Shapley Mappings and the Cumulative Value for n-Person Games with Fuzzy Coalitions

Dan Butnariu
Department of Mathematics
University of Haifa
31 905 Haifa, Israel

Tomáš Kroupa
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 4
182 08 Prague, Czech Republic

Abstract

In this paper we prove the existence and uniqueness of a solution concept for n-person games with fuzzy coalitions, which we call the Shapley mapping. The Shapley mapping, when it exists, associates to each fuzzy coalition in the game an allocation of the coalitional worth satisfying the efficiency, the symmetry, and the null-player conditions. It determines a "cumulative value" that is the "sum" of all coalitional allocations and for whose computation we provide an explicit formula.

Keywords: coalition, cumulative value, fuzzy coalition, n-person cooperative game, Shapley value
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1 Introduction

Let $N := \{1, 2, \ldots, n\}$ be the set whose elements are called *players*. As usual, by a *coalition* we mean a subset of N. A *fuzzy coalition* is a vector

$A = (A(1), \ldots, A(n))$ with coordinates $A(i)$ contained in the interval $[0, 1]$ (cf. [1, 2]). The number $A(i)$ is called the *membership degree of player* i to the fuzzy coalition A. We denote by $\mathcal{P}$ the set of all coalitions and by $\mathcal{F}$ the set of all fuzzy coalitions. When referring to coalitions we do not notationally distinguish between a coalition S and its indicator vector $(S(1), \ldots, S(n))$, where the coordinates $S(i)$ are either one or zero depending on whether i belongs or not to S. In this way we can view $\mathcal{P}$ as a subset of $\mathcal{F}$. A fuzzy coalition A can be also seen as a partition of the set of players into coalitions

$$A_t := \{i \in N : A(i) = t\}, \quad t \in [0, 1],$$

such that all players belonging to A_t for some $t \in [0, 1]$ have the same degree of membership to A. Clearly, all but at most n coalitions A_t are nonempty.

A *(characteristic function n-person cooperative) game* is a function $u : \mathcal{P} \to \mathbb{R}$ such that $u(\emptyset) = 0$. The function u associates to each coalition S its *worth* $u(S)$, measuring the utility of forming coalition S. We presume that, besides coalitions, formation of fuzzy coalitions in the game u is also possible: the worth of a fuzzy coalition A is the aggregated worth of the coalitions A_t weighted by a quantity $\psi(t)$ which depends on the degree of membership t. In other words, the worth of a fuzzy coalition A in the game u is given by

$$u^{\psi}(A) = \sum_{t \in [0,1]} \psi(t) u(A_t). \tag{1}$$

Note that the sum occurring here is well-defined since all but finitely many terms of it are zero. In this context it is natural to assume that the function $\psi : [0, 1] \to \mathbb{R}$ is such that the coalition A_1 of fully fledged members of A gets its full worth while the coalition A_0 of players who are not members of A does not contribute to the worth $u^{\psi}(A)$. Therefore, all over this paper we make the following assumption:

Assumption 1: $(\psi(t) = 0 \Leftrightarrow t = 0)$ and $(\psi(1) = 1)$.

A function $\psi : [0, 1] \to \mathbb{R}$ with this property is called a *weight function.*

In what follows, a function $v : \mathcal{F} \to \mathbb{R}$ satisfying $v(\varnothing) = 0$ is called a *fuzzy game* – cf. [1, 2]. We denote by $\mathcal{G}[\psi]$ the set of fuzzy games v satisfying

$$v(A) = \sum_{t \in [0,1]} \psi(t) v(A_t). \tag{2}$$

It is easy to see that, if $v \in \mathcal{G}[\psi]$, then the restriction u of v to $\mathcal{P}$ is a game such that $v = u^{\psi}$. The game u is called *the underlying game of* v. Clearly, a fuzzy game $v \in \mathcal{G}[\psi]$ and its underlying game u completely determine each other. Also, observe that $\mathcal{G}[\psi]$ is a linear space with the usual operations induced from $\mathbb{R}$.

A first question we are dealing with in this paper is whether, in games in which formation of fuzzy coalitions is possible and the worth of each fuzzy coalition is determined according to (2), there are ways of "fairly" distributing the worth of all fuzzy coalitions among the players. Of course, the answer to this question essentially depends on the meaning of "fairness". In order to make this precise, recall (cf. [9, 10]) that if v is a fuzzy game and if A is a fuzzy coalition, then the fuzzy coalition B is called a *v-carrier of A* if for every $t \in (0,1]$ we have: $B_t \subseteq A_t$, and for every fuzzy coalition C such that $C_t \subseteq A_t$, the equality

$$v(B_t \cap C_t) = v(C_t).$$

holds true. As usual, for every permutation π of N, every $A \in \mathcal{F}$, and any fuzzy game v, we denote $\pi A := A \circ \pi^{-1}$ and $\pi v(A) := v(\pi^{-1}A)$. Clearly, if v belongs to $\mathcal{G}[\psi]$, then the function $\pi v : A \mapsto \pi v(A)$ from $\mathcal{F}$ to $\mathbb{R}$ is still a fuzzy game in $\mathcal{G}[\psi]$. With these in mind we can define the following notion which describes a concept of fairness according to which each fuzzy coalition allocates its worth to its members obeying the principles intrinsic to the Shapley value, that is, efficiency, null-players get nothing, symmetry, and linearity (see [13]).

