THE REALIZATION SPACE OF A Π-ALGEBRA: A MODULI PROBLEM IN ALGEBRAIC TOPOLOGY

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Contents

1. Introduction

A Π-*algebra* A is a graded group $\{A_n\}_{n\geq 1}$ with all of the primary algebraic structure possessed by the collection of homotopy groups of a pointed connected topological space. In particular, A_n is abelian for $n \geq 2$, and there are Whitehead product and composition maps which satisfy appropriate identities (see 4.1). The basic example of a Π-algebra is the homotopy Π-algebra $\pi_* X$ of a space X.

Given an abstract Π-algebra A, it is tempting to ask whether it has any topological significance. Is it possible to find a space X such that A is isomorphic to $\pi_* X$? If such an X exists, is it unique up to weak equivalence? These questions and others can be studied by looking at the *moduli space* $\mathcal{TM}(A)$ of topological realizations of A, or the *realization space* of A. This is defined to be the nerve or classifying space of the category whose objects are the topological spaces X with $\pi_* X \simeq A$ and whose morphisms are the weak equivalences between

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these spaces. Up to homotopy $\mathcal{TM}(A)$ can be identified (2.1) as a disjoint union

$$\coprod_{\langle X\rangle} \bar{W}\mathrm{Aut}^{\mathrm{h}}(X) \ ,$$

indexed by homotopy equivalence classes of CW-complexes X with $\pi_*X \simeq A$, where $\bar{W}\mathrm{Aut}^{\mathrm{h}}(X)$ is the classifying space of the simplicial monoid of self homotopy equivalences of X. The Π-algebra A can be realized as π_*X for some X if and only if $\mathcal{TM}(A)$ is nonempty; the realization is unique up to weak equivalence if and only if $\mathcal{TM}(A)$ is connected.

In this paper we study $\mathcal{TM}(A)$. The first step is to construct partial moduli spaces $\mathcal{TM}_n(A)$, $n \geq 0$, which fit into a tower

$$\cdots \to \mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A) \to \cdots \to \mathcal{TM}_1(A) \to \mathcal{TM}_0(A)$$

whose homotopy limit is equivalent to $\mathcal{TM}(A)$. We then approach the partial moduli spaces inductively, and show that $\mathcal{TM}_n(A)$ is tied to $\mathcal{TM}_{n-1}(A)$ by a simple homotopy fibre square (9.6, 9.7). The conclusion is that the spaces $\mathcal{TM}_n(A)$ are relatively accessible, and in fact have a surprisingly cohomological flavor. In analyzing them we are doing a type of homotopical deformation theory; the obstructions and choices at each level lie in the Quillen cohomology groups of A, which are the analogues for a Π-algebra of the Hochschild cohomology groups of an associative ring or the André-Quillen cohomology groups of a commutative ring.

One of the motivations for this paper is that we expect our study of the realization space of a Π-algebra to serve as a blueprint for the study of other moduli problems of a similar type. For that reason we have tried to keep our constructions and arguments as conceptual as possible. There are several lessons that might be learned from the paper. One is the usefulness of working with moduli spaces as a whole, rather than with their sets of components, if only because the moduli spaces tend to fit into fibration sequences and fibre squares. This is not a new lesson, but it comes through pretty clearly in what we do. Another point is the power and flexibility that can be gained by working with simplicial resolutions of objects (in our case simplicial resolutions of spaces) instead of with the objects themselves. Finally, on a much more technical level, suppose that F is a functor from finite sets to sets or spaces. The reader might be interested in how prolonging F to the category of simplicial sets can be interpreted as taking a homotopy coend (5.10); this explains to the authors a family of connectivity formulas (e.g. 5.1) which otherwise can seem mysterious.

We will now discuss our results in more detail.

The partial moduli spaces. We first describe how the partial moduli spaces $\mathcal{TM}_n(A)$ arise. Any space X has a *spherical resolution* $S(X)$; this is a simplicial space whose realization is equivalent to X, and each of whose simplicial constituents $S(X)[n]$ is equivalent to a wedge of spheres. In fact there is a model category structure on the category of simplicial spaces in which the cofibrant objects are spherical; the resolution $S(X)$ is obtained by treating X as a constant simplicial space and taking a cofibrant model for it. This is analogous to a standard procedure in homological algebra. There is a model category structure on the category of nonnegatively graded chain complexes in which the cofibrant objects are the chain complexes of projective modules. A projective resolution of a module M is then obtained by treating M as a chain complex concentrated in degree 0 and taking a cofibrant model for it.

Suppose now that A is a Π-algebra. Rather than directly trying to build a space X which realizes A, we try to build the resolution $S(X)$. This gives some added flexibility, because inside the category of simplicial spaces there are various types of Postnikov stages; we concentrate on one of these types, the *horizontal Postnikov stages* $\hat{P}_*$, and attempt to construct $S(X)$ inductively by building its Postnikov sections $\hat{P}_n S(X)$. It turns out that there is a simple algebraic condition that a simplicial space Y has to satisfy in order to be of the form $\hat{P}_n S(X)$ for some space X realizing A; if Y satisfies this property, we say that it is a *potential n-stage* for A. The partial moduli space $\mathcal{TM}_n(A)$ is then defined to be the moduli space of all potential n-stages for A, i.e., the nerve of the category whose objects are the simplicial spaces which are potential n-stages for A, and whose maps are the weak equivalences between these simplicial spaces.

Analyzing the partial moduli spaces. A *module* M over the Π-algebra A is defined to be an abelian Π-algebra with a certain kind of action by A, or equivalently as an abelian group object in the category of Π-algebras over A. Associated to such a module M are cohomology groups $H^n(A; M)$, $n \geq 0$. These cohomology groups can be described in terms of the homotopy groups of certain simplicial sets $\mathcal{H}^n(A; M)$ obtained by mapping A into Eilenberg-Mac Lane objects. The group $H^n(A; M)$ is given by $\pi_0 \mathcal{H}^n(A; M)$, and more generally there are isomorphisms

$$\pi_i \mathcal{H}^n(A; M) \simeq H^{n-i}(A; M) \ .$$

By functoriality the discrete group $\mathrm{Aut}(A, M)$ of automorphisms of the pair (A, M) acts on $\mathcal{H}^n(A; M)$, and we let $\hat{\mathcal{H}}^n(A; M)$ denote the Borel construction of this action. The group $\mathrm{Aut}(A, M)$ fixes the basepoint of

$\hat{\mathcal{H}}^n(A;M)$ (which corresponds to the zero element of $H^n(A;M)$), and this gives a natural map $\bar{W}\mathrm{Aut}(A,M) \to \hat{\mathcal{H}}^n(A;M)$. The A-modules that are interesting for our purposes are shifted copies $\Omega^m A$ of A itself. Our main result is the following one, which is a recast version (9.7) of Theorem 9.6. It provides an inductive approach to understanding the partial moduli spaces $\mathcal{TM}_n(A)$.

1.1**. Theorem.** *Suppose that A is a Π-algebra. Then $\mathcal{TM}_0(A)$ is equivalent to $\bar{W}\mathrm{Aut}(A)$, and for each $n \geq 1$ there is a homotopy fibre square*

$$\begin{array}{ccc} \mathcal{TM}_n(A) & \longrightarrow & \bar{W}\mathrm{Aut}(A,\Omega^n A) \\ \downarrow & & \downarrow \\ \mathcal{TM}_{n-1}(A) & \longrightarrow & \hat{\mathcal{H}}^{n+2}(A;\Omega^n A) \end{array} .$$

It follows immediately from the theorem that the homotopy fibre of $\mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A)$ over any point of $\mathcal{TM}_{n-1}(A)$ is equivalent to the generalized Eilenberg-Mac Lane space $\Omega\mathcal{H}^{n+2}(A;\Omega^n A) \sim \mathcal{H}^{n+1}(A;\Omega^n A)$. This space has nontrivial homotopy groups only in dimensions 0 through $n+1$, and so the tower $\{\mathcal{TM}_n(A)\}$ is a type of modified Postnikov system for $\mathcal{TM}(A)$. This tower is better than the usual Postnikov system for $\mathcal{TM}(A)$ in that the successive fibres depend in an explicit cohomological way on A. The tower also leads to an obstruction theory for finding a point in $\mathcal{TM}(A) \sim \mathrm{holim}\,\mathcal{TM}_n(A)$, i.e., an obstruction theory for finding a topological realization of A.

1.2**. Theorem.** *Suppose that A is a Π-algebra, and that Y is a potential $(n-1)$-stage for A. Then there is an associated element o_Y in $H^{n+2}(A;\Omega^n A)$, well-defined up to the action of $\mathrm{Aut}(A;\Omega^n A)$ on this group, such that Y lifts up to weak equivalence to a potential n-stage for A if and only if $o_Y = 0$.*

This theorem is proved by noticing that $\pi_0\hat{\mathcal{H}}^{n+2}(A;\Omega^n A)$ is the orbit space of the action of $\mathrm{Aut}(A,\Omega^n A)$ on $H^{n+2}(A;\Omega^n A)$; by 1.1, the path component P of $\mathcal{TM}_{n-1}(A)$ corresponding to Y is the image of a component of $\mathcal{TM}_n(A)$ if and only if the image of P in $\hat{\mathcal{H}}^{n+2}(A;\Omega^n A)$ lies in the component corresponding to the zero element of $H^{n+2}(A,\Omega^n A)$.

Interpretation of the partial moduli spaces. It is natural to ask about the conceptual nature of the partial moduli spaces $\mathcal{TM}_n(A)$. Since a vertex of $\mathcal{TM}_n(A)$ is just a simplicial space with is a potential n-stage for A, this amounts to asking what topological information relevant to the problem of realizing A is contained in such a Y. To begin with, the realization of Y is a connected space $X\langle 0,n{+}1\rangle$ with $\pi_i X\langle 0,n{+}1\rangle = A_i$ for $i \leq n+1$ and vanishing homotopy in higher dimensions; this is just the $(n+1)$'st (ordinary) Postnikov stage of a potential realization of

A. But there is more. Suppose that a and b are positive integers with $b > a$ and $b - a \le n$. With some simple manipulation (9.9) it is possible to extract from Y spaces $X\langle a, b\rangle$ with

$$\pi_i X\langle a, b\rangle = \begin{cases} A_i & a \le i \le b \\ 0 & \text{otherwise} \end{cases}.$$

This $X\langle a, b\rangle$ is the b'th ordinary Postnikov stage of the $(a-1)$'st connective cover of a potential realization of A. The various $X\langle a, b\rangle$ obtained in this way are as compatible as they can be when a and b vary; for instance $X\langle a, b-1\rangle$ is the $(b-1)$'st Postnikov stage of $X\langle a, b\rangle$. We interpret this to mean that giving a potential n-stage Y for A amounts among other things to threading the constituents of A together by k-invariants in such a way that the threads only reach a depth of n-dimensions. These threads create genuine spaces which realize each block of groups from A which is n dimensions or less in extent. As the threads grow in length one dimension at time (if possible, since by 1.2 there may be obstructions) the blocks of homotopy which achieve geometric expression also expand. In the limit, we obtain a space X with $\pi_* X = A$.

Organization of the paper. Section 2 contains a general discussion of moduli spaces, and §3 analyzes Postnikov theory for ordinary topological spaces in terms of moduli. Sections 4 and 6 treat the Postnikov theory of simplicial Π-algebras; this is what leads to the construction of our algebraic invariants. There is a detour in §5 to prove a general relative connectivity theorem that gives information about homotopy pushouts in the category of simplicial Π-algebras. Sections 7 and 8 look at simplicial spaces and their Postnikov theory, and §9 contains proofs of the main results.

1.3. **Notation.** We use the language of simplicial model categories ([19] [12] [15] [13]); if $\mathbf{C}$ is a simplicial model category and X and Y are objects of $\mathbf{C}$, then $\mathrm{Map}(X, Y)$ denotes the simplicial set of maps in $\mathbf{C}$ from X to Y. All of our model categories have functorial factorizations, in that a map $X \to Y$ can be naturally factored as a cofibration followed by an acyclic fibration, or as an acyclic cofibration followed by a fibration. The notation $\mathrm{Map}^{\mathrm{h}}(X, Y)$ denotes the derived mapping complex obtained by finding a functorial cofibrant model $X' \to X$ for X, a functorial fibrant model $Y \to Y'$ for Y, and forming $\mathrm{Map}(X', Y')$; the set $\pi_0 \mathrm{Map}^{\mathrm{h}}(X, Y)$ of derived homotopy classes of maps is denoted $[X, Y]$. In the same way, $\mathrm{Aut}^{\mathrm{h}}(X)$ is the simplicial monoid of self homotopy equivalences of some cofibrant/fibrant object weakly equivalent to X in a functorial way. Homotopy pushouts and pullbacks are constructed as usual [12, §10]; since the model categories have functorial factorization, we can take the homotopy pushouts and pullbacks to be functorial.

