in\mathbb{N}}$ of $\{\nu_f(E, \theta_{i_k}(\omega))\}_{k\in\mathbb{N}}$ such that $\mu(\Omega_0) = 1$ and

$$\lim_{k\to\infty} \nu_f(E, \theta_{j_k}(\omega)) = 0, \tag{29}$$

for all $\omega \in \Omega_0$ (see Proposition 24.18 in [22]). Suppose, by contradiction, that there exists $\omega_0 \in \Omega_0$ such that

$$\overline{\lim}_{k\to\infty} \theta_{j_k}(\omega_0) > 0.$$

Then, for some positive real number ε_0 and for some subsequence $\left\{\theta_{j_{h_k}}(\omega_0)\right\}_{k\in\mathbb{N}}$ of $\{\theta_{j_k}(\omega_0)\}_{k\in\mathbb{N}}$ we have $\theta_{j_{h_k}}(\omega_0) \geq \varepsilon_0 > 0$ for all $k \in \mathbb{N}$. Applying Lemma 2.4(iii) we deduce that

$$\nu_f(E, \theta_{j_{h_k}}(\omega_0)) \geq \nu_f(E, \varepsilon_0) > 0,$$

for all $k \in \mathbb{N}$ and this contradicts (29). Hence, $\lim_{k\to\infty} \theta_{j_k}(\omega) = 0$, for almost all $\omega \in \Omega$. This, (28) and condition $(C2)$ taken together imply that, for almost all $\omega \in \Omega$, we have

$$g(\omega, x^*) \leq \underline{\lim}_{k\to\infty} g(\omega, x^{j_k}) \leq \overline{M} \lim_{k\to\infty} \theta_{j_k}(\omega) = 0,$$

that is, x^* is a solution of (1). □

2.7. We proceed to prove the statement (II) of our theorem.

Lemma. *Under the assumptions of (II) any weak accumulation point x^* of an OBPM generated sequence $\{x^k\}_{k\in\mathbb{N}}$ is a solution of problem (1) and the sequence of functions $\{\max[0, g(\cdot, x^k)]\}_{k\in\mathbb{N}}$ converges in mean to zero.*

Proof. Observe that (26) still holds with the functions $\theta_k(\omega)$ given by (25). Note that, according to (22), for almost all $\omega \in \Omega$ and for all $k \in \mathbb{N}$, the vectors $\Pi^f_{H_k(\omega)}(x^k)$ are contained in the bounded set $R^f_0(z)$ defined at (24). Since the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded (see Lemma 2.5), it results that there exists a real number $a > 0$ such that, for all $k \in \mathbb{N}$,

$$\theta_k(\omega) \le \|\Pi^f_{H_k(\omega)}(x^k)\| + \|x^k\| \le a, \text{ a.e.}$$

Hence, for each $k \in \mathbb{N}$,

$$\nu_f(E, \theta_k(\omega)) \ge \eta_E(a, \theta_k(\omega)), \text{ a.e.}$$

Combining this inequality with (26) and with Jensen's inequality, we deduce that

$$0 \le \lim_{k\to\infty} \eta_E\left(a, \int_\Omega \theta_k(\omega)d\mu(\omega)\right) \le \lim_{k\to\infty} \int_\Omega \eta_E(a, \theta_k(\omega))d\mu(\omega) = 0.$$

Since $\eta_E(a, \cdot)$ is strictly increasing, this cannot happen unless

$$\lim_{k\to\infty} \int_\Omega \theta_k(\omega) = 0.$$

Taking into account (28), which still holds in the current circumstances, we obtain that the sequence of functions $\left\{\max\left[0, g(\cdot, x^k)\right]\right\}_{k\in\mathbb{N}}$ converges in mean to zero. □

3. APPLICABILITY IN PARTICULAR BANACH SPACES

3.1. One of the most practically interesting features of the OBPM is that, under the assumption of Theorem 2.1(II), it produces sequences $\left\{x^k\right\}_{k\in\mathbb{N}}$ which not only accumulate weakly to solutions of the given stochastic convex feasibility problem, but also have the property that, along them, the size of the constraint violations converges in mean to zero. This section is aimed at showing that, in the most usual Banach spaces, i. e., in Hilbert spaces, Lebesgue spaces and Sobolev spaces, the main requirement of Theorem 2.1(II) is satisfied in the sense that there are numbers $r \in (1, \infty)$ such that the function $f(x) = \|x\|^r$ has the following property:

(*) *For any nonempty, bounded set $E \subset X$, there exists a function $\eta_E : [0, \infty)\times[0, \infty) \to [0, \infty)$ such that, for each real number $a > 0$, the function $\eta_E(a, \cdot)$ is strictly increasing, continuous, convex, and $\nu_f(E, t) \ge \eta_E(a, t)$ whenever $t \in [0, a)$.*

Combining this fact with the easy implementability of OBPM pointed out in Section 1, we observe that this algorithm is a potentially useful tool for solving stochastic convex feasibility problems and, in particular, optimization and equilibrium problems, variational inequalities, operator equations, etc. (see [9]).