Definition 1. *A* Shapley mapping *is a linear function $\mathbf{\Phi} : \mathcal{G}[\psi] \to (\mathbb{R}^N)^{\mathcal{F}}$ satisfying the following conditions for any $v \in \mathcal{G}[\psi]$ and any $A \in \mathcal{F}$:*

Axiom 1 *(Coalitional Efficiency) For every v-carrier $B \in \mathcal{F}$ of A we have*

$$\sum_{i \in N : B(i) > 0} \mathbf{\Phi}_i(v)(A) = v(B). \tag{3}$$

Axiom 2 *(Non-Member) If $A(j) = 0$, then $\mathbf{\Phi}_j(v)(A) = 0$.*

Axiom 3 *(Symmetry) If π is a permutation of N, then*

$$\mathbf{\Phi}_{\pi i}(\pi v)(\pi A) = \mathbf{\Phi}_i(v)(A), \quad i = 1, \dots, n. \tag{4}$$

Note that a Shapley mapping, if it exists, associates to each fuzzy game $v \in \mathcal{G}[\psi]$ and to any fuzzy coalition A a vector $\Phi(v)(A) = (\Phi_1(v)(A), \dots, \Phi_n(v)(A))$, that satisfies the basic principles of efficiency, null-player condition, symmetry and linearity characterizing the Shapley value ([13]) when extrapolated to fuzzy coalitional levels. The following result, whose detailed proof is given in Section 2, shows that these principles uniquely determine a Shapley mapping on $\mathcal{G}[\psi]$.

Theorem 1. *There exists a unique Shapley mapping* $\mathbf{\Phi} : \mathcal{G}[\psi] \to (\mathbb{R}^N)^{\mathcal{F}}$ *and it is given by the following formula:*

$$\mathbf{\Phi}_i(v)(A) = \begin{cases} \psi(r) \sum\limits_{S \in \mathcal{P}_i(A_r)} \frac{(|S|-1)!(|A_r|-|S|)!}{|A_r|!}(v(S) - v(S \setminus \{i\})), & \textit{if } A(i) = r > 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{5}$$

where

$$\mathcal{P}_i(A_r) = \{R \subseteq N | i \in R \textit{ and } R \subseteq A_r\}.$$

A second problem we are addressing in this paper concerns the expected total allocation $\Phi_i(v)$ of player i in the cooperative process in which fuzzy coalitions allocate to their members their worth. Precisely, we consider

$$\Phi_i(v) := \int_{\mathcal{F}} \Phi_i(v)(A) dA, \tag{6}$$

where the integral over the set of fuzzy coalitions $\mathcal{F}$ is taken with respect to the Lebesgue measure. It is interesting to know whether the total-payoff vector $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$, which we call the *cumulative value* of the fuzzy game v, is well-defined and, if possible, to estimate its coordinates. The following result, that is proved in Section 3, contains an answer to that question.

Theorem 2. *If the weight function ψ is bounded and (Lebesgue) integrable, then, for any $v \in \mathcal{G}[\psi]$, the cumulative value $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$, given by* (6) *is well defined and we have*

$$\Phi_i(v) = v(\{i\}) \int_0^1 \psi(t) dt, \tag{7}$$

for each $i \in N$.

As noted above, the Shapley mapping models a scheme of allocating each coalition's worth to its member following some "fairness criteria". Theorem

2 essentially says that if a cooperative game u is extended to a fuzzy game (i.e., to a game in which formation of fuzzy coalitions is possible) according to the rule (1) (or, equivalently, (2)), then the scheme underlying the Shapley mapping is no more and no less than a procedure through which each player is re-evaluating his personal worth by taking into account the "weight" of his membership degrees to fuzzy coalitions. Along this procedure a weight function ψ with average value $\int_0^1 \psi(t)dt > 1$ favors players i with positive individual worth $v(\{i\})$, while a weight function ψ with average value $\int_0^1 \psi(t)dt < 1$ favors players with negative worth $v(\{i\})$.

Using (7) it is easy to deduce that the function $v \mapsto \Phi(v)$ is a *semi-value* on $\mathcal{G}[\psi]$, that is, it has the null-player property, it is symmetric and linearly dependent on v. Moreover, on the linear subspace of $\mathcal{G}[\psi]$ consisting of all games having the property

$$\Big(\sum_{i\in N} v(\{i\})\Big) \int_0^1 \psi(t)\, dt = v(N), \tag{8}$$

the cumulative value of v is also efficient, that is,

$$\sum_{i\in N} \Phi_i(v) = v(N), \tag{9}$$

and, thus, the function $v \mapsto \Phi(v)$ is a value.