We will make use of Eilenberg-Mac Lane objects in various categories, and we will try to make notational distinctions between them. We use $\bar{W}G$ for the classifying simplicial set of a group or simplicial monoid G [17, §21]. The notations $\mathrm{B}G(M,n)$, $K_\Lambda(M,n)$, and $B_\Lambda(M,n)$ specify twisted Eilenberg-Mac Lane objects in, respectively, the category of pointed spaces (3.1), simplicial Π-algebras (6.1), and simplicial spaces (8.1). Here G is a group, Λ is a Π-algebra, M is a module over G or Λ, and n denotes the dimension in which M sits as a homotopy object. We will also need various coproducts: $\coprod$ is a generic coproduct, $\sqcup$ is the coproduct of sets or unpointed spaces, $\vee$ the coproduct for pointed spaces, and $*$ the coproduct for Π-algebras.

1.4. *Simplicial objects.* A simplicial object X in a category $\mathbf{C}$ is a functor from Δ^{op} to $\mathbf{C}$, where Δ is the simplicial category [17]. Equivalently, X is a collection $X[n]$, $n \geq 0$ of objects of $\mathbf{C}$, together with face maps $d_i : X[n] \to X[n-1]$ and degeneracy maps $s_i : X[n] \to X[n+1]$ which satisfy the standard simplicial identities. Note that we write $X[n]$ to distinguish the simplicial grading of X from a possible internal grading associated to the individual objects of $\mathbf{C}$. We identify $\mathbf{C}$ with the category of constant simplicial objects in $\mathbf{C}$, i.e., simplicial objects in which the face and degeneracy maps are identities.

1.5. *Simplicial disks and spheres.* Our basic reference for simplicial sets and their model category structure is [13]. It is convenient to have fixed models for simplicial disks and spheres. The standard simplicial model for the n-sphere is $\mathrm{c}S^n = \Delta_n/\partial\Delta_n$ (the letter "c" stands for combinatorial). It is natural to take as a model for the n-disk the combinatorial simplex Δ_n itself, so that the sphere $\mathrm{c}S^n$ is obtained from the disk by collapsing out the boundary. This convention is slightly awkward, because the boundary $\partial\Delta_n$ is not combinatorially isomorphic to $\mathrm{c}S^{n-1}$ (although these two complexes are weakly equivalent). To avoid this awkwardness we let Δ_n^0 be the contractible subcomplex of Δ_n obtained by taking the union of all faces of the top-dimensional simplex except the 0'th face, and we take as our simplicial model for the n-disk the quotient $\mathrm{c}D^n = \Delta_n/\Delta_n^0$. The inclusion of the 0'th face in Δ_n induces a map $\Delta_{n-1} \to \mathrm{c}D^n$ which is constant on $\partial\Delta_{n-1}$ and passes to an inclusion $\mathrm{c}S^{n-1} \to \mathrm{c}D^n$. This gives a cofibration sequence of pointed simplicial sets

$$\mathrm{c}S^{n-1} \to \mathrm{c}D^n \to \mathrm{c}S^n \ .$$

2. Moduli spaces

Here we define moduli spaces, and recall some of the properties of moduli spaces which arise from model categories. For our purposes, a moduli space is always the nerve [3, XI.2] of some category. The reader may be worried by the fact that the categories we consider in this connection are usually large, in the sense that the collection of objects forms a proper class instead of a set. The nerve of such a category is not strictly speaking a simplicial set. There are two ways to deal with this. One is to notice that the nerves we make use of are *homotopically small* [5] and so determine well-defined ordinary homotopy types. Another is to restrict in each case to a small subcategory of the category in question, a subcategory which is still large enough to have a nerve of the correct homotopy type; e.g., in the case of a model category $\mathbf{C}$, restrict to some small model subcategory of $\mathbf{C}$ containing some desired set of objects. The issues here are routine, and we will suppress them in order to avoid cluttering the exposition.

2.1**. Moduli spaces for objects.** A *category with weak equivalences* is a pair $(\mathbf{C}, \mathbf{W})$ consisting of a category $\mathbf{C}$ together with a subcategory $\mathbf{W}$ which contains all of the isomorphisms of $\mathbf{C}$. The morphisms of $\mathbf{W}$ are called weak equivalences. The basic examples are model categories, which come equipped with such subcategories of weak equivalences as part of the model category structure. Two objects X and Y of $\mathbf{C}$ are said to be weakly equivalent if they are related by the equivalence relation generated by the existence of a weak equivalence $f : X \to Y$.

If X is an object of a category with weak equivalences, the *moduli space* $\mathcal{M}(X)$ is defined to be the nerve of the subcategory of $\mathbf{C}$ consisting of all objects weakly equivalent to X together with the weak equivalences between them. By definition $\mathcal{M}(X)$ is connected. The main general theorem about it is the following.

2.2**. Theorem.** [7, 2.3] *Suppose that* $\mathbf{C}$ *is a simplicial model category and that* X *is an object of* $\mathbf{C}$*. Then there is a natural weak equivalence* $\mathcal{M}(X) \sim \bar{W}\mathrm{Aut}^{\mathrm{h}}(X)$.

If $\{X_\alpha\}$ is a set of objects in a category with weak equivalences, then $\mathcal{M}\{X_\alpha\}$ denotes the nerve of the category consisting of all objects weakly equivalent to one of the X_α's, together with the weak equivalences between these objects.

2.3**. Moduli spaces for diagrams.** Suppose that $\mathbf{C}$ is a category with weak equivalences and that $\mathbf{D}$ is some small category. The functor category $\mathbf{C}^{\mathbf{D}}$ is in a natural way a category with weak equivalences, where a natural transformation between functors is a weak equivalence if for each object in **D** it gives a weak equivalence in **C**. For instance, if **D** is a category with two objects and one nonidentity map between them, we obtain the *category of arrows* in **C**. Given a map $f : X \to Y$ in **C**, we let $\mathcal{M}(X \xrightarrow{f} Y) = \mathcal{M}(f)$ denote the moduli space of f inside the category of arrows. More generally, $\mathcal{M}(X \rightsquigarrow Y)$ denotes the moduli space of all arrows $X' \to Y'$, where X' is weakly equivalent to X and Y' is weakly equivalent to Y. If **C** is a model category, X is cofibrant, and Y is fibrant, then $\mathcal{M}(X \rightsquigarrow Y)$ is $\coprod_{\langle f\rangle} \mathcal{M}(f)$, where f ranges over weak equivalence classes of maps $X \to Y$. The indexing set here is not quite homotopy classes of maps (see 2.10).

If **C** is a category with some specified notion of homotopy groups or homotopy objects π_i, $i \geq 0$, then for convenience we let $\mathcal{M}(X \looparrowright Y)$ denote the moduli space of arrows $f : X' \to Y'$, where X' is weakly equivalent to X, Y' is weakly equivalent to Y, and f induces isomorphisms on π_i for all i with the property that $\pi_i X$ and $\pi_i Y$ are *both* nontrivial. Note that $\mathcal{M}(X \looparrowright Y)$ is a (possibly empty) union of components of $\mathcal{M}(X \rightsquigarrow Y)$.

We use similar notation for moduli spaces of pairs of arrows. For instance $\mathcal{M}(X \rightsquigarrow Y \looparrowleft Z)$ denotes the moduli space of all diagrams $U \to V \leftarrow W$ in which U, V and W are weakly equivalent to X, Y and Z respectively, and the map $W \to V$ has the appropriate isomorphism property on homotopy.

2.4**. Function spaces as moduli.** We also need to express derived function complexes as moduli spaces. If X and Y are two objects of a model category **C**, let $\mathcal{M}_{\mathrm{Hom}}(X, Y)$ denote the nerve of the category whose objects are diagrams $X \leftarrow U \to V \leftarrow Y$ in which the maps $U \to X$ and $Y \to V$ are weak equivalences. The morphisms are commutative diagrams

$$\begin{array}{ccccccc} X & \xleftarrow{\sim} & U & \longrightarrow & V & \xleftarrow{\sim} & Y \\ {\scriptstyle =}\downarrow & & {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle =} \\ X & \xleftarrow{\sim} & U' & \longrightarrow & V' & \xleftarrow{\sim} & Y \end{array} \tag{2.5}$$

in which the indicated maps are identities or weak equivalences.

2.6**. Theorem.** [6, 4.7] [5, 1.1] *Suppose that* **C** *is a simplicial model category and that* X *and* Y *are objects of* **C**. *Then* $\mathcal{M}_{\mathrm{Hom}}(X, Y)$ *is in a natural way weakly equivalent to the simplicial set* $\mathrm{Map}^{\mathrm{h}}(X, Y)$.

2.7. *Remark.* One can consider a similar category whose objects are the smaller diagrams $X \xleftarrow{\sim} U \to Y$; this is the full subcategory of the above given by diagrams in which the map $Y \to V$ is required to be the identity. We denote the nerve of this category $\mathcal{M}^{\mathrm{f}}_{\mathrm{Hom}}(X, Y)$. If Y is a fibrant object of **C**, then the inclusion $\mathcal{M}^{\mathrm{f}}_{\mathrm{Hom}}(X,Y)\to\mathcal{M}_{\mathrm{Hom}}(X,Y)$ is a weak equivalence. This follows from the arguments of [6, 7.2].

2.8**. Relationships between moduli spaces.** Suppose that X and Y are two objects of a model category **C**. There is a map $\mathcal{M}_{\mathrm{Hom}}(X,Y)\to\mathcal{M}(X\rightsquigarrow Y)$ given by the functor which sends a diagram $X\leftarrow U\to V\leftarrow Y$ to the diagram $U\to V$. The composite of this with the obvious projection $\mathcal{M}(X\rightsquigarrow Y)\to\mathcal{M}(X)\times\mathcal{M}(Y)$ is again given by a functor, and this is connected to the constant functor with value (X,Y) by a chain of two natural transformations. This induces a map from $\mathcal{M}_{\mathrm{Hom}}(X,Y)$ to the homotopy fibre of the projection.

2.9**. Theorem.** *Suppose that X and Y are two objects of a model category* **C**. *The sequence*

$$\mathcal{M}_{\mathrm{Hom}}(X,Y)\to\mathcal{M}(X\rightsquigarrow Y)\xrightarrow{p}\mathcal{M}(X)\times\mathcal{M}(Y)$$

is a homotopy fibre sequence, in the sense that the natural map from $\mathcal{M}_{\mathrm{Hom}}(X,Y)$ to the homotopy fibre of p is a weak equivalence.

Proof. This follows from Quillen's Theorem B [18], given the observation, immediate from 2.6, that weak equivalences $X\to X'$ and $Y'\to Y$ induce a weak equivalence $\mathcal{M}_{\mathrm{Hom}}(X,Y)\to\mathcal{M}_{\mathrm{Hom}}(X',Y')$.

2.10. *Remark.* Theorem 2.9 indicates that in the model category case the set which indexes the components of $\mathcal{M}(X\rightsquigarrow Y)$ is the set of homotopy classes of maps from X to Y, modulo the action on the one hand of the self homotopy equivalences of X and on the other of the self homotopy equivalences of Y.

2.11. *Remark.* The proof of 2.9 gives many other similar results. For instance, given three objects X, Y, Z in an appropriate model category, there is a natural homotopy fibre sequence

$$\mathcal{M}_{\mathrm{Hom}}(X,Y)\to\mathcal{M}(X\rightsquigarrow Y\leftsquigarrow Z)\to\mathcal{M}(X)\times\mathcal{M}(Y\leftsquigarrow Z)\ .$$

3. Postnikov systems for spaces

In this section we sketch an approach to the Postnikov theory of pointed topological spaces which is based on the use of moduli spaces. Our object is to establish some notation and provide a context for what we do later on. We assume that the spaces are pointed and usually (for convenience) that they are connected. The category of pointed topological spaces has its usual model category structure [19, II.3] [12, §8] in which weak equivalences are weak homotopy equivalences, fibrations are Serre fibrations, and cofibrations are retracts of relative cell complexes.