Our argumentation below is based on the fact, established in [11, Formula (12)], that if X is an uniformly convex Banach space, then there exists a real number $K > 0$ such that, for all $z \in X$,

$$\nu_f(z, t) \ge rK \left(\frac{t}{2}\right)^r \int_0^1 \tau^{r-1}\delta_X\left(\frac{\tau t}{2\left(\|z\| + \tau t\right)}\right) d\tau, \tag{30}$$

where $\delta_X : [0,2] \to [0,1]$ stands for the modulus of uniform convexity of X.

3.2. We first show that in Hilbert spaces condition (*) holds.

Proposition. *If X is a Hilbert space and $r \in (1,\infty)$, then the function $f(x) = \|x\|^r$ has the property (*).*

Proof. Recall (see, for instance, [2]) that the modulus of uniform convexity of the Hilbert space X has the property that $\delta_X(t) \geq ct^2$ for some constant $c > 0$. Let E be a bounded set in X and let $M > 0$ be of upper bound of E. From this and from (30) we deduce that, for $a > 0$ and $t \in (0,a]$

$$\nu_f(z,t) \geq rK\left(\frac{t}{2}\right)^r \int_0^1 \tau^{r-1}\delta_X\left(\frac{\tau t}{2\left(\|z\|+\tau t\right)}\right) d\tau$$

$$\geq crK\left(\frac{t}{2}\right)^{r+2} \int_0^1 \frac{\tau^{r+1}}{\left(M+\tau a\right)^2}\, d\tau,$$

for all $z \in E$. Hence, $\nu_f(E,t) \geq \beta(a,r)t^{r+2}$, where $\beta(a,r)$ is a positive real number. □

3.3. The next result shows that condition (*) holds in Lebesgue spaces no matter how the number $r > 1$ is chosen.

Proposition. *Suppose that $p, r \in (1,+\infty)$. In the Lebesgue space $\mathcal{L}^p$ the function $f : \mathcal{L}^p \to \mathbb{R}$, defined by $f(x) = \|x\|_p^r$, satisfies the condition (*).*

Proof. Let t be a positive real number, E a nonempty and bounded subset of $\mathcal{L}^p$, and denote by M a upper bound of E. It follows from (30) that, for any $z \in E$,

$$\nu_f(z,t) \geq rK\left(\frac{t}{2}\right)^r \cdot \int_0^1 \tau^{r-1}\delta_p\left(\frac{\tau t}{2\left(M+\tau t\right)}\right) d\tau. \tag{31}$$

because the modulus of uniform convexity δ_p of $\mathcal{L}^p$ is increasing on $[0,2]$. Recall (see, for instance, [16, Theorem 1, pp. 69]) that

$$\delta_p(t) \geq \begin{cases} c_p t^2 & \text{if } p \in (1,2), \\ \\ c_p t^p & \text{if } p \geq 2, \end{cases} \tag{32}$$

where c_p is some positive constant (depending on p). Define the function $\eta_E : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ by

$$\eta_E(a,t) = \begin{cases} \beta_1(r,p,a)t^{r+2} & \text{if } p \in (1,2), \\ \\ \beta_2(r,p,a)t^{r+p} & \text{if } p \geq 2, \end{cases}$$

with

$$\beta_1(r,p,a) := \frac{rc_pK}{2^{r+2}} \int_0^1 \frac{\tau^{r+1}}{\left(M+\tau a\right)^2} d\tau \tag{33}$$

and

$$\beta_2(r,p,a) := \frac{rc_pK}{2^{r+p}} \cdot \int_0^1 \frac{\tau^{r+p-1}}{(M+\tau a)^p} d\tau. \tag{34}$$