The concepts of fuzzy coalition and the possibility of extending games to games with fuzzy coalitions naturally emerged from the works of R. Aumann and L. S. Shapley where "ideal set" and "ideal set functions" (fuzzy coalitions and fuzzy games, respectively) are technical tools in the study of games with infinitely many players [7]. However, it was J.-P. Aubin who not only introduced notions of fuzzy coalitions and fuzzy games but also studied them per se (see [1, 2, 3, 4, 5, 6]). The notion of Shapley mapping studied in this article was introduced in [10], where the existence of a Shapley mapping was proved for a particular class of fuzzy games. The existence theorem for Shapley mapping given above (Theorem 1) as well as the form of the cumulative value given in Theorem 2 essentially depend on the specific way in which the worth of each fuzzy coalition is aggregated from the worth of its level sets in formula (2). There are other meaningful ways of embedding games into fuzzy games. The oldest among them, as far as we know, is Owen's multilinear extension [12] which can be seen as a fuzzy game extending a game (see [11], Section 19). More recently, M. Tsurumi et al. [14] proposed another way of extending a game to a fuzzy game and have shown that by using their extension, which is more regular than the one given by (2), one can also obtain Shapley mappings on a class of necessarily continuous games. As one can see from the examples from Section 4, fuzzy games defined by (2) need not be continuous. The notion of cumulative value introduced here measures the pay-off each player should expect from his participation in the extended fuzzy game. It is an interesting open question whether different rules of aggregating fuzzy games from games and, in particular, that of [14] and other mentioned above, leads to well-defined cumulative values and whether it is possible to estimate them.

We have noted above that the vector $\Phi(v)$ is a semi-value on $\mathcal{G}[\psi]$, which is even a value on some subspace of $\mathcal{G}[\psi]$. It is natural to ask how this new value relates with the other value concepts already discussed in literature. In Section 4 we point out that the cumulative value exists for some fuzzy games for which the other existing value concepts need not be defined.

2 Proof of Theorem 1

We start our proof by observing that if $v \in \mathcal{G}[\psi]$ and if B is a v-carrier of A, then $v(A_t) = v(B_t)$ for all $t \in (0,1]$ and, therefore, $v(A) = v(B)$ because of (2). We follow Shapley [13] and to any non-empty coalition S we associate the *simple game* $w_S : \mathcal{P} \to \{0,1\}$ defined by

$$w_S(A) = \begin{cases} 1, & \text{if } S \subseteq A, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

and the number

$$c_S(v) = \sum_{B\in\mathcal{P}:B\subseteq S} (-1)^{|S|-|B|} v(B). \tag{11}$$

It is known (see Lemma 3 in [13]) that the set $\mathcal{G}_0$ of all simple games is a basis in the linear space $\mathcal{G}$ of all games, and, if u is a game, then it can be uniquely written as

$$u = \sum_{S\in\mathcal{P}:S\neq\emptyset} c_S(u) w_S. \tag{12}$$

We prove below that there exists a unique Shapley mapping on $\mathcal{G}[\psi]$. Our proof consists of a sequence of lemmata.

Lemma 1. *If* $v \in \mathcal{G}[\psi]$, *then*

$$v = \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v) w_S^{\psi}. \tag{13}$$

Proof. Let $v \in \mathcal{G}[\psi]$ and $A \in \mathcal{F}$. Then, applying (12) to the restriction of v to $\mathcal{P}$, we obtain:

$$\begin{aligned} v(A) &= \sum_{t \in [0,1]} \psi(t) v(A_t) = \sum_{t \in [0,1]} \psi(t) \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v) w_S(A_t) \\ &= \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v) \sum_{t \in [0,1]} \psi(t) w_S(A_t) = \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v) w_S^{\psi}(A). \end{aligned}$$

□

Observe that, according to (1), for every fuzzy coalition A and for every non-empty coalition S the fuzzy game w_S^{ψ} can be represented as

$$w_S^{\psi}(A) = \begin{cases} \psi(r), & \text{if } S \subseteq A_r \text{ for some } r \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

We denote

$$\mathcal{G}_0[\psi] = \{ w_S^{\psi} \mid S \in \mathcal{P}, S \neq \emptyset \}.$$

Note that, according to Lemma 1, $\mathcal{G}_0[\psi]$ is a basis $\mathcal{G}[\psi]$. With these remarks and notations in mind we can state the following result.

Lemma 2. *Let* $\mathbf{\Phi} : \mathcal{G}_0[\psi] \to (\mathbb{R}^N)^{\mathcal{F}}$ *be the function defined by*

$$\mathbf{\Phi}_i(w_S^{\psi})(A) = \begin{cases} \frac{\psi(r)}{|S|}, & \textit{if } i \in S \subseteq A_r \textit{ for some } r \in (0,1], \\ 0, & \textit{otherwise.} \end{cases} \tag{14}$$

The function $\mathbf{\Phi}$ *satisfies the Coalitional Efficiency, the Non-Member, and the Symmetry Axiom given in Definition 1 for games in* $\mathcal{G}_0[\psi]$.