Postnikov systems. Attaching an $(n+2)$-cell to a space X by a map $f : S^{n+1} \to X$ has no effect on the homotopy of X in dimensions $\le n$, and clearly kills off the class represented by f in $\pi_{n+1}X$. Now attach cells of dimension $(n+2)$ and greater to X by all possible attaching maps to obtain an inclusion $X \subset X_1$, repeat the process to obtain $X_1 \subset X_2$, repeat again, etc., and let $P_nX = \cup_k X_k$. There is a map $X \to P_nX$ which induces isomorphisms on π_i for $i \le n$, and $\pi_i P_n X \simeq 0$ for $i > n$. The construction of P_nX is functorial in X and preserves weak equivalences, and so it induces a map $\mathcal{M}(P_nX) \to \mathcal{M}(P_{n-1}X)$.

3.1. *Eilenberg-Mac Lane objects.* If G is a group, we say that a space X is *of type* $\mathrm{B}G$ if $\pi_1 X$ is isomorphic to G and the higher homotopy of X vanishes. Suppose that M is a G-module. We say that a map $X \to Y$ is *of type* $\mathrm{B}G(M,n)$, $n \ge 2$, if X is of type $\mathrm{B}G$, $\pi_1 Y \simeq G$, $\pi_n Y \simeq M$ (as a G-module), all other homotopy groups of Y vanish, and the map $X \to Y$ gives an isomorphism on π_1. Sometimes we say for short that the target space Y is of type $\mathrm{B}G(M,n)$.

The difference construction. Suppose that $f : Y \to X$ is a map of spaces. Consider the pushout C of the diagram $X' \leftarrow Y' \to (P_1X)'$ obtained by using some functorial construction to replace Y by a cofibrant space and the two maps $Y \to X$ and $Y \to P_1X$ by cofibrations. There is a commutative diagram

$$
\begin{array}{ccccc}
Y & \xleftarrow{\sim} & Y' & \longrightarrow & (P_1X)' \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_n(f)} \\
X & \xleftarrow{\sim} & X' & \longrightarrow & P_{n+1}C
\end{array}
\tag{3.2}
$$

We denote the vertical map on the right by $\Delta_n(f)$; its source is $\Delta_n^{\mathrm{s}}(f)$ and its target is $\Delta_n^{\mathrm{t}}(f)$.

The following is easy to prove by calculating that, in the above situation, if $X \to Y$ is surjective on π_1 then the universal cover of C is the homotopy cofibre of the map $\tilde{X} \to \tilde{Y}$, where $\tilde{Y}$ is the universal cover of Y and $\tilde{X}$ is the pullback of the cover $\tilde{Y}$ to X.

3.3. **Proposition.** *Suppose that $f : Y \to X$ is a map of spaces whose homotopy fibre F is $(n-1)$-connected, $n \ge 1$. Let $M = \pi_n F$ and if $n = 1$ assume that M is abelian. Then M is naturally a G-module for $G = \pi_1 F$, and $\Delta_n(f)$ is a map of type* $\mathrm{B}G(M, n+1)$. *If $\pi_i F$ vanishes except for $i = n$, then the right-hand square in 3.2 is a homotopy fibre square.*

Existence and uniqueness of Eilenberg-Mac Lane objects. It is easy to construct spaces of type $\mathrm{B}G$ by hand (take a wedge of circles indexed by a set of generators for G, attach a 2-cell for each relation between the generators, and apply the functor P_1) or by taking the geometric realization of $\bar{W}G$. A simple argument gives that these spaces are unique up to weak equivalence. We let $\mathrm{B}G$ denote a generic cofibrant space of this type. It follows from obstruction theory or covering space theory that $\mathrm{Aut}^{\mathrm{h}}(\mathrm{B}G)$ is homotopically discrete and that its group of components is $\mathrm{Aut}(G)$. Another way to express this is to say that the moduli space of all spaces of type $\mathrm{B}G$ is weakly equivalent to $\bar{W}\mathrm{Aut}(G)$. The next proposition extends this to higher Eilenberg-Mac Lane objects.

If G is a group and M is a G-module, we write $\mathrm{Aut}(G, M)$ for the group of pairs (α, β), where α is an automorphism of G and β is an α-linear automorphism of M. This is the same as the group of automorphisms of the split short exact sequence

$$0 \longrightarrow M \longrightarrow G \rtimes M \leftrightarrows G \longrightarrow 0\,.$$

3.4. **Proposition.** *Let G be a group, M a G-module, and $n \geq 2$ and integer. Then the moduli space of all maps of type* $\mathrm{B}G(M,n)$ *is weakly equivalent to* $\bar{W}\mathrm{Aut}(G,M)$.

3.5. *Remark.* In particular the moduli space is nonempty and connected, and so spaces or maps of type $\mathrm{B}G(M,n)$ exist and are unique up to weak equivalence. We denote a generic space of this type by $\mathrm{B}G(M,n)$.

Sketch of proof. Let $\mathcal{M}_n$, $n \geq 2$, denote the moduli space of all maps $X \to Y$ of type $\mathrm{B}G(M,n)$. There is a map $\mathcal{M}_n \to \mathcal{M}_{n+1}$ induced by the functor which sends $X \to Y$ to $\Delta_n(X \to P_1X)$. There is also a map $\mathcal{M}_{n+1} \to \mathcal{M}_n$ induced by the functor which sends $X \to Y$ to the homotopy pullback of $X \to Y \leftarrow X$. Both composite functors are connected to the respective identity functors by chains of natural transformations, and so these maps of moduli spaces are weak equivalences. Similar constructions give a weak equivalence $\mathcal{M}_2 \sim \mathcal{M}(\mathrm{B}(G \rtimes M) \leftrightarrows \mathrm{B}G)$, where this last denotes the moduli space of maps $U \to V$ with a section $V \to U$, such that U and V have no higher homotopy groups, and such that on the level of π_1 the map $U \to V$ with its section gives a diagram of groups isomorphic to $G \rtimes M \leftrightarrows G$. Now compute directly that this last moduli space is weakly equivalent to $\bar{W}\mathrm{Aut}(G,M)$. □

3.6. *Cohomology of spaces.* Consider a space $\mathrm{B}G(M,n)$, $n \geq 2$. Then $P_1\,\mathrm{B}G(M,n) \sim \mathrm{B}G$, and so we write the map from this space to its first Postnikov stage as $\mathrm{B}G(M,n) \to \mathrm{B}G$. Given another space X over $\mathrm{B}G$ (i.e. with a map $X \to \mathrm{B}G$), we define $H^n_G(X;M)$ by

$$H^n_G(X;M) \simeq [X, \mathrm{B}G(M,n)]_{\mathrm{B}G}$$

where the symbol on the right denotes derived (1.4) homotopy classes of maps from X to $\mathrm{B}G(M,n)$ in the model category of spaces over

BG [12, 3.11]. Let $\mathcal{H}^n_G(X;M)$ denote $\mathrm{Map}^{\mathrm{h}}_{\mathrm{B}G}(X, \mathrm{B}G(M,n))$, so that $H^n_G(X;M)$ is π_0 of this space. The homotopy fibre squares

$$\begin{array}{ccc} \mathrm{B}G(M,n-1) & \longrightarrow & \mathrm{B}G \\ \downarrow & & \downarrow \\ \mathrm{B}G & \longrightarrow & \mathrm{B}G(M,n) \end{array}$$

give natural weak equivalences $\Omega\mathcal{H}^n_G(X;M) \sim \mathcal{H}^{n-1}_G(X;M)$, so that there are isomorphisms

$$\pi_i\mathcal{H}^n_G(X;M) \simeq \begin{cases} H^{n-i}_G(X;M) & 0 \le i \le n-2 \\ 0 & i > n \end{cases} .$$

We use this formula to define $H^i_G(X;M)$ for $i = 0, 1$; because we are working with pointed maps these turn out to be what would normally be called reduced twisted cohomology groups.

Classification of Postnikov stages. Suppose that X is a space with $X \sim P_{n-1}X$, $n \ge 2$, and that M is a module over $G = \pi_1 X$. If Y is a space, we write $Y \sim X + (M,n)$ if $P_nY \sim Y$, $P_{n-1}Y \sim X$, and $\pi_nY \simeq M$ as a module over G, where this module isomorphism is realized with respect to some isomorphism $\pi_1Y \simeq G$. We write $\mathcal{M}(X + (M,n))$ for the moduli space of all spaces Y of this type.

3.7. **Proposition.** *Suppose that X is a space with $X \sim P_{n-1}X$, $n \ge 2$ and that M is a module over $G = \pi_1X$. Then there is a natural weak equivalence of moduli spaces*

$$\mathcal{M}(X + (M,n)) \sim \mathcal{M}(X \looparrowright \mathrm{B}G(M,n+1) \looparrowleft \mathrm{B}G) .$$

3.8. *Remark.* The arrows $\looparrowright$ on the right indicate maps which induce isomorphisms on appropriate homotopy groups (2.3); in this case it is just isomorphisms on π_1.

Proof. There is a functor in one direction which given a space $Y \sim X + (M,n)$ constructs the diagram $(P_{n-1}Y)' \to \Delta^{\mathrm{t}}_n(f) \leftarrow \Delta^{\mathrm{s}}_n(f)$ from 3.2, where f is the map $Y \to P_{n-1}Y$. There is a functor in the other direction which given $U \to V \leftarrow W$ of type $X \looparrowright \mathrm{B}G(M,n+1) \looparrowleft \mathrm{B}G$ constructs the space $Y \sim X + (M,n)$ which is the homotopy pullback of $U \to V \leftarrow W$. Both composites are connected to the corresponding identity functors by chains of natural transformations, and so they induce weak equivalences on the moduli spaces. □

3.9. *Interpretation.* Let X, G and M be as above. According to 3.7, 3.4, and 2.11, there is a fibration sequence

(3.10) $\mathrm{Map}^{\mathrm{h}}_1(X, \mathrm{B}G(M,n+1)) \to \mathcal{M}(X + (M,n)) \to \mathcal{M}(X) \times \bar{W}\Gamma$.

where $\Gamma = \mathrm{Aut}(G, M)$ and the object on the left is the union of the components of $\mathrm{Map}^{\mathrm{h}}(X, \mathrm{B}G(M, n))$ giving maps which induce isomorphisms on π_1. It is easy to identify this subcomplex as $\sqcup_\alpha \mathcal{H}^G_X(n + 1; M_\alpha)$, where α runs through the isomorphisms $\pi_1 X \to G$ and M_α is the module over $\pi_1 X$ determined by M and α. Each space $Y \sim X + (M, n)$ determines an element of

$$\pi_0\left(\sqcup_\alpha \mathcal{H}^G_X(n + 1; M_\alpha)\right) \simeq \sqcup_\alpha H^{n+1}_G(X; M_\alpha)$$

modulo the action of $\pi_0 \mathrm{Aut}^{\mathrm{h}}(X) \times \mathrm{Aut}(G, M)$ on this set; this is the k-invariant $k_n(Y)$, in its genuinely invariant form. Correspondingly, each k-invariant gives rise to a space Y. Note that 3.7 not only classifies spaces of type $X + (M, n)$, but also determines their self-equivalences.

The reader might want to compare fibration 3.10 with the corresponding fibration

$$\mathrm{Map}^{\mathrm{h}}_0(X, \mathrm{B}\gamma(M, n + 1))_{\mathrm{u}} \to \mathcal{M}_{\mathrm{u}}(X + (M, n)) \to \mathcal{M}_{\mathrm{u}}(X)$$

from [9]. Here $\gamma = \mathrm{Aut}(M)$, $\mathrm{Map}^{\mathrm{h}}_0(\text{–}, \text{–})_{\mathrm{u}}$ denotes an appropriate set of components of the space of unpointed maps, and $\mathcal{M}_{\mathrm{u}}$ is the unpointed moduli space. The appearance of the extra factor in the base of the our fibration 3.10 is explained by the fact that for us the target of the k-invariant map is $\mathrm{B}G(M, n+1)$, $G = \pi_1 X$, while in [9] it is $\mathrm{B}\gamma(M, n+1)$, $\gamma = \mathrm{Aut}(M)$; the extra factor allows for potential automorphisms of M which are not induced by elements of G.

4. Π-algebras and their modules

Here we explore Π-algebras, simplicial Π-algebras, and modules over them. This is in preparation for a discussion in §6 of their cohomology.