This function η_E has the properties required by condition (*) when $p \in (1,2)$. Indeed, combining (31) and (32) we obtain

$$\nu_f(z,t) \geq \frac{rc_pK}{2^{r+2}} t^{r+2} \cdot \int_0^1 \frac{\tau^{r+1}}{(M+\tau t)^2} d\tau,$$

for any $z \in E$. Let a be a positive real number and suppose that $t \in (0,a]$. Then, taking the infimum for $z \in E$ in the last inequality, we get that

$$\nu_f(E,t) \geq \beta_1(r,p,a) t^{r+2}, \tag{35}$$

for all $t \in (0,a]$, where $\beta_1(r,p,a)$ is given at (33). A similar argument shows that the function η_E defined above satisfies condition (*) in the case $p \geq 2$. □

3.4. Now we are going to show that, in Sobolev spaces $\mathcal{W}^{m,p}$, the p power of the norm has property (*). To make notations precise, recall (see [1]) that, for an open, bounded, convex, nonempty subset $S \subset \mathbb{R}^n$, the Sobolev space $\mathcal{W}^{m,p} := \mathcal{W}^{m,p}(S)$ consists of all $u := u(t_1, ..., t_n) \in \mathcal{L}^p(S)$ such that, for each multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ for which $|\alpha| := \sum_{i=1}^n \alpha_i \leq m$, the (generalized) derivative

$$\mathfrak{D}^\alpha u := \frac{\partial^{|\alpha|} u}{\partial t_1^{\alpha_1} ... \partial t_n^{\alpha_n}}$$

exists and belongs to $\mathcal{L}^p(S)$. Provided with the norm

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|\mathfrak{D}^\alpha u\|_p^p \right)^{1/p}, \tag{36}$$

the space $\mathcal{W}^{m,p}$ is uniformly convex and smooth (cf. [1, Theorem 3.5] and [13, Corollary 4.5, pp. 27 and Theorem 3.5, pp. 22]).

Proposition. *Suppose that $p \in (1,+\infty)$ and m is a positive integer. In the Sobolev space $\mathcal{W}^{m,p}$ the function $f(x) = \|x\|_{m,p}^p$ has the property (*).*

Proof. Let t be a positive real number and let E a nonempty and bounded subset of $\mathcal{W}^{m,p}$. Fix $u \in E$ and choose $w \in \mathcal{W}^{m,p}$ be such that $\|u - w\|_{m,p} = t$. There exists a multi-index $\alpha_0 \in \mathbb{N}^n$ with $0 \leq |\alpha_0| \leq m$ such that

$$\|\mathfrak{D}^{\alpha_0} u - \mathfrak{D}^{\alpha_0} w\|_p \geq \frac{t}{\gamma^{1/p}}, \tag{37}$$

where γ is the cardinality of the set of multi-indexes α with $0 \leq |\alpha| \leq m$. Indeed, if we assume, by contradiction, that for any such multi-index α, we have

$$\|\mathfrak{D}^\alpha u - \mathfrak{D}^\alpha w\|_p < \frac{t}{\gamma^{1/p}},$$

then it follows from (36) that $\|u-w\|_{m,p} < t$, and this contradicts the equality $\|u-w\|_{m,p} = t$. Observe that, according to (36), we have

$$\begin{aligned} D_f(w,u) &= \|w\|_{m,p}^p - \|u\|_{m,p}^p - \langle J_p(u), w-u\rangle & (38) \\ &= \sum_{0\le|\alpha|\le m} \int_S [|\mathfrak{D}^\alpha w|^p - |\mathfrak{D}^\alpha u|^p]\, dx - \langle J_p(u), w-u\rangle, & (39) \end{aligned}$$

where J_p is the duality mapping of weight pt^{p-1} in $\mathcal{W}^{m,p}$. Applying Asplund's theorem we deduce that

$$\begin{aligned} \langle J_p(u), v\rangle &= \frac{d}{dt}\left(\|u+tv\|_{m,p}^p\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\sum_{0\le|\alpha|\le m}\int_S |\mathfrak{D}^\alpha u + t\mathfrak{D}^\alpha v|^p\, dx\right)\Bigg|_{t=0} \\ &= p\sum_{0\le|\alpha|\le m}\int_S \left[|\mathfrak{D}^\alpha u|^{p-1}\, sign\,(\mathfrak{D}^\alpha u)\,\mathfrak{D}^\alpha v\right] dx. \end{aligned}$$