Proof. Let S be a non-empty coalition and A be a fuzzy coalition. We claim that $\mathbf{\Phi}$ satisfies the Coalitional Efficiency Axiom. In order to prove this claim, let B be w_S^{ψ}-carrier of the fuzzy coalition A. Then there exists only one $r \in (0,1]$ such that $S \subseteq B_r \subseteq A_r$. Note that, according to (14), if $i \notin S$ then $\mathbf{\Phi}_i(w_S^\psi)(A) = 0$. Consequently, we have

$$\begin{aligned}\sum_{i \in N : B(i) > 0} \mathbf{\Phi}_i(w_S^\psi)(A) &= \sum_{i \in S} \mathbf{\Phi}_i(w_S^\psi)(A) = \sum_{i \in S} \frac{\psi(r)}{|S|} \\ &= \psi(r) = w_S^\psi(B),\end{aligned}$$

proving our claim. We prove next that $\mathbf{\Phi}$ satisfies the Symmetry Axiom too. Let π be a permutation of N. We obviously have

$$\pi w_S^\psi(A) = w_S^\psi(\pi^{-1}A) = \begin{cases} \psi(r), & \text{if } S \subseteq \pi^{-1}A_r \text{ for some } r \in (0,1], \\ 0, & \text{otherwise,} \end{cases}$$

and since $S \subseteq \pi^{-1}A_r$ if and only if $\pi S \subseteq A_r$, it results that

$$\pi w_S^\psi(A) = w_{\pi S}^\psi(A). \tag{15}$$

If $i \in N$, then

$$\mathbf{\Phi}_{\pi i}(\pi w_S^\psi)(\pi A) = \mathbf{\Phi}_{\pi i}(w_{\pi S}^\psi)(\pi A),$$

where, by (14), we have that

$$\mathbf{\Phi}_{\pi i}(w_{\pi S}^\psi)(\pi A) = \begin{cases} \frac{\psi(r)}{|\pi S|}, & \text{if } \pi i \in \pi S \subseteq \pi A_r \text{ for some } r \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Since $|\pi S| = |S|$ and the condition $\pi i \in \pi S$ and $\pi S \subseteq \pi A_r$ is equivalent to $i \in S$ and $S \subseteq A_r$, respectively, we have

$$\mathbf{\Phi}_{\pi i}(\pi w_S^\psi)(\pi A) = \mathbf{\Phi}_i(w_S^\psi)(A).$$

Observe that if $A(j) = 0$ for some $j \in N$, then we have $\mathbf{\Phi}_j(w_S^\psi)(A) = 0$ by (14). Hence, the Non-Member Axiom is also verified. □

Lemma 3. *The function $\mathbf{\Phi}$ defined by* (14) *is the only function from $\mathcal{G}_0[\psi]$ to $(\mathbb{R}^N)^{\mathcal{F}}$ which simultaneously satisfies the Coalitional Efficiency, the Non-Member, and the Symmetry Axiom for any $v \in \mathcal{G}_0[\psi]$.*

Proof. Suppose that $\mathbf{\Phi'} : \mathcal{G}_0[\psi] \to (\mathbb{R}^N)^{\mathcal{F}}$ is another function satisfying the Coalitional Efficiency and Symmetry Axioms for every fuzzy game in $\mathcal{G}_0[\psi]$. Let A be a fuzzy coalition and S be a non-empty coalition. We distinguish

two possible cases.
Case 1: $S \subseteq A_r$ for some $r \in (0,1]$. Note that the fuzzy coalition B defined by

$$B(i) = \begin{cases} A(i), & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

is a w_S^ψ-carrier of A. Therefore we have

$$\sum_{i \in S} \mathbf{\Phi}'_i(w_S^\psi)(A) = \sum_{i \in N: B(i) > 0} \mathbf{\Phi}'_i(w_S^\psi)(A) = w_S^\psi(B) = \psi(r) \neq 0. \qquad (16)$$

Fix $i, j \in S$ such that $i \neq j$. Let π be a permutation of N such that:

$$\pi k = \begin{cases} j, & \text{if } k = i, \\ i, & \text{if } k = j, \\ k, & \text{otherwise.} \end{cases}$$

Observe that $\pi S = S$ and $\pi A = A$. Consequently, by the Symmetry Axiom and (15) we obtain

$$\mathbf{\Phi}'_i(w_S^\psi)(A) = \mathbf{\Phi}'_j(\pi w_S^\psi)(\pi A) = \mathbf{\Phi}'_j(w_{\pi S}^\psi)(\pi A) = \mathbf{\Phi}'_j(w_S^\psi)(A).$$