4.1. Π-algebras. Let Π be the full sub-category of the homotopy category of pointed spaces closed under isomorphism and containing the wedges of spheres

$$S^{n_1} \vee \ldots \vee S^{n_k}$$

with $n_i \geq 1$. A Π-*algebra* is a product-preserving functor

$$\Lambda : \Pi^{\mathrm{op}} \longrightarrow \mathcal{S} \ ,$$

or equivalently a contravariant functor $\Pi \to \mathcal{S}$ which takes wedges to products. This condition and the Hilton-Milnor Theorem imply that Λ is determined by the sets $\Lambda_n = \Lambda(S^n)$, $n \geq 1$ and the following additional data:

(1) a group structure on Λ_n which is abelian for $n > 1$;
(2) composition maps $\Pi(S^n, S^k) \times \Lambda_k = \pi_n(S^k) \times \Lambda_k \to \Lambda_n$;
(3) Whitehead product maps $[\ ,\ ] : \Lambda_n \times \Lambda_k \to \Lambda_{n+k-1}$;

(4) a Λ_1-module structure on each abelian group Λ_n, $n > 1$.

There are relations among these structures; for example, (4) is redundant, since for $x \in \Lambda_1$ and $a \in \Lambda_n$,

$$ax = [a, x] + a$$

where $+$ is the group operation on Λ_n. The relations are classical, but are complicated to write down [4]. We omit them, as the exact formulas are unnecessary for our purposes. But recall that composition is not additive: if $\{\omega\}$ is a basis for the free Lie algebra over $\mathbb{Z}$ on two generators, then for $x, y \in \Lambda_k$, $k > 1$, and $\alpha \in \pi_k S^n$, we have

$$(x + y) \circ \alpha = x \circ \alpha + y \circ \alpha + \sum_{\omega} \omega(x, y) \circ H_\omega(\alpha) \tag{4.2}$$

where the sum is over elements ω of length greater than 1, we write $\omega(x, y)$ for the corresponding iterated Whitehead product, and H_ω is the associated higher Hopf invariant [22, §XI.8.5]. We may at times take Λ to be the graded group $\{\Lambda_n\}$ together with this additional structure; however, we will often stipulate Π-algebras by displaying the functor

$$U \mapsto \Lambda(U)$$

from Π^{op} to the category of sets. In particular, we will often write U for an object in the category Π. Π-algebras form a category, in which the morphisms are natural transformations of functors.

4.3. *Example.* If X is a pointed space, there is a Π-algebra $\pi_* X$ given by the functor which sends $U \in \Pi$ to the set $[U, X]$ of homotopy classes of pointed maps from U to X. Note that $\pi_*(X)_n = \pi_n X$, and that this functor does not include $\pi_0 X$. The Π-algebra $\pi_* X$ captures the homotopy groups of X and all of the primary operations tying these groups together. The construction $\pi_*(-)$ gives a functor from the homotopy category of pointed spaces to the category of Π-algebras.

The category of Π-algebras is a category of universal algebras and has all limits and colimits. We write 0 for the trivial object, which can be described as $\pi_* X$ for X a one-point space. This object is both initial and terminal, and the category of Π-algebras is *pointed* in the sense that the unique map from the initial object to the terminal object is an isomorphism.

4.4. **Simplicial Π-algebras.** As usual, a *simplicial Π-algebra* A is a simplicial object (1.4) in the category of Π-algebras. The Π-algebra $A[n]$ is the portion of A in simplicial degree n, and $A[n]_i$ is the group (abelian if $i > 1$) which is the i'th constituent of the Π-algebra $A[n]$. We write A_i for the associated simplicial group which in simplicial dimension n contains the group $A[n]_i$. Each simplicial group A_i has homotopy groups $\pi_* A_i$, which can be computed from the associated normalized (Moore) complex $N(A_i)$ [17, 17.3, 22.1]. We let $\pi_* A$ denote the collection of all of these homotopy groups.

4.5. *Model category structure.* By Quillen [19, §II.4], there is a standard simplicial model category structure on the category of simplicial Π-algebras. In this structure, a map $f : A \to B$ is a weak equivalence if and only if it is a weak equivalence of graded simplicial groups, i.e., if and only if $\pi_* A \to \pi_* B$ is an isomorphism. Every object is fibrant, and a map $A \to B$ is a fibration if for each i the induced map $N(A_i) \to N(B_i)$ is surjective in degrees 1 and above. A map is a cofibration if and only if it is a retract of a map which is "free" in the sense of [19, §II.4]. To define these free maps, note that the forgetful functor from Π-algebras to graded sets has a left adjoint F with

$$F(V_*) \cong \pi_*(\vee_n \vee_{x \in V_n} S^n) \simeq *_n *_{x \in V_n} \pi_* S^n.$$

Then a morphism $A \to B$ of simplicial Π-algebras is *free* if for each $n \geq 0$ there is a graded set $V_n \subset B[n]$, closed under the degeneracy maps in B, such that

$$B[n] \cong A[n] * F(V_n).$$

Suppose that A is a simplicial Π-algebra and K is a simplicial set. The simplicial structure on the category of simplicial Π-algebras is given by letting $K \otimes A$ be the simplicial object with $(K \otimes A)[n] = *_{s \in K[n]} A[n]$.

4.6. *Cells.* Suppose in the above situation that K is a pointed simplicial set. In this case we write $K \bar{\otimes} A = (K \bar{\otimes} A)/(* \bar{\otimes} A)$, where the quotient is taken in the category of simplicial Π-algebras. The pairs $(\mathrm{c}D^{i+1} \bar{\otimes} \pi_* S^j, \mathrm{c}S^i \bar{\otimes} \pi_* S^j)$, $i \geq 0$, $j \geq 1$, are called *cells*, and a simplicial Π-algebra is *cellular* if it can be constructed from a trivial simplicial Π-algebra by attaching cells, perhaps transfinitely often. Any cellular simplicial Π-algebra is cofibrant, any simplicial Π-algebra has a functorial cellular approximation, and any cofibrant simplicial Π-algebra is a retract of a cellular one.

Cells are attached to A by elements in $\pi_* A$, in that $[\mathrm{c}S^n \bar{\otimes} \pi_* S^j, A]$ is isomorphic to $\pi_n A_j$. Note that in fact for each $n \geq 0$, $\pi_n A$ is a Π-algebra, given as a functor (4.1) by the formula

$$(\pi_n A)(U) = [\mathrm{c}S^n \bar{\otimes} \pi_* U, A], \quad U \in \Pi. \tag{4.7}$$

4.8. **Abelian Π-algebras; modules.** A Π-algebra M is *abelian* if the map $M \times M \to M$ given in each gradation by group multiplication is a map of Π-algebras. This is equivalent to saying that M admits the structure of an abelian group object in the category of Π-algebras, or more concretely to saying that all of the Whitehead products in

M vanish [1]. The full-subcategory of Π-algebras consisting of abelian Π-algebras is an abelian category.

As in any category of universal algebras, the notion of *module* is a relativization of this concept.

4.9. **Definition.** Given a Π-algebra Λ, a Λ-*module* is an abelian group object in the category of Π-algebras over Λ.

More explicitly, a Λ-module amounts to a split short exact sequence of Π-algebras

$$0 \longrightarrow M \longrightarrow E_M \leftrightarrows \Lambda \longrightarrow 0 \tag{4.10}$$

in which M is an abelian Π-algebra. A morphism of Λ-modules is a map of split sequences which is the identity on Λ. We will sometimes identify a Λ-module with M and leave the short exact sequence understood; in particular, we usually write $M \to N$ for a morphism of Λ-modules. Since the graded constituents of a Π-algebra are already groups, it is easy to see that an abelian group object in the category of Π-algebras over Λ is the same as a group object in this category.

Modules via actions. A Λ-module M gives rise to a type of action of Λ on M. To see this, observe that the splitting of $E_M \to \Lambda$ determines, for each $U \in \Pi$, an isomorphism of sets

$$E_M(U) \cong \Lambda(U) \times M(U).$$

This means that for each map $f : V \to U$ in Π, the morphism $E_M(f) : E_M(U) \to E_M(V)$ is determined by an action map

$$\phi_f : \Lambda(U) \times M(U) \to M(V) \tag{4.11}$$

subject to the conditions

(1) $\phi_f(0, x) = M(f)(x)$, and
(2) $\phi_{g\circ f}(a, x) = \phi_g(\Lambda(f)a, \phi_f(a, x))$.

It is even possible to go in the other direction. Given maps 4.11 subject to the indicated conditions, we can form a Π-algebra $\Lambda \rtimes M$ which lies in a split sequence

$$0 \longrightarrow M \longrightarrow \Lambda \rtimes M \leftrightarrows \Lambda \longrightarrow 0$$

and so define a Λ-module structure on M. If M began life as a Λ-module, there is an isomorphism of Π-algebras $E_M \cong \Lambda \rtimes M$, making the evident diagram of split sequences commute.

4.12. *Modules via split cofibration sequences.* A *split cofibration sequence* in a pointed model category $\mathbf{C}$ is a diagram

$$A \leftrightarrows B \longrightarrow C$$

in **C** such that the objects involved are cofibrant, $A \to B \to C$ is a cofibration sequence, and the left-hand maps exhibit A as a retract of B. Suppose that there are functors $\phi, \psi : \Pi \to \mathbf{C}$ which take on cofibrant values and preserve coproducts up to weak equivalence. Then there are Π-algebras M_X and Λ_X associated to any object X of **C** and given by the formulas

$$\Lambda_X(U) = [\phi U, X] \qquad M_X(U) = [\psi U, X]\ .$$

In order to show that M_X is in a natural way a module over Λ_X it is enough to prove that M_X is abelian for each X, and to construct objects $\psi_+ U$ which fit into split cofibration sequences

$$\phi U \leftrightarrows \psi_+ U \longrightarrow \psi U$$

which are natural in U. For the split sequences encoding the module structure (4.10) can be constructed by mapping this split cofibration sequence into objects X of **C**. Note that in order to show that M_X is an abelian Π-algebra for each X, it is enough by Yoneda's lemma to show that ψU is a cogroup object in the homotopy category of **C** in a way which is natural in U.

4.13. *Examples.* A Π-algebra Λ is not a module over itself, unless Λ is abelian. However, we may define new Π-algebras $\Omega^n \Lambda$ by the functor on Π^{op}

$$U \mapsto \Lambda(S^n \wedge U).$$

This mimics topology: $\Omega^n \pi_* X \cong \pi_* \Omega^n X$. For $n \geq 1$, $\Omega^n \Lambda$ is a Λ-module. To see this, define a Π-algebra $\Omega^n_+ \Lambda$ by

$$U \mapsto \Lambda(S^n_+ \wedge U)$$

where the $(-)_+$ denotes adding a disjoint basepoint. Then there is a split sequence

$$0 \longrightarrow \Omega^n \Lambda \longrightarrow \Omega^n_+ \Lambda \leftrightarrows \Lambda \longrightarrow 0$$

which gives a canonical Λ-module structure on $\Omega^n \Lambda$. These module structures are central to what follows in this paper; they arise from the fact that in the homotopy category of pointed spaces, S^n_+ for $n \geq 1$ is a cogroup object in the category of spaces under S^0. Note that if X is a space then $\Omega^n_+ \pi_* X$ is naturally isomorphic to the homotopy Π-algebra of the space of all (not necessarily pointed) maps $S^n \to X$.

If we have a morphism $M \to N$ of Λ-modules, the ordinary kernel K is a Λ-module; the necessary total space E_K for the split sequence is the pull-back of $E_M \to E_N \overset{s}{\leftarrow} \Lambda$. If M is a Λ-module, so is $\Omega_+ M$;

the total space of the split sequence is defined by the pull-back square

$$\begin{array}{ccc} E_{\Omega_+ M} & \longrightarrow & \Omega_+ E_M \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{s} & \Omega_+ \Lambda \end{array} .$$

Consequently, if M is a Λ-module, ΩM is a Λ module: it is the kernel of $\Omega^n_+ M \to M$. It is easy to check that the Λ-module structure on $\Omega^n \Lambda$ described above is the same as that obtained inductively by starting with the given Λ-module structure on $\Omega\Lambda$ and making the identification $\Omega^n\Lambda \simeq \Omega(\Omega^{n-1}\Lambda)$.