This and (38) imply

$$\begin{aligned} D_f(w,u) &= \sum_{0\le|\alpha|\le m}\int_S [|\mathfrak{D}^\alpha w|^p - |\mathfrak{D}^\alpha u|^p \\ &\quad -p\,|\mathfrak{D}^\alpha u|^{p-1}\,(\mathfrak{D}^\alpha w - \mathfrak{D}^\alpha u)\, sign\,(\mathfrak{D}^\alpha u)\big]\, dx, \end{aligned}$$

that is,

$$D_f(w,u) = \sum_{0\le|\alpha|\le m} D_{\|\cdot\|_p^p}\left(\mathfrak{D}^\alpha w, \mathfrak{D}^\alpha u\right).$$

Consequently, for the multi-index α_0 satisfying (37), we have

$$\begin{aligned} D_f(w,u) &\ge D_{\|\cdot\|_p^p}\left(\mathfrak{D}^{\alpha_\circ} w, \mathfrak{D}^{\alpha_\circ} u\right) \\ &\ge \nu_{\|\cdot\|_p^p}\left(\mathfrak{D}^{\alpha_\circ} u, \left\|\mathfrak{D}^{\alpha_\circ} w - \mathfrak{D}^{\alpha_\circ} u\right\|_p\right) \\ &\ge \nu_{\|\cdot\|_p^p}\left(\mathfrak{D}^{\alpha_\circ} u, \frac{t}{\gamma^{1/p}}\right), \end{aligned}$$

where $\nu_{\|\cdot\|_p^p}$ denotes the modulus of total convexity of the function $\|\cdot\|_p^p$ on $\mathcal{L}^p(S)$ and the last inequality holds because the function $\nu_{\|\cdot\|_p^p}(\mathfrak{D}^{\alpha_\circ}u,\cdot)$ is nondecreasing (cf. [9, Proposition 1.2.2]). According to (36), and taking into account that E is bounded by some real number $M>0$, we have that $\left\|\mathfrak{D}^\alpha u\right\|_p \le M$, whenever α is a multi-index with $0\le|\alpha|\le m$. Take F to be the closed ball of center zero and radius M in $\mathcal{L}^p(S)$. Then, $\mathfrak{D}^{\alpha_\circ}u \in F$ and we have

$$D_f(w,u) \ge \nu_{\|\cdot\|_p^p}\left(\mathfrak{D}^{\alpha_\circ} u, \frac{t}{\gamma^{1/p}}\right) \ge \nu_{\|\cdot\|_p^p}\left(F, \frac{t}{\gamma^{1/p}}\right). \qquad (40)$$

Fix a positive real number a and define

$$\eta_E(a,t)=\begin{cases}\beta_{1,p}(a)\left(\frac{t}{\gamma^{1/p}}\right)^{p+2} & \text{if } p\in(1,2),\\ \\ \beta_{2,p}(a)\left(\frac{t}{\gamma^{1/p}}\right)^{2p} & \text{if } p\geq 2.\end{cases}$$

where $\beta_{i,p}(a):=\beta_i(p,p,a)$, for $i=1,2$, are given by (33) and (34), respectively. Then, using (40), one deduces that this function η_E satisfies the condition (*) because, as shown in the proof of Proposition 2.2, we have that

$$\nu_{\|\cdot\|_p^p}\left(F,\frac{t}{\gamma^{1/p}}\right)\geq\begin{cases}\beta_1(p,p,a)\left(\frac{t}{\gamma^{1/p}}\right)^{p+2} & \text{if } p\in(1,2),\\ \\ \beta_2(p,p,a)\left(\frac{t}{\gamma^{1/p}}\right)^{2p} & \text{if } p\geq 2,\end{cases}$$

for all $t\in(0,a]$. □

3.5. A restrictive condition involved in Theorem 2.1 concerns the existence of a solution z of the given stochastic convex feasibility problem (1) such that the function $D_f(z,\cdot)$ is convex when $f(x)=\|x\|^r$ for some real number $r>1$. Instances in which this may happen in $\mathbb{R}^n$ were discussed in [4]. If X is a Hilbert space and if $r=2$, then this condition is satisfied by any solution z of the problem (recall that we presume that the problem is consistent). That is so because, in this case, $D_f(z,x)=\|z-x\|^2$. In general, i.e., if X is not a Hilbert space or $r\neq 2$, then this condition holds whenever $z=0$ is a solution of the given problem (note that $D_f(0,x)=(r-1)\|x\|^r$). In such situations and depending on how r is chosen, it may happen that $z=0$ is the only element of X for which $D_f(z,\cdot)$ is convex. This is the case if $X=\ell^p$ and $r=p>2$. Indeed, in this specific case, for the twice differentiable function $D_f(z,\cdot)$ with $f(x)=\|x\|_p^p$, $p>2$, to be convex in ℓ^p it is necessarily and sufficient (cf. [24, Theorem 2.1.7]) that

$$\left\langle [D_f(z,\cdot)]''(x)h,h\right\rangle\geq 0,$$

whenever $x,h\in\ell^p$. Also, we have

$$\left\langle [D_f(z,\cdot)]''(x)h,h\right\rangle=p(p-1)\sum_{n=0}^{\infty}|x_n|^{p-4}\left[(p-1)x_n^2-(p-2)x_nz_n\right]h_n^2.$$