This, together with (16), implies

$$\mathbf{\Phi}'_i(w_S^\psi)(A) = \frac{\psi(r)}{|S|},$$

for every $i \in S$. Hence in this situation we have

$$\mathbf{\Phi}'_i(w_S^\psi)(A) = \mathbf{\Phi}_i(w_S^\psi)(A) \qquad (17)$$

for any $i \in S$. Now, take $k \notin S$ and denote $T = S \cup \{k\}$. The fuzzy coalition B' defined by

$$B'(i) = \begin{cases} A(i), & \text{if } i \in T, \\ 0, & \text{otherwise,} \end{cases}$$

is a w_S^ψ-carrier of A and, therefore, by the Coalitional Efficiency Axiom and (16) we get

$$\sum_{i \in S} \mathbf{\Phi}'_i(w_S^\psi)(A) = w_S^\psi(B').$$

By consequence we have

$$w_S^\psi(B') = \sum_{i \in N: B'(i) > 0} \mathbf{\Phi}'_i(w_S^\psi)(A) = \sum_{i \in T} \mathbf{\Phi}'_i(w_S^\psi)(A)$$

and hence $\mathbf{\Phi}'_k(w_S^\psi)(A) = 0$. Combining this fact with (17), we conclude that $\mathbf{\Phi}(w_S^\psi)(A) = \mathbf{\Phi}'(w_S^\psi)(A)$.
Case 2: $S \nsubseteq A_r$ for every $r \in (0,1]$. In this case, it can be easily verified that any $B \in \mathcal{F}$ such that $B_t \subseteq A_t$ for all $t \in (0,1]$ is a w_S^ψ-carrier of A. Let $j \in N$. If $A(j) = 0$, then $\mathbf{\Phi}'_j(w_S^\psi)(A) = 0$ due to the Non-Member Axiom. If $A(j) \neq 0$, then we define the fuzzy coalition C as follows:

$$C(i) = \begin{cases} A(j), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $C_t \subseteq A_t$ for all $t \in (0,1]$ because $C_t = \{j\}$, if $t = A(j)$, and $C_t = \emptyset$, otherwise. If B is a fuzzy coalition such that $B_t \subseteq A_t$ for all $t \in (0,1]$, then there is no $t \in (0,1]$ such that $S \subseteq B_t$ because otherwise $S \subseteq A_t$. Consequently, we have

$$w_S(B_t \cap C_t) = 0 = w_S(B_t),$$

showing that the fuzzy coalition C is a w_S^ψ-carrier of A. Hence, by the Coalitional Efficiency Axiom and (14), we get

$$\mathbf{\Phi}'_j(w_S^\psi)(A) = \sum_{i \in N: C(i) > 0} \mathbf{\Phi}'_i(w_S^\psi)(A) = w_S^\psi(C) = 0 = \mathbf{\Phi}_j(w_S^\psi)(A).$$

Summarizing, in both cases $\mathbf{\Phi}'$ and $\mathbf{\Phi}$ coincide which contradicts our initial assumption. □

According to Lemma 1 it is possible to extend the function $\mathbf{\Phi}$ defined by (14) from $\mathcal{G}_0[\psi]$ to $\mathcal{G}[\psi]$ by letting

$$\mathbf{\Phi}_i(v)(A) = \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v)\, \mathbf{\Phi}_i(w_S^\psi)(A). \tag{18}$$

Lemma 4. *The function* $\mathbf{\Phi}$ *defined by* (18) *is the unique Shapley mapping over* $\mathcal{G}[\psi]$.

Proof. The fact there is no more than one Shapley mapping over $\mathcal{G}[\psi]$ results from Lemma 1 and Lemma 3 because they show that all Shapley mappings coincide on the set of generators $\mathcal{G}_0[\psi]$ of $\mathcal{G}[\psi]$. It remains to show that the function $\mathbf{\Phi}$ defined by (18) is indeed a Shapley mapping on $\mathcal{G}[\psi]$, that is, it satisfies the axioms given in Definition 1. The mapping $\mathbf{\Phi}$ is linear because, for each non-empty subset S of N, the function $v \mapsto c_S(v)$ is linearly dependent on $v \in \mathcal{G}[\psi]$.

In order to show that $\mathbf{\Phi}$ satisfies the Coalitional Efficiency Axiom, assume that $B \in \mathcal{F}$ is a v-carrier of a fuzzy coalition A. Hence, we have

$$\begin{aligned}\sum_{i \in N: B(i)>0} \mathbf{\Phi}_i(v)(A) &= \sum_{i \in N: B(i)>0} \sum_{S \in \mathcal{P}: S \neq \emptyset} c_S(v)\, \mathbf{\Phi}_i(w_S^{\psi})(A) \\ &= \sum_{i \in N: B(i)>0} \sum_{S \in \mathcal{P}_i(A_{B(i)})} c_S(v) \frac{\psi(B(i))}{|S|} \\ &\sum_{i \in N: B(i)>0} \psi(B(i)) \sum_{S \in \mathcal{P}_i(A_{B(i)})} c_S(v) \frac{1}{|S|} \\ &= \sum_{t \in [0,1]} \psi(t) \sum_{i \in B_t} \sum_{S \in \mathcal{P}_i(A_t)} c_S(v) \frac{1}{|S|}.\end{aligned}$$