4.14. *Homotopy group modules.* For $n \geq 1$, $\mathrm{c}S^n$ is a cogroup object in the homotopy category of pointed simplicial sets, and there is a split cofibration sequence

$$\mathrm{c}S^0 \leftrightarrows \mathrm{c}S^n_+ \longrightarrow \mathrm{c}S^n \qquad (4.15)$$

of pointed simplicial sets, where $(-)_+$ denotes adding a disjoint basepoint. Tensoring this with $\pi_* U$ for $U \in \Pi$ (4.7) gives the structure necessary (4.12) to show that for any simplicial Π-algebra A, $\pi_n A$ is abelian for $n \geq 1$ and is naturally a module over $\pi_0 A$.

5. Relative connectivity of pushouts

In this section we give a partial calculation of the homotopy type of the homotopy pushout of a diagram of simplicial Π-algebras (5.1). This is along the lines of [21, 1.10, 3.6], but we work in more generality and remove some simple connectivity hypotheses.

To express the result we will introduce a slightly unorthodox notion of connectivity. If $f : A \to B$ is a map of simplicial sets, the *cellular connectivity* of f, denoted $\kappa(f)$ (or $\kappa(B, A)$ if f is understood), is the greatest integer n such that f can be obtained up to weak equivalence by taking A (or a fibrant representative) and attaching cells of dimension n and above. If f is a weak equivalence, then $\kappa(f) = \infty$. More precisely, $\kappa(f) = n$ if and only if all of the homotopy fibres of f are $(n-2)$-connected, and at least one of the homotopy fibres is not $(n-1)$-connected. The numbers here are potentially confusing. One rough way to remember them is to keep in mind that if A and B are 1-connected and A is a subcomplex of B, then $\kappa(B, A)$ is the lowest dimension in which B/A has nontrivial homology (or homotopy).

If $f : A \to B$ is a map of simplicial Π-algebras or of graded simplicial sets, we let $\kappa(B, A)$ denote the minimum value of the numbers $\kappa(B_n, A_n)$, $n \geq 1$. In the statement of the following proposition the symbol $\cup^{\mathrm{h}}$ denotes homotopy pushout in the category of graded simplicial sets, while $*^{\mathrm{h}}$ is homotopy pushout in the category of simplicial Π-algebras.

5.1. **Proposition.** *Suppose that $B \leftarrow A \to C$ is a two-source of simplicial Π-algebras. Then*

$$\kappa(B *^{\mathrm{h}}_A C, B \cup^{\mathrm{h}}_A C) \geq \kappa(B, A) + \kappa(C, A) .$$

We will deduce 5.1 from some very general observations. A *finite graded set* is a graded set which is finite in every gradation and empty in all but a finite number of gradations. Consider a functor F from the category of finite graded sets to the category of graded simplicial sets. There is a standard way to prolong F to a functor on the category of all graded sets by setting

$$F(T) = \mathrm{colim}_{S \subset T} F(S) \ , \tag{5.2}$$

where the colimit is taken over the category of finite graded subsets of T. The functor F can be further prolonged to a functor on the category of graded simplicial sets by setting

$$F(X) = \mathrm{diag}(n \mapsto F(X[n]) \ . \tag{5.3}$$

Here diag is the *diagonal* or *realization* functor from the category of bisimplicial sets to the category of simplicial sets [13, IV.1]. The argument of diag in the above formula is a graded bisimplicial set, but the diagonal is to be taken gradation by gradation. In each of the following statements the functor F involved is prolonged like this to a functor on the category of graded simplicial sets.

5.4. **Proposition.** *Any functor F from finite graded sets to graded simplicial sets respects cellular connectivity, in the sense that for any map $X \to Y$ of graded simplicial sets there is an inequality*

$$\kappa(F(Y), F(X)) \geq \kappa(Y, X) \ .$$

5.5. **Proposition.** *Any functor F from finite graded sets to graded simplicial sets preserves homotopy pushouts in the stable range, in the sense that for any two-source $Y \leftarrow X \to Z$ of graded simplicial sets there is an inequality*

$$\kappa\left(F(Y \cup^{\mathrm{h}}_X Z),\ F(Y) \cup^{\mathrm{h}}_{F(X)} F(Z)\right) \geq \kappa(Y, X) + \kappa(Z, X) .$$

We also need the following lemma, which can be proved by the same sort of gluing argument used in the proof of [13, IV.1.7].

5.6. **Lemma.** *Suppose that $X \to Y$ is a map of bisimplicial sets, and that n is an integer such that $\kappa(Y[i], X[i]) \geq n$ for all $i \geq 0$. Then $\kappa(\mathrm{diag}(Y), \mathrm{diag}(X)) \geq n$.*

Proof of 5.1. This is similar to the second part of the proof of [21, 3.6]. First, some background. Let F denote the free functor from graded sets to Π-algebras, prolonged degreewise to be a functor from graded simplicial sets to simplicial Π-algebras. For any simplicial Π-algebra D there is a bar resolution $\mathcal{B}(D)$ [21, 3.2]; this is a bisimplicial Π-algebra, i.e. a simplicial object in the category simplicial Π-algebras, with $\mathcal{B}(D)[n] = F^{n+1}(D)$. Let $\bar{D} = \operatorname{diag}(\mathcal{B}(D))$. By [21, 3.2], $\bar{D}$ is a cofibrant simplicial Π-algebra; more generally, if $D \to D'$ is a map of simplicial Π-algebras which is an injection of underlying graded simplicial sets, then the maps $\mathcal{B}(D)[n] \to \mathcal{B}(D')[n]$ and the diagonal map $\bar{D} \to \bar{D}'$ are both cofibrations. There is a natural weak equivalence $\bar{D} \to D$.

Now for the proof. By adjusting the objects up to weak equivalence, we can assume that the maps $A \to B$ and $A \to C$ are cofibrations of simplicial Π-algebras and hence injections on underlying graded simplicial sets. The simplicial Π-algebra $\bar{A}$ is cofibrant and the induced maps $\bar{A} \to \bar{B}$ and $\bar{A} \to \bar{C}$ are cofibrations; hence there are weak equivalences

$$\begin{aligned} B *^{\mathrm{h}}_A C &\sim \bar{B} *_{\bar{A}} \bar{C} = \operatorname{diag}\left(\mathcal{B}(B) *_{\mathcal{B}(A)} \mathcal{B}(C)\right) \\ B \cup^{\mathrm{h}}_A C &\sim \bar{B} \cup^{\mathrm{h}}_{\bar{A}} \bar{C} = \operatorname{diag}\left(\mathcal{B}(B) \cup_{\mathcal{B}(A)} \mathcal{B}(C)\right) \ . \end{aligned} \tag{5.7}$$

Let $U = \mathcal{B}(A)$, $V = \mathcal{B}(B)$, $W = \mathcal{B}(C)$. By 5.4 and induction on n, there are inequalities

$$\kappa(V[n], U[n]) = \kappa(F^{n+1}(B), F^{n+1}(A)) \geq \kappa(B, A)$$
$$\kappa(W[n], U[n]) = \kappa(F^{n+1}(C), F^{n+1}(A)) \geq \kappa(C, A)$$

and hence by 5.5 inequalities

$$\kappa\left((V *_U W)[n], (V \cup_U W)[n]\right) \geq \kappa(B, A) + \kappa(C, A) \ .$$

Note in this connection that because of the fact that F (as a left adjoint) preserves colimits, there is a natural isomorphism $(V *_U W)[n] \simeq F((V \cup_U W)[n-1])$. The result follows from 5.6 and 5.7. □

For the sake of clarity we will prove 5.5 and 5.4 in the ungraded case (i.e. with the word "graded" deleted from the statements); the modifications necessary to pass to the graded case are notational.

Suppose that $\mathbf{D}$ be a small category and that F and G are respectively covariant and contravariant functors from $\mathbf{D}$ to simplicial sets. We denote the *coend* [16, IX.6] of the bifunctor $G \times F$ by $G \times_{\mathbf{D}} F$. This is the coequalizer of a more or less evident pair of maps

$$\coprod_{d \to d'} G(d') \times F(d) \rightrightarrows \coprod_{d} G(d) \times F(d)$$

where the coproduct in the range is indexed by the objects in $\mathbf{D}$ and the coproduct in the domain by the arrows. This coequalizer diagram is the low degree part of the bisimplicial set $B(F,\mathbf{D},G)$ (cf. [14, §3]) with

$$B(F,\mathbf{D},G)[k] = \coprod_{d_0\to\cdots\to d_k} G(d_k)\times F(d_0)\ , \tag{5.8}$$

where the coproduct is indexed by the k-simplices of the nerve of $\mathbf{D}$. We will denote the diagonal of this bisimplicial set by $G\times^{\mathrm{h}}_{\mathbf{D}} F$ and call it the *homotopy coend* of the bifunctor $G\times F$. There is an obvious map

$$G\times^{\mathrm{h}}_{\mathbf{D}} F \to G\times_{\mathbf{D}} F. \tag{5.9}$$

Let $\mathcal{F}$ be the category of finite sets. Suppose that F is a functor from finite sets to simplicial sets, prolonged as in 5.2 and 5.3 to a functor of simplicial sets. As remarked in [21, 1.1], this prolonged functor can be expressed by the formula

$$F(X) = X^*\times_{\mathcal{F}} F$$

where X is a simplicial set and X^* is the contravariant functor on $\mathcal{F}$ which sends S to X^S. The observation we begin with is that this coend is actually equivalent to the corresponding homotopy coend.

5.10. **Proposition.** *Suppose that F is a functor from finite sets to simplicial sets. Then for any simplicial set X the natural map*

$$X^*\times^{\mathrm{h}}_{\mathcal{F}} F \to X^*\times_{\mathcal{F}} F = F(X)$$

is a weak equivalence.

Proof. We consider the map 5.9 for an arbitrary contravariant functor G from $\mathcal{F}$ to sets or simplicial sets. It is easy to see that the map is a weak equivalence if G is representable, that is, if G has the form $\mathrm{Hom}(-,T)$ for some object T of $\mathcal{F}$; in this case both domain and range are equivalent to $F(T)$ [14, 3.1(5)]. Since filtered colimits preserve weak equivalences [3, XII.3.6] and all of the constructions in question commute with filtered colimits, the map 5.9 is clearly an equivalence if G is a filtered colimit of representable functors. It now follows from a diagonal argument that 5.9 is a weak equivalence if each of the functors $G(-)[n]$ is a filtered colimit of representable functor; to obtain this use [13, IV.1.7] and the fact that 5.9 is the diagonal of a map of bisimplicial sets which in degree n contains the map $G(-)[n]\times^{\mathrm{h}}_{\mathcal{F}} F\to G(-)[n]\times_{\mathcal{F}} F$. But observe that any set is the filtered colimit of its finite subsets, so that the functor on $\mathcal{F}$ sending S to $X[n]^S = \mathrm{Hom}_{\mathcal{S}}(S, X[n])$ is indeed a filtered colimit of representable functors. □

The following is an exercise in elementary homotopy theory.

5.11. **Lemma.** *Suppose that $Y \leftarrow X \rightarrow Z$ is a two-source of simplicial sets in which the maps are injective (so that the homotopy pushout agrees with the ordinary pushout). Then for any $n \geq 0$ there are inequalities*

$$\kappa(Y^n, X^n) \geq \kappa(Y, X)$$
$$\kappa((Y \cup_X Z)^n,\, Y^n \cup_{X^n} Z^n) \geq \kappa(Y, X) + \kappa(Z, X)$$

Proof of 5.4 (ungraded case). By 5.10, $\kappa(F(Y), F(X))$ is the same as the cellular connectivity of the map $X^* \times^{\mathrm{h}}_{\mathcal{F}} F \to Y^* \times^{\mathrm{h}}_{\mathcal{F}} F$. This map can be realized as the diagonal of a map of bisimplicial sets (5.8) which in degree k is constructed as a disjoint union of maps of the form $X^S \times F(T) \to Y^S \times F(T)$. It follows from the first inequality of 5.11 that $\kappa(Y^S \times F(T), X^S \times F(T)) \geq \kappa(Y, X)$. Since taking disjoint unions does not lower cellular connectivity, the desired result follows from 5.6. □

Proof of 5.5 (ungraded case). We can assume that $X \to Y$ and $X \to Z$ are injections, so that the pushout of the two-source is the same as the homotopy pushout. By 5.10, $\kappa\left(F(Y \cup^{\mathrm{h}}_X Z),\, F(Y) \cup^{\mathrm{h}}_{F(X)} F(Z)\right)$ is the same as the cellular connectivity of the map

$$(Y^* \times^{\mathrm{h}}_{\mathcal{F}} F) \cup_{X^* \times^{\mathrm{h}}_{\mathcal{F}} F} (Z^* \times^{\mathrm{h}}_{\mathcal{F}} F) \to (Y \cup_X Z)^* \times^{\mathrm{h}}_{\mathcal{F}} F\,.$$

By definition (5.8) and inspection, this map is realized as the diagonal of a map of bisimplicial sets which in degree k is constructed as a disjoint union of maps of the form

$$(Y^S \cup_{X^S} Z^S) \times F(T) \to (Y \cup_X Z)^S \times F(T)\,.$$

It follows from the second inequality of 5.11 that the cellular connectivity of this last map is at least $\kappa(Y, X) + \kappa(Z, X)$, and as in the proof above the desired result is now a consequence of 5.6. □

6. Postnikov systems for simplicial Π-algebras

In this section we study Postnikov systems for simplicial Π-algebras in a way which is largely parallel to the study of Postnikov systems for topological spaces in §3. In the course of this we develop a notion of cohomology for simplicial Π-algebras. This differs from the notion of cohomology for Π-algebras considered by the second author and Kan in [8] in that more general coefficients are allowed. In [8] the coefficients are "strongly abelian" Π-algebras in which both Whitehead products and compositions are trivial; here we accept arbitrary abelian

Π-algebras, in which the Whitehead products vanish but compositions may be nontrivial.