If the last sum is nonnegative for all $x,h\in\ell^p$, then it is also nonnegative for all $h=e^k\in\ell^p$, where $e_n^k=1$, if $n=k$, and $e_n^k=0$, otherwise. This implies that, for each $n\in\mathbb{N}$, we have

$$|x_n|^{p-4}\left[(p-1)x_n^2-(p-2)x_nz_n\right]\geq 0,$$

for all $x_n\in\mathbb{R}\setminus\{0\}$. This cannot happen unless $z_n=0$, for all $n\in\mathbb{N}$. Hence, $z=0$ is the only element of ℓ^p such that $D_f(z,\cdot)$, with $f(x)=\|x\|_p^p$, is convex when $p>2$.

3.6. In order to illustrate de behavior of *OBPM*, we consider the following stochastic feasibility problem: Find $x\in\mathcal{L}^p$ such that

$$\int_\Omega K(\omega,t,x(t))dt\leq c(\omega),\ \text{a.e.}(\Omega),$$

where $\Omega = [a, b]$ and c, $K(\cdot, t, u) : \Omega \to \mathbb{R}$ are measurable and, for each $\omega \in \Omega$ and $t \in [a, b]$, the function $K(\omega, t, \cdot)$ is convex on $\mathbb{R}$. We apply the *OBPM* to the particular case where $p = 4/3$, $a = 0$, $b = 1$, $c(\omega) = 3e^{\omega}$ and

$$K(\omega, t, x(t)) = e^{\omega^2 t} x^{4/3}(t).$$

In this situation, we define the function g as follows:

$$g(\omega, x) = \int_0^1 e^{\omega^2 t} x^{4/3}(t) dt - 3e^{\omega}.$$

Note that this function g satisfies conditions $(C1)$-$(C3)$ and that $z = 0$ is a solution of the problem.

We start the *OBPM* procedure with $x^0 \equiv 5^{3/4}$ and $r = 1.5$. We have

$$g(\omega, x^0) = (5/\omega^2) \cdot (e^{\omega^2} - 1) - 3e^{\omega} > 0,$$

for all $\omega \in (0, 1]$ and, then,

$$\int_{\Omega} \max\left[0, g(\omega, x^0)\right] d\mu(\omega) = 0.80612. \tag{41}$$

We observe that

$$J_r x = \begin{cases} r\|x\|_p^{r-p}|x|^{p-2}x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$J_r^* \xi = \begin{cases} r^{\frac{1}{1-r}}\|\xi\|_q^{\frac{r}{r-1}-q}|\xi|^{q-2}\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where $q = p/(p-1) = 4$. Since $g(\omega, \cdot)$ is differentiable, we necessarily have $\Gamma_k(\omega) = g'(\omega, \cdot)(x^k)$, for all k . Thus, we obtain that

$$x^{k+1} = r^{\frac{1}{1-r}} \int_{\Omega} \|y^k\|_q^{\frac{r}{r-1}-q}|y^k|^{q-2}y^k d\mu(\omega),$$

where

$$y^k := s_k(\omega)g'(\omega, \cdot)(x^k) + r\|x^k\|_p^{r-p}|x^k|^{p-2}x^k$$

and $s_k(\omega)$ is a solution of the equation

$$\int_{\Omega} g'(\omega, \cdot)(x^k)\left[r^{\frac{1}{1-r}}\|y^k\|_q^{\frac{r}{r-1}-q}|y^k|^{q-2}y^k - x^k\right] d\mu(\omega) = -g(\omega, x^k).$$

Computing x^1 according to the formula above for the given x^0, we obtain the corresponding averaged constraint violation

$$\int_{\Omega} \max\left[0, g(\omega, x^1)\right] d\mu(\omega) = 0.235225.$$

Comparing that to (41), we note that, after just a single step, we have a significant reduction of the averaged constraint violation.

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