Let $t \in (0,1]$ and let u_t be the game with the set of players A_t defined by $u_t(T) = v(T)$, for all $T \in \mathcal{P}(A_t)$, where $\mathcal{P}(A_t)$ is the set of all sub-coalitions of A_t. Let $\varphi(u_t) \in \mathbb{R}^{A_t}$ be the Shapley value (as defined in [13]) associated to the game u_t. According to [13], the coordinates of $\varphi(u_t)$ are given by

$$\varphi_i(u_t) = \sum_{S \in \mathcal{P}_i(A_t)} c_S(v) \frac{1}{|S|}, \tag{19}$$

for all $i \in A_t$. Note that if the fuzzy coalition B is a v-carrier of A, then, for each $t \in (0,1]$, the coalition B_t is a u_t-carrier of A_t. Due to the efficiency and null-player properties of the Shapley value $\varphi(u_t)$, we have

$$\sum_{i \in B_t} \varphi_i(u_t) = u_t(B).$$

Combined with (19), this implies that

$$\sum_{i \in B_t} \sum_{S \in \mathcal{P}_i(A_t)} c_S(v) \frac{1}{|S|} = u_t(B_t) = v(B_t).$$

Consequently, we have that

$$\begin{aligned} v(B) &= \sum_{t\in(0,1]} \psi(t)v(B_t) \\ &= \sum_{t\in(0,1]} \psi(t) \sum_{i\in B_t} \varphi_i(u_t) \\ &= \sum_{t\in(0,1]} \psi(t) \sum_{i\in B_t} \sum_{S\in\mathcal{P}_i(A_t)} c_S(v)\frac{1}{|S|} \\ &= \sum_{t\in(0,1]} \sum_{i\in B_t} \sum_{S\in\mathcal{P}_i(A_t)} c_S(v)\frac{\psi(A(i))}{|S|} \\ &= \sum_{t\in(0,1]} \sum_{i\in B_t} \mathbf{\Phi}_i(v)(A) = \sum_{i\in N: B(i)>0} \mathbf{\Phi}_i(v)(A). \end{aligned}$$

Hence, the Coalitional Efficiency Axiom is satisfied.

Now, we show that the function $\mathbf{\Phi}$ also satisfies the Symmetry Axiom. To this end, observe that, according to Lemma 1,

$$\pi v(A) = \sum_{S\in\mathcal{P}: S\neq\emptyset} c_S(\pi v) w_S^{\psi}(A),$$

for every $A \in \mathcal{F}$. According to (18), for any $i \in N$, we have that

$$\begin{aligned} \mathbf{\Phi}_{\pi i}(\pi v)(\pi A) &= \sum_{S\in\mathcal{P}: S\neq\emptyset} c_S(\pi v)\, \mathbf{\Phi}_{\pi i}(w_S^{\psi})(\pi A) \\ &= \sum_{S\in\mathcal{P}: S\neq\emptyset} c_{\pi^{-1}S}(v)\, \mathbf{\Phi}_{\pi i}(\pi(\pi^{-1} w_S^{\psi}))(\pi A) \\ &= \sum_{S\in\mathcal{P}: S\neq\emptyset} c_{\pi^{-1}S}(v)\, \mathbf{\Phi}_{i}(\pi^{-1} w_S^{\psi})(A) \\ &= \sum_{S\in\mathcal{P}: S\neq\emptyset} c_{\pi^{-1}S}(v)\, \mathbf{\Phi}_{i}(w_{\pi^{-1}S}^{\psi})(A) \\ &= \sum_{S\in\mathcal{P}: S\neq\emptyset} c_S(v)\, \mathbf{\Phi}_{i}(w_S^{\psi})(A) = \mathbf{\Phi}_i(v)(A), \end{aligned}$$

where the second equality is true because $c_S(\pi v) = c_{\pi^{-1}S}(v)$ and the third equality results from the symmetry of $\mathbf{\Phi}$ over $\mathcal{G}_0[\psi]$ (see Lemma 2). The last two equalities show that $\mathbf{\Phi}_{\pi i}(\pi v)(\pi A) = \mathbf{\Phi}_i(v)(A)$ for every $i \in N$ and $A \in \mathcal{F}$ and, hence, the Symmetry Axiom is verified.

Finally, we show that the Non-Member Axiom is satisfied. If $A(j) = 0$ for some $j \in N$, then $\mathbf{\Phi}_i(w_S^\psi)(A) = 0$, for every non-empty coalition S, and thus $\mathbf{\Phi}_i(v)(A) = 0$ by (18). $\square$

In order to complete the proof of Theorem 1, we still have to prove the following result.

Lemma 5. *The function* $\mathbf{\Phi}$ *defined by* (18) *is also given by formula* (5).