Postnikov systems. Suppose that X is a simplicial Π-algebra. Attaching an $(n+2)$-cell $\mathrm{c}D^{n+2}\bar{\otimes}\pi_*S^k$ to X via a map $f : \mathrm{c}S^{n+1}\bar{\otimes}\pi_*S^k \to X$ has no effect on $\pi_i X$ for $i \le n$, and clearly kills of the class represented by f (4.7) in $(\pi_{n+1}X)_k$. Now attach cells of dimension $(n+2)$ and greater to X by all possible attaching maps to obtain an inclusion $X \subset X_1$, repeat the process to obtain $X_1 \subset X_2$, repeat again, etc., and let $P_n X = \cup_j X_j$. There is a map $X \to P_n X$ which induces isomorphisms on π_i for $i \le n$, and $\pi_i P_n X \simeq 0$ for $i > n$. The construction of $P_n X$ is functorial in X, and there is a natural map $P_n X \to P_{n-1}X$ which respects the inclusions of X in these two simplicial Π-algebras.

6.1. *Eilenberg-Mac Lane objects.* If Λ is a Π-algebra, we say that a simplicial Π-algebra X is *of type* K_Λ if $\pi_0 X \simeq \Lambda$ and the higher homotopy of X is trivial. Suppose that M is a Λ-module. We say that a map $X \to Y$ is *of type* $\mathrm{B}\Lambda(M,n)$ $n \ge 1$, if X is of type K_Λ, $\pi_0 Y \simeq \Lambda$, $\pi_n Y \simeq M$ (as a Λ-module), all other homotopy of Y is trivial, and the map $X \to Y$ gives an isomorphism on π_0. Sometimes we will say for short that the target Y is of type $K_\Lambda(M,n)$.

The difference construction. Suppose that $f : Y \to X$ is a map of simplicial Π-algebras. Consider the pushout C of the diagram $X' \leftarrow Y' \to (P_0X)'$ obtained by using some functorial construction to replace Y by a cofibrant object and the two maps $Y \to X$ and $Y \to P_0 X$ by cofibrations. There is a commutative diagram

(6.2)
$$\begin{array}{ccccc} Y & \xleftarrow{\sim} & Y' & \longrightarrow & (P_0X)' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_n(f)} \\ X & \xleftarrow{\sim} & X' & \longrightarrow & P_{n+1}C \end{array}$$

in which the vertical map on the right is $\Delta_n(f)$. The source $(P_0X)'$ of $\Delta_n(f)$ is $\Delta_n^{\mathrm{s}}(f)$, and the target $P_{n+1}C$ is $\Delta_n^{\mathrm{t}}(f)$.

6.3. **Proposition.** *Suppose that $f : Y \to X$ is a map of simplicial Π-algebras which is an isomorphism on π_0 and whose homotopy fibre F is $(n-1)$-connected, $n \ge 1$. Let $M = \pi_n F$. Then M is naturally a Λ-module for $\Lambda = \pi_0 X$ and $\Delta_n(f)$ is a map of type $K_\Lambda(M, n+1)$. If $\pi_i F$ vanishes except for $i = n$, then the right-hand square in 6.2 is a homotopy fibre square.*

We need a modified form of 3.3. A map $f : A \to B$ of connected simplicial sets is *simple* if its homotopy fibre is connected and $\pi_1 A$ acts trivially on the homotopy groups of the homotopy fibre.

6.4. **Proposition.** *Let $f : A \to B$ be a simple map of connected simplicial sets with homotopy fibre F. Assume that $\pi_i F$ is trivial for $i < n$, $n \geq 1$, and let $M = \pi_n F$. Let Γ be the mapping cone of f, and $P_{n+1}\Gamma$ its $(n+1)$'st Postnikov stage in the category of simplicial sets. Then $P_{n+1}\Gamma$ is a simplicial set of type $K(M, n+1)$. If the homotopy of F vanishes except in dimension n, then the sequence $A \to B \to P_{n+1}\Gamma$ is a homotopy fibre sequence.*

Proof of 6.3. This follows from 5.1 and 6.4. Clearly $\kappa((P_0X)', Y') \geq 2$ and $\kappa(X', Y') \geq 2$. Let $\Gamma = (P_0X)' \cup_{Y'} X'$; this is a homotopy pushout in the category of graded simplicial sets. By 5.1, $\kappa(C, \Gamma) \geq 2$ (here C is from 6.2). It is easy to see that up to weak equivalence applying P_{n+1} to a simplicial Π-algebra commutes with taking the underlying graded simplicial set. But $\pi_0\Gamma \simeq \pi_0 C \simeq \Lambda$, and it follows from 6.4 that $\pi_i P_{n+1} C$ vanishes except for the fact that it is isomorphic to Λ if $i = 0$ and to M if $i = n+1$. Thus M is naturally a Λ-module (4.14) and $P_{n+1}C$ is of type $K_\Lambda(M, n+1)$. (This last deduction involves applying 6.4 componentwise to a map $Y' \to X'$ of graded disconnected simplicial sets which is an isomorphism on π_0; note that P_0X is homotopically discrete, so that Γ is essentially obtained by taking componentwise mapping cones of $Y' \to X'$. The map $Y' \to X'$ is componentwise simple because Y' and X', as simplicial Π-algebras, are actually graded simplicial groups.) The final statement again follows from 6.4, since taking the homotopy pullback of a two-sink of simplicial Π-algebras commutes up to weak equivalence with passing to underlying graded simplicial sets. □

Existence and uniqueness of Eilenberg-Mac Lane objects. The Π-algebra Λ, considered as a constant simplicial object, is of type K_Λ. Moreover, if X is any simplicial Π-algebra of type K_Λ then the natural map from X to its Π-algebra of components gives a weak equivalence $X \sim \Lambda$. It is easy to deduce from this that the moduli space of all simplicial Π-algebra's of type K_Λ is connected and weakly equivalent to $\bar{W}\mathrm{Aut}(\Lambda)$. We will denote a generic simplicial Π-algebra of this type by K_Λ.

6.5. **Proposition.** *Let Λ be a Π-algebra and M a Λ-module. Then for each $n \geq 1$ the moduli space of all maps of type $K_\Lambda(M, n)$ is weakly equivalent to $\bar{W}\mathrm{Aut}(\Lambda, M)$.*

6.6. *Remark.* In particular, the moduli space is nonempty and connected, so objects or maps of type $K_\Lambda(M, n)$ are unique up to weak equivalence. We will denote a generic simplicial Π-algebra of this type by $K_\Lambda(M, n)$.

Proof. Let $\mathcal{M}_n$ be the moduli space of maps of type $K_\Lambda(M, n)$. As in the proof of 3.4, the difference construction 6.3 gives weak equivalences

$\mathcal{M}_n \to \mathcal{M}_{n+1}$, $n \geq 1$. Let $\mathcal{M}_0$ be the moduli space $\mathcal{M}(K_{\Lambda \rtimes M} \overset{\longleftarrow}{\longrightarrow} K_\Lambda)$, i.e, the moduli space of maps $U \to V$ of simplicial Π-algebras with a section $V \to U$ such that U and V have trivial higher homotopy and on π_0 the map with its section gives a diagram of Π-algebras isomorphic to $\Lambda \rtimes M \overset{\longleftarrow}{\longrightarrow} \Lambda$. It is easy to see that $\mathcal{M}_0$ is weakly equivalent to $\bar{W}\mathrm{Aut}(\Lambda, M)$. The functor which assigns to a map $U \to V$ of type $K_\Lambda(M, n)$ the homotopy pullback of $U \to V \leftarrow U$ gives a map $\mathcal{M}_1 \to \mathcal{M}_0$, but in contrast to the situation in the proof of 3.4, the difference construction does not give an inverse. Instead we proceed as follows. Given $U \overset{\longleftarrow}{\longrightarrow} V$ of type $K_{\Lambda \rtimes M} \overset{\longleftarrow}{\longrightarrow} K_\Lambda$, write $\Lambda' = \pi_0 V$, $\Lambda' \rtimes M' = \pi_0 U$ and form the map $\Lambda' \to \Lambda' \rtimes \bar{W} M'$ of type $K_\Lambda(M, 1)$. This construction is functorial and gives a map $\mathcal{M}_0 \to \mathcal{M}_1$. The composite $\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_0$ is clearly an equivalence because the underlying functor is connected to the identity by natural transformations. The same is true of the other composite; the key observation is this. Suppose that $U \to V$ is a map of type $K_\Lambda(M, 1)$, which we can assume to be a fibration, and let U^*_V be the simplicial object which in simplicial degree n contains the n-fold fibre power of U over V. The diagonal of this bisimplicial Π-algebra maps to V by a weak equivalence, but it also maps to the simplicial Π-algebra obtained by applying π_0 degreewise; this simplicial Π-algebra is exactly $\pi_0 V \rtimes \bar{W} \pi_1 V$. □

6.7. *Cohomology of* Π-*algebras.* We follow 3.6. Consider an Eilenberg-Mac Lane object $K_\Lambda(M, n)$, $n \geq 1$. Then $P_0 K_\Lambda(M, n) \sim K_\Lambda$, and so we write the map from this object to its zeroth Postnikov stage as $K_\Lambda(M, n) \to K_\Lambda$. Given another simplicial Π-algebra X over K_Λ, we define $H^n_\Lambda(X; M)$ by

$$H^n_\Lambda(X; M) = [X, K_\Lambda(M, n)]_{K_\Lambda}$$

where the symbol on the right denotes derived homotopy classes of maps in the category of simplicial Π-algebras over K_Λ. Let $\mathcal{H}^n_\Lambda(X; M)$ denote $\mathrm{Map}^{\mathrm{h}}_{K_\Lambda}(X, K_\Lambda(M, n))$, so that the set of components of this space is $H^n_\Lambda(X; M)$. As in 3.6, there are isomorphisms

$$\pi_i \mathcal{H}^n_\Lambda(X; M) \simeq \begin{cases} H^{n-i}_\Lambda(X; M) & 0 \leq i \leq n-1 \\ 0 & i > n \end{cases} .$$

We use this formula to define $H^0_\Lambda(X; M)$.

Classification of Postnikov stages. Suppose that X is a simplicial Π-algebra with $X \sim P_{n-1} X$, $n \geq 1$ and that M is a module over $\pi_0 X$. If Y is a simplicial Π-algebra, we write $Y \sim X + (M, n)$ if $P_n Y \sim Y$, $P_{n-1} Y \sim X$, and $\pi_n Y$ is isomorphic to M as a module over $\pi_0 Y$, where the isomorphism is realized by some isomorphism $\pi_0 X \to \pi_0 Y$. We write $\mathcal{M}(X+(M,n))$ for the moduli space of all simplicial Π-algebras of type $X+(M,n)$.

The following result is proved in the same way as 3.7, with 6.3 replacing 3.2 in the argument.

6.8. **Theorem.** *Suppose that X is a simplicial Π-algebra with $X \sim P_{n-1}X$, $n \geq 1$. Let $\Lambda = \pi_0 X$, and let M be a module over Λ. Then there is an natural weak equivalence*

$$\mathcal{M}(X+(M,n)) \sim \mathcal{M}(X \looparrowright K_\Lambda(M,n+1) \looparrowleft K_\Lambda) \ .$$

6.9. *Remark.* The arrows $\looparrowright$ on the right indicate maps which induce isomorphisms on appropriate homotopy groups (2.3); in this case it is just isomorphisms on π_0. The remarks at the beginning of 3.9 can be repeated almost verbatim here.

7. Simplicial spaces and the spiral exact sequence

In [10] and [11], Kan and Stover and the second author of this paper developed a model category structure on the category of simplicial pointed topological spaces which is adapted to making spherical resolutions of ordinary spaces that mirror resolutions of their homotopy Π-algebras. In this section we spell out what we need from these papers and extend the theory in some ways (7.13). All of our topological spaces have basepoints; we sometimes take this for granted and refer to “spaces” instead of to “pointed spaces”.

7.1. **The Reedy model structure.** To begin with, the category of simplicial spaces acquires a Reedy model category structure [20] [10, 2.4] [15, 5.2.5] from the usual model category structure (§3) on the category of pointed spaces. A map $X \to Y$ of simplicial spaces is a *Reedy weak equivalence* if $X[n] \to Y[n]$ is a weak equivalence for all $n \geq 0$, a *Reedy fibration* if $X[0] \to Y[0]$ is a fibration and, for all $n \geq 1$, the map

$$X[n] \to Y[n] \times_{M_n Y} M_n X$$

is a fibration. Here $M_n X$ is the nth matching space:

$$M_n X = \lim_{\phi:[m]\to[n]} X[m]$$

where ϕ runs over injections in the ordinal number category with $m < n$. Cofibrations are defined symmetrically: $X \to Y$ is a *Reedy cofibration* if $X[0] \to Y[0]$ is a cofibration and for $n \geq 1$,

$$X[n] \bigvee_{L_n X} L_n Y \to Y[n]$$

is a cofibration. Here L_nX is the latching space

$$L_nX = \operatorname*{colim}_{\psi:[n]\to[m]} X[m]$$

where ψ runs over the surjections in the ordinal number category with $m < n$. This Reedy model structure has the desirable property that the geometric realization functor $X \mapsto |X|$ preserves weak equivalences between cofibrant objects [11, §4].

7.2. **The E_2 model structure.** The E_2 model category structure is built from the Reedy model category structure. If X is a simplicial pointed space, we let π_*X denote the simplicial Π-algebra obtained by applying the functor π_* degreewise to X.

7.3. **Definition.** Define a morphism $f : X \to Y$ of simplicial pointed spaces to be

(1) an *E_2-equivalence* if $\pi_*(f)$ is a weak equivalence of simplicial Π-algebras (4.4);
(2) an *E_2-fibration*, if f is a Reedy fibration and $\pi_*(f)$ is a fibration of simplicial Π-algebras (4.4); and
(3) an *E_2-cofibration* if f is a retract of an S^1-free map; here f is *S^1-free* if there is a CW complex $Z_n \subseteq Y[n]$ which has the homotopy type of a wedge of spheres S^k, $k \geq 1$, and

$$(X[n] \bigvee_{L_nX} L_nY) \vee Z_n \to Y[n]$$

is an acyclic cofibration.

The category of simplicial spaces has a standard simplicial structure in the sense of Quillen [19, §II.2]; if K is a simplicial set and X is simplicial space, then $K \otimes X$ is the simplicial space with

$$(K \otimes X)[n] = \vee_{x\in K[n]} X[n] \ .$$

The Reedy model category structure on simplicial spaces does *not* extend to a simplicial model category structure with respect to this simplicial structure: if $X \to Y$ is a Reedy cofibration and $K \to L$ is a cofibration of simplicial sets, then

$$X \otimes L \vee_{X\otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is Reedy acyclic if $X \to Y$ is a Reedy weak equivalence, but pretty evidently need not be a Reedy weak equivalence if $K \to L$ is a weak equivalence of simplicial sets. The main result of [10] is:

7.4. **Proposition.** *With notions of E_2-equivalence, E_2-fibration, and E_2-cofibration just given, and with the simplicial structure described* *above, the category of simplicial spaces becomes a cofibrantly generated simplicial model category.*

From now on, when we refer to cofibrations, fibrations, and weak equivalences between simplicial spaces, we will unless otherwise specified be referring to the E_2-model structure.

7.5. *Remark.* Note that an object is E_2-fibrant if and only if it is Reedy fibrant. If X is E_2-cofibrant, it is also Reedy cofibrant, although not vice versa (cf 7.8).

7.6. *The functor π_* preserves homotopy pushouts.* If $f : X \to Y$ is an E_2-cofibration, then $\pi_*(f)$ is a cofibration of simplicial Π-algebras. Suppose that $X \leftarrow Y \to Z$ is a two-source of simplicial pointed spaces in which the objects are E_2-cofibrant and the maps are E_2-cofibrations, and let C be the pushout of the square. Then π_*C is the pushout of $\pi_*X \leftarrow \pi_*Y \to \pi_*Z$ (in each simplicial degree, the pushout process just involves wedging on spheres). It follows that the functor π_* from simplicial spaces to simplicial Π-algebras preserves homotopy pushouts.

7.7. *The functor π_* often preserves homotopy fibres.* Let $f : X \to Y$ be an E_2-fibration with fibre F. If $\pi_*(f)$ is surjective, then clearly the fibre of $\pi_*(f)$ is exactly π_*F. By 4.4 and the definition of E_2-fibration, $\pi_*(f)$ is surjective if and only if the map $\pi_0\pi_*X \to \pi_0\pi_*Y$ is surjective. It follows that for such maps f, the functor π_* preserves (homotopy) fibres.

7.8. *Cells.* If X is a simplicial space and K is a simplicial set with basepoint $*$, we define $K\bar{\otimes}X$ to be the quotient $(K \otimes X)/(* \otimes X)$. The *bigraded spheres* $S^{i,j}$ are defined by $S^{i,j} = \mathrm{c}S^i\bar{\otimes}S^j$, and the corresponding disks by $D^{i,j} = \mathrm{c}D^i\bar{\otimes}S^j$. Say that a simplicial space is *cellular* if it is constructed from the trivial simplicial space by attaching cells $(D^{i+1,j}, S^{i,j})$, $i \geq 0$, $j \geq 1$. Then any cellular simplicial space is E_2-cofibrant, any simplicial space has a functorial cellular approximation, and any cofibrant simplicial space is a retract of a cellular one.

7.9**. Homotopy groups and the spiral exact sequence.** If X is a Reedy cofibrant simplicial space, there is a first quadrant (homology) spectral sequence converging to $\pi_*|X|$ with $E^2_{i,j} = \pi_i\pi_jX$ [2] [11, 8.3]. This explains the term "E_2 model category structure": a map $X \to Y$ of simplicial spaces is an E_2 weak equivalence if and only if it induces an isomorphism on these E_2-pages. We will write $\hat{\epsilon}_iX = \pi_i\pi_*X$ for the i'th column of this E_2-term. By 4.5 and 4.14, $\hat{\epsilon}_iX$ is a Π-algebra which for $i \geq 1$ is naturally a module over $\hat{\epsilon}_0X$. By definition, a map $X \to Y$ is an E_2 weak equivalence if and only if it induces isomorphisms $\hat{\epsilon}_*X \simeq \hat{\epsilon}_*Y$.

The notion of cellular simplicial space (7.8) suggests another notion of homotopy; if X is a simplicial space we define $\pi_{i,j}X$, $i \geq 0$, $j \geq 1$ by

$$\pi_{i,j}X = \pi_i \operatorname{Map}^{\mathrm{h}}(S^j, X) \simeq [S^{i,j}, X]$$

where the symbol on the right denotes derived homotopy classes of maps in the E_2 model category. These are the *bigraded homotopy groups* of X. Let $\hat{\pi}_i X = \pi_{i,*}X$. The objects $\hat{\pi}_i X$ $(i \geq 0)$ have formal properties very similar to those of $\hat{\epsilon}_i X$.

7.10. **Proposition.** *Suppose that X is a simplicial space. Then $\hat{\pi}_i X$ is a Π-algebra, which for $i \geq 1$ is a module over $\hat{\pi}_0 X$. A map $X \to Y$ of simplicial spaces is a weak equivalence if and only if it induces isomorphisms $\hat{\pi}_i X \to \hat{\pi}_i Y$, $i \geq 0$.*

Proof. It is easy to see that $\hat{\pi}_i X$ is exhibited as a Π-algebra by the functor which sends $U \in \Pi$ to $\pi_i \operatorname{Map}^{\mathrm{h}}(U, X) = [\mathrm{c}S^i \bar{\otimes} U, X]$. The module structure arises (4.12) from the fact that for $i \geq 1$, $\mathrm{c}S^i_+$ is a cogroup object in the homotopy category of simplicial sets under $\mathrm{c}S^0$ with $\mathrm{c}S^i_+ / \mathrm{c}S^0 \simeq \mathrm{c}S^i$. The last statement is from [11, 5.3]. □

The objects $\hat{\epsilon}_i X$ and $\hat{\pi}_i X$ are related by a long exact sequence, called the *spiral exact sequence.*

7.11. **Proposition.** [11, 7.2, 8.1] *Suppose that X is a simplicial space. Then there is a natural isomorphism $\hat{\pi}_0 X \simeq \hat{\epsilon}_0 X$ of Π-algebras, as well as a long exact sequence of Π-algebras*

$$\cdots \to \hat{\epsilon}_{n+1} X \to \Omega \hat{\pi}_{n-1} X \to \hat{\pi}_n X \to \hat{\epsilon}_n X \to \cdots \to \hat{\pi}_1 X \to \hat{\epsilon}_1 X \to 0 \ .$$

7.12. **Structure of the spiral exact sequence.** All of the constituents of the spiral exact sequence are naturally modules over $\hat{\pi}_0 X$: $\hat{\pi}_n X$ by 7.10, $\Omega \hat{\pi}_{n-1} X$ by 7.10 and 4.13, and $\hat{\epsilon}_n X$ by 4.14 and the isomorphism $\hat{\epsilon}_0 X \simeq \hat{\pi}_0 X$ given by 7.11. In the rest of this section we will prove the following proposition.

7.13. **Proposition.** *With respect to the module structures described above, the spiral exact sequence 7.11 is an exact sequence of $\hat{\pi}_0 X$-modules.*

This will be proved in stages.

7.14. **Proposition.** *The homomorphisms $\hat{\pi}_i X \to \hat{\epsilon}_i X$ from 7.11 are maps of modules over $\hat{\pi}_0 X$.*

Proof. By definition [11] these homomorphisms are obtained from the isomorphisms $\pi_*(\mathrm{c}S^i \bar{\otimes} U) \simeq \mathrm{c}S^i \bar{\otimes} \pi_* U$, $U \in \Pi$; these give maps

$$(\hat{\pi}_i X)(U) = [\mathrm{c}S^i \bar{\otimes} U, X] \to [\mathrm{c}S^i \bar{\otimes} \pi_* U, \pi_* X] = (\hat{\epsilon}_i X)(U) \ .$$

For $i = 0$ we obtain the isomorphism $\hat{\pi}_0 X \simeq \hat{\epsilon}_0 X$. Let Q be the split cofibration sequence from 4.15. Then the corresponding maps

$[Q\bar{\otimes}U, X] \to [Q\bar{\otimes}\pi_*U, \pi_*X]$ provide morphisms of split sequences (4.10) which show that $\hat{\pi}_i X \to \hat{\epsilon}_i X$ is a map of $\hat{\pi}_0 X$-modules.

To go any further, we need more information about how to represent the constituents of the spiral exact sequence in the E_2 homotopy category. This information is in [11, 7.4], but we have to examine it in some detail because we need a relative version.

If X is a space, the *pointed cylinder* IX is the pushout of the diagram $* \leftarrow * \times I \to X \times I$, where $I = [0,1]$; the *cone* CX is then $(IX)/(X \times 1)$. There is a natural inclusion $X \to CX$ given by $x \mapsto (x, 0)$, and the quotient CX/X is the *suspension* ΣX.

If X is a simplicial space, we write $\hat{D}^n X = \mathrm{c}D^n \bar{\otimes} X$ and $\hat{\Sigma}^n X = \mathrm{c}S^n \bar{\otimes} X$. It is easy to see [10, 4.1] that $\hat{D}^n X$ is always E_2-contractible, in the sense that it is E_2 weakly equivalent to a trivial simplicial space with one point in each simplicial degree.

The representing objects. Suppose that $U \in \Pi$, and that $n \geq 2$ is an integer. We wish to construct a simplicial space $\tilde{\Sigma}^{n-2}\Sigma U$ by considering the following diagram

$$\begin{array}{ccccc} \hat{\Sigma}^{n-2}U & \longrightarrow & \hat{\Sigma}^{n-2}CU & \longrightarrow & \hat{\Sigma}^{n-2}\Sigma U \\ {\scriptstyle =}\downarrow & & {\scriptstyle \sim}\downarrow & & {\scriptstyle \sim}\downarrow \\ \hat{\Sigma}^{n-2}U & \longrightarrow & \hat{D}^{n-1}CU & \longrightarrow & \tilde{\Sigma}^{n-2}\Sigma U \end{array} .$$

The top row is a sequence of simplicial spaces which in each simplicial degree gives a cofibration sequence of spaces, and $\tilde{\Sigma}^{n-2}\Sigma U$ is defined so that the same is true of the bottom row. (These are *not* E_2-cofibration sequences; for instance, the left hand horizontal maps do not induce injections on π_*. In spite of the notation, $\tilde{\Sigma}^{n-2}\Sigma U$ is a functor of U, not of ΣU.) It is clear that the vertical arrows are Reedy equivalences, and therefore E_2-equivalences; in effect, $\tilde{\Sigma}^{n-2}\Sigma U$ is obtained from $\hat{\Sigma}^{n-2}\Sigma U$ by wedging on some number of copies of CU in each simplicial degree. The following is clear from the definitions (4.13).

7.15. **Proposition.** *If X is a simplicial space, the Π-algebra $\Omega\hat{\pi}_{n-2}X$ is represented by the functor*

$$U \mapsto [\tilde{\Sigma}^{n-2}\Sigma U, X] \simeq [\hat{\Sigma}^{n-2}\Sigma U, X] .$$

Notice that there is a natural map

$$\beta : \hat{\Sigma}^{n-1}U = \hat{D}^{n-1}U/\hat{\Sigma}^{n-2}U \to \hat{D}^{n-1}CU/\hat{\Sigma}^{n-2}U = \tilde{\Sigma}^{n-2}\Sigma U \ .$$

Now we construct a simplicial space $\psi^n U$ by considering the following diagram

$$\begin{array}{ccccc} \hat{\Sigma}^{n-1}U & \xrightarrow{\alpha} & \hat{D}^n U & \longrightarrow & \hat{\Sigma}^n U \\ {\scriptstyle\beta}\downarrow & & \downarrow & & \downarrow \\ \tilde{\Sigma}^{n-2}\Sigma U & \xrightarrow{\gamma} & \psi^n U & \longrightarrow & \hat{\Sigma}^n U \end{array} \tag{7.16}$$

The object $\psi^n U$ is defined so that the left hand square is a pushout square. Since the map α is an E_2-cofibration and both of the objects on the left are E_2-cofibrant, the rows of this diagram are E_2-cofibre sequences.

7.17. **Proposition.** [11, 7.5] *For any simplicial space X and integer $n \geq 2$, the Π-algebra $\hat{\epsilon}_n X$ is given by the functor*

$$U \mapsto [\psi^n U, X] \ .$$

7.18. *Remark.* The functor $\hat{\epsilon}_n X$ is representable by $U \mapsto \psi^n U$ for $n \geq 2$, and by $U \mapsto \hat{\Sigma}^0 U$ for $n = 0$. It does not appear that $\hat{\epsilon}_1 X$ is representable in a similar way.

Now we can prove 7.13. The terminal homomorphism $\hat{\pi}_1 X \to \hat{\epsilon}_1 X$ is a $\hat{\pi}_0 X$-module map by 7.14; this proposition also handles the other maps $\hat{\pi}_n X \to \hat{\epsilon}_n X$. Suppose $n \geq 2$. According to [11], the homomorphism $\hat{\epsilon}_n X \to \Omega\hat{\pi}_{n-2} X$ is induced (via 7.15) by the map γ in 7.16, and the homomorphism $\Omega\hat{\pi}_{n-2} X \to \hat{\pi}_{n-1} X$ is similarly induced by β. Now let F be one of the functors of U which appears in 7.16, or the functor given by $U \mapsto \hat{\Sigma}^{n-2}\Sigma U$. Let $C(F)$ be the pointed simplicial space $F(S^0)$; true, S^0 is not an object of Π, but the construction of $F(S^0)$ still makes sense. For each one of these functors F it is clear that there are isomorphisms

$$F(U) \simeq C(F) \wedge U$$

where the object on the right is obtained by taking the simplicial space $C(F)$ and smashing it in each degree with U. To each F there is naturally associated a split diagram

$$S^0 \leftrightarrows C(F)_+ \longrightarrow C(F)$$

where $C(F)_+$ is obtained by adding a disjoint basepoint in each degree to $C(F)$. Smashing these diagrams with $U \in \Pi$ and mapping into X produces the maps of split sequences (4.10) required to show that the homomorphisms in question are maps of modules over $\hat{\pi}_0 X$ (cf. 4.12).

8. Postnikov systems for simplicial spaces

In this section we set up a theory of Postnikov systems for simplicial spaces, which is parallel to the Postnikov theories in §3 and §6. The new ingredient is 8.15, which essentially gives a functorial relationship between geometric k-invariants for simplicial spaces and algebraic k-invariants for the associated simplicial Π-algebras.

Postnikov systems. Suppose that X is a simplicial space. Attaching a cell (see 7.8) $(D^{n+2,k}, S^{n+1,k})$ of horizontal dimension $(n+2)$ to X via a map $f : S^{n+1,k} \to X$ has no effect on $\hat{\pi}_i X$ for $i \leq n$, and clearly kills off the class represented by f in $\pi_{n+1,k}X$. Now attach cells of horizontal dimension $(n+2)$ and greater to X by all possible attaching maps and perform a functorial fibrant replacement to obtain an inclusion $X \subset X_1$, repeat the process to obtain $X_1 \subset X_2$, repeat again, etc., and let $\hat{P}_n X = \cup_j X_j$. (We use the notation $\hat{P}_n X$ to distinguish this construction from $P_n X$, which is the result of applying the topological Postnikov construction P_n in each dimension to the simplicial space X. The "functorial fibrant replacement" involves taking an object Z and finding a functorial acyclic cofibration $Z \to Z'$ such that Z' is fibrant; it is necessary here because in the E_2 model category not every object is fibrant.) There is a map $X \to \hat{P}_n X$ which induces isomorphisms on $\hat{\pi}_i$ for $i \leq n$, and $\hat{\pi}_i \hat{P}_n X$ is trivial for $i > n$. The construction of $\hat{P}_n X$ is functorial in X, and there is a natural map $\hat{P}_n X \to \hat{P}_{n-1} X$ which respects the inclusions of X in these two simplicial spaces.

8.1. *Eilenberg-Mac Lane objects.* If Λ is a Π-algebra, we say that a simplicial space X is *of type* B_Λ if $\hat{\pi}_0 X \simeq \Lambda$ and $\hat{\pi}_i X$ is trivial for $i > 0$. Suppose that M is a Λ-module. We say that a map $X \to Y$ is *of type* $B_\Lambda(M, n)$ $n \geq 1$, if X is of type B_Λ, $\hat{\pi}_0 Y \simeq \Lambda$, $\hat{\pi}_n Y \simeq M$ (as a Λ-module), all other homotopy of Y is trivial, and the map $X \to Y$ gives an isomorphism on $\hat{\pi}_0$. Sometimes we will say for short that the target Y is of type $B_\Lambda(M, n)$.

8.2. *Remark.* Recall that taking homotopy groups gives a functor π_* from simplicial spaces to simplicial Π-algebras. Let $f : X \to Y$ be a map of type $B_\Lambda(M, n)$. It turns out that $\pi_*(f)$ is *not* in general a map of type $K_\Lambda(M, n)$. In fact, according to the spiral exact sequence, there are isomorphisms

$$\pi_i \pi_* X \simeq \begin{cases} \Lambda & i = 0 \\ \Omega\Lambda & i = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_i \pi_* Y \simeq \pi_i \pi_* X \times \begin{cases} M & i = n \\ \Omega M & i = n+2 \\ 0 & \text{otherwise} \end{cases} .$$

The difference construction. Suppose that $f : Y \to X$ is a map of simplicial spaces. Consider the pushout C of the diagram $X' \leftarrow Y' \to (P_0X)'$ obtained by using some functorial construction to replace Y by an E_2-cofibrant space and the two maps $Y \to X$ and $Y \to P_0X$ by E_2-cofibrations. There is a commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\sim} & Y' & \longrightarrow & (\hat{P}_0X)' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_n(f)} \\ X & \xleftarrow{\sim} & X' & \longrightarrow & \hat{P}_{n+1}C \end{array} \tag{8.3}$$

in which the vertical map on the right is denoted $\Delta_n(f)$. The source $(\hat{P}_0X)'$ of $\Delta_n(f)$ is $\Delta_n^{\mathrm{s}}(f)$, and the target $\hat{P}_{n+1}C$ is $\Delta_n^{\mathrm{t}}(f)$.

8.4**. Proposition.** *Suppose that $f : Y \to X$ is a map of simplicial Π-algebras which is an isomorphism on $\hat{\pi}_0$ and whose homotopy fibre F has $\hat{\pi}_iF$ trivial for $i < n$ $(n \geq 1)$. Let $M = \hat{\pi}_nF$. Then M is naturally a Λ-module for $\Lambda = \hat{\pi}_0X$ and $\Delta_n(f)$ is a map of type $B_\Lambda(M, n+1)$. If $\hat{\pi}_iF$ vanishes except for $i = n$, then the right-hand square in 8.3 is a homotopy fibre square.*

Proof. This is very much along the lines of the proof of 6.3. Let $F_\pi \sim \pi_*F$ be the homotopy fibre of $\pi_*Y' \to \pi_*X'$. By the spiral exact sequence, $\pi_iF_\pi = \hat{\epsilon}_iF$ is trivial for $i < n$ and isomorphic to M for $i = n$. Diagram 8.3 gives a homotopy pushout diagram

$$\begin{array}{ccc} \pi_*Y' & \longrightarrow & \pi_*(\hat{P}_0X)' \\ \downarrow & & \downarrow \\ \pi_*X' & \longrightarrow & \pi_*C \end{array}$$

Let F'_π be the homotopy fibre of the right-hand map. The techniques in the proof of 6.3, which involve using 5.1 to relate a homotopy pushout of simplicial Π-algebras to the corresponding homotopy pushout of simplicial sets, show that the map $\pi_iF_\pi \to \pi_iF'_\pi$ is an isomorphism for $i \leq n$. Let F' be the homotopy fibre of $(P_0X)' \to C$, so that $F'_\pi = \pi_*F'$. Again, the spiral exact sequence gives that $\hat{\pi}_iF'$ is trivial for $i < n$ and isomorphic to M for $i = n$. A homotopy exact sequence argument shows that $\Delta_n(f)$ is of type $\mathrm{B}\Lambda(M, n+1)$ for an appropriate action of Λ on M. It is straightforward to check the homotopy pullback condition. □

8.5. *Mapping into Eilenberg-Mac Lane objects.* We wish to study spaces of maps from simplicial spaces into Eilenberg-Mac Lane objects. Consider an Eilenberg-Mac Lane map $f : B_\Lambda \to B_\Lambda(M, n)$ with $n > 1$; we can assume that the target is fibrant. It follows from 6.3 that if $n > 1$

then $\Delta_{n-1}(\pi_* f)$ is a map of type $K_\Lambda(M,n)$ (note that the difference construction here is taken in the category of simplicial Π-algebras). Assigning to a diagram $X \xleftarrow{\simeq} U \to B_\Lambda(M,n)$ of simplicial spaces the associated diagram $\pi_* X \xleftarrow{\simeq} \pi_* U \to \Delta_n^{\mathrm{t}}(\pi_* f) \sim K_\Lambda(M,n)$ gives a natural map (cf. 2.7)

$$(8.6) \qquad \Phi_n(X) : \mathcal{M}^{\mathrm{f}}_{\mathrm{Hom}}(X, B_\Lambda(M,n)) \to \mathcal{M}^{\mathrm{f}}_{\mathrm{Hom}}(\pi_* X, K_\Lambda(M,n))\,.$$

8.7. **Proposition.** *The map $\Phi_n(X)$ is a weak equivalence of simplicial sets for all simplicial spaces X and all $n \geq 2$.*

8.8. *Remark.* By a slightly more elaborate construction, it is possible to