Proof. Let $i \in N$ and denote $r = A(i)$. We distinguish two complementary cases.
Case 1. If $r = 0$, then $\mathbf{\Phi}_i(w_S^\psi)(A) = 0$ for any non-empty coalition S and thus $\mathbf{\Phi}_i(v)(A) = 0$ because of (18).
Case 2. If $r > 0$, then

$$\mathbf{\Phi}_i(v)(A) = \sum_{S \in \mathcal{P}_i(A_r)} c_S(v)\, \mathbf{\Phi}_i(w_S^\psi)(A) = \psi(r) \sum_{S \in \mathcal{P}_i(A_r)} c_S(v) \frac{1}{|S|}.$$

It follows from ([13], formula (13)) that the last sum above is exactly

$$\sum_{S \in \mathcal{P}_i(A_r)} c_S(v) \frac{1}{|S|} = \sum_{S \in \mathcal{P}_i(A_r)} \frac{(|S|-1)!(|A_r|-|S|)!}{|A_r|!} (v(S) - v(S \setminus \{i\}))$$

and this completes the proof. $\square$

3 Proof of Theorem 2

In this section we assume, in addition to Asumption 1, that the function ψ is bounded and (Lebesgue) integrable on $[0, 1]$. We begin our proof of Theorem 2 with the following result implicitly showing that the cumulative value is well-defined.

Lemma 6. *For every* $i \in N$ *and every* $v \in \mathcal{G}[\psi]$*, the function* $\mathbf{\Phi}_i(v)(\cdot)$ *is integrable over* $\mathcal{F}$*.*

Proof. Since $\mathcal{F}$ is a space $\mathcal{F}$ of finite Lebesgue measure it is sufficient to show that for every $i \in N$ and $v \in \mathcal{G}[\psi]$, the function $\mathbf{\Phi}_i(v)(.)$ is bounded and measurable over $\mathcal{F}$. By formula (18), in order to prove boundedness and measurability of $\mathbf{\Phi}_i(v)(.)$, it is enough to show that, for every $S \in \mathcal{P}$, $S \neq \emptyset$, the function $\mathbf{\Phi}_i(w_S^\psi)(.)$ is bounded and measurable over $\mathcal{F}$. Let $A \in \mathcal{F}$. According to (14), we have that

$$\left|\mathbf{\Phi}_i(w_S^\psi(A))\right| \leq \frac{|\psi(r)|}{|S|} \leq |\psi(r)|,$$

for some $r \in (0,1]$. This shows that the function $\mathbf{\Phi}_i(w_S^\psi)(\cdot)$ is bounded because so is ψ. For proving measurability of $\mathbf{\Phi}_i(w_S^\psi)(\cdot)$, define the function $\delta : [0,1]^2 \to [0,1]$ by

$$\delta(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 = x_2, \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

and observe that δ is measurable over $[0,1]^2$ since it is the characteristic function of the closed subset $\{(x_1, x_2) \in [0,1]^2 \mid x_1 = x_2\}$ of $[0,1]^2$. We claim that the following formula holds true for every $A \in \mathcal{F}$, every $i \in N$, and every non-empty coalition S:

$$\mathbf{\Phi}_i(w_S^\psi)(A) = \frac{\psi(A(i))}{|S|} S(i) \prod_{j \in S} \delta(A(i), A(j)). \tag{21}$$

In order to prove this, let $i \in N$ be fixed. We can distinguish two cases. First, if $i \in S \subseteq A_r$ for some $r \in (0,1]$, then $r = A(i)$ and $\delta(A(i), A(j)) = 1$ for each $j \in S$. This implies that the right-hand side of (21) is exactly $\frac{\psi(A(i))}{|S|}$, that is, the equality in (21) holds (see (14)). Second, we consider the situation when $i \notin S$ or there is no $r \in (0,1]$ such that $S \not\subseteq A_r$. In this case, $S(i) = 0$ or there exists $j \in S$ such that $A(i) \neq A(j)$. Therefore, in this situation we have

$$S(i) \prod_{j \in S} \delta(A(i), A(j)) = 0,$$

and formula (21) holds because of (14). By (21) we deduce that $\mathbf{\Phi}_i(w_S^\psi)(.)$ is a product of the measurable functions $\psi(A(i))$ and of $\prod_{j \in S} \delta(x_i, x_j)$ which is the characteristic function of the closed subset of $\mathcal{F}$ defined by $\{A \in \mathcal{F} \mid A(j) = A(i), \text{for every } j \in S\}$, and therefore $\mathbf{\Phi}_i(w_S^\psi)(.)$ is measurable. □

Now we are going to establish formula (7) and, in this way, to complete the proof of Theorem 2.

Lemma 7. *If $i \in N$ and $v \in G[\psi]$, then $\Phi_i(v)$ is given by (7).*

Proof. Denote by $\mathcal{F}_1$ the set of all fuzzy coalitions A such that $A(k) \neq A(l)$ for any pair of players $k, l \in N$ with $k \neq l$. For any $j \in \{2, 3, \dots, n\}$ and for any set of j pairwise different numbers $\{i_1, \dots, i_j\}$ contained in N, let $\mathcal{F}_j(i_1, \dots, i_j)$ be the set of fuzzy coalitions A such that $A(i_l) = A(i_1)$ for $l = 1, \dots, j$. Denote by $\mathcal{F}_j$ the union of all sets $\mathcal{F}_j(i_1, \dots, i_j)$. Clearly, $\mathcal{F}_j(i_1, \dots, i_j)$ is included in the intersection of the set $\mathcal{F}$ with a hyperplane in $\mathbb{R}^n$ and, therefore, it has Lebesgue measure zero. Consequently, the set $\mathcal{F}_j$ has also Lebesgue measure zero because it is a finite union of sets with this property. Obviously, we also have that

$$\mathcal{F} = \bigcup_{l=1}^{n} \mathcal{F}_l$$

and the sets $\mathcal{F}_1$ and $\mathcal{F}' := \bigcup_{l=2}^{n} \mathcal{F}_l$ are disjoint. So, we deduce that

$$\begin{aligned} \Phi_i(v) &= \int_{\mathcal{F}} \mathbf{\Phi}_i(v)(A)dA \\ &= \int_{\mathcal{F}_1} \mathbf{\Phi}_i(v)(A)dA + \int_{\mathcal{F}'} \mathbf{\Phi}_i(v)(A)dA, \end{aligned}$$

where the last integral is null because the set $\mathcal{F}'$ has Lebesgue measure zero. This implies

$$\Phi_i(v) = \int_{\mathcal{F}_1} \mathbf{\Phi}_i(v)(A)dA. \tag{22}$$

Now, let $A \in \mathcal{F}_1$. The level sets A_t are either empty or singletons and thus there are n mutually different numbers $t_l \in [0, 1]$ such that A_{t_l} are singletons. This implies that, if S is a coalition such that $i \in S \subseteq A_t$, we necessarily have $S = A_t = \{i\}$. Hence, the only non-zero term occuring in the summation contained in (5) is that corresponding to the coalition $S = \{i\}$ and we have

$$\mathbf{\Phi}_i(v)(A) = v(\{i\})\psi(A(i)).$$

This and (22) show that

$$\begin{aligned} \Phi_i(v) &= v(\{i\}) \int_{\mathcal{F}_1} \psi(A(i))dA = v(\{i\}) \int_{\mathcal{F}} \psi(A(i))dA \\ &= v(\{i\}) \int_0^1 \psi(t)dt, \end{aligned}$$

where the last equality results from Fubini's Theorem. □

4 Comments and Examples

It was mentioned in Section 1 that there are various ways of defining fair allocation in games with fuzzy coalitions. The oldest among them, as far as the knowledge of authors goes, is Aubin's concept of value studied in the series of papers [1, 2, 3, 4, 5, 6]. Conceptually speaking, Aubin's *multivalue* (see [6]) is essentially different from our Shapley mapping concept. The multivalue is a point-to-set mapping whose selectors may be seen as values of the fuzzy games. By contrast, a Shapley mapping associates to each fuzzy game the set of allocations that fuzzy coalitions may make in accordance with the Shapley's fairness criteria. It is worth noting that Aubin's multivalue is not defined for all fuzzy games in $\mathcal{G}[\psi]$. This is shown by the following example.

Example 1. *Let $N = \{1, 2\}$ and $\psi(t) = t$. The game v defined by $v(\{1\}) = 1$, $v(\{2\}) = 1, v(\{1, 2\}) = 3$ extends by the formula* (1) *to a fuzzy game that is not continuous at* $(1, 1)$*, and thus its multivalue is not defined. According to Theorem 1 and Theorem 2, the Shapley mapping gives $\mathbf{\Phi}_i(v)(A) = A(i)$, $i = 1, 2$, and the cumulative value $\Phi(v)$ is the vector $(\frac{1}{2}, \frac{1}{2})$, respectively.*

On the other hand, the class of fuzzy games $\mathcal{G}[\psi]$ essentially depends on the aggregation rule (1). Therefore, there are fuzzy games for which Aubin's multivalue exists, but they are not contained in $\mathcal{G}[\psi]$. Here is an example of such a game.

Example 2. *Let $N = \{1, \ldots, n\}$ and*

$$v(A) = \max_{i \in N} A(i).$$

While the Aubin's multivalue of this fuzzy game is defined and equals the subgradient of v at N, it is clear that v can not be expressed in the form (2) *and, therefore, it is not contained in $\mathcal{G}[\psi]$.*

The difference mentioned above are not only conceptual but also technical. The necessary condition for the existence of Aubin's multivalue is continuity of a fuzzy game in the open neighborhood of the coalition N. This property is not necessarily shared by all the fuzzy games in $\mathcal{G}[\psi]$. Hence the cumulative value may exist for fuzzy games which are discontinuous on the diagonal for which Aubin's multivalue is not defined at all.

In historical order another value concept was introduced in [9, 10]. In fact, the notion of Shapley mapping discussed in our paper is a generalization of that concept. Among other approaches of extending games there are fuzzy games studied by M. Tsurumi et al. [14], which are rather continuous and belong to generalized sharing games with side payments of Aubin (Definition 13.4 in [6]). Systematic research of alternative value concepts in games with fuzzy coalitions was carried out by S. Tijs and his collaborators (see [8] and the references therein). Inspired by Owen's multilinear extension [12], they study the so-called *diagonal value* for continuously differentiable fuzzy games, and also the class of *compromise values* that are defined for fuzzy games with non-empty Aubin's *core* [6]. For these games a cumulative value may even exist and, if this is the case, it is natural to ask how such cumulative value relates to Aubin's multivalue concept.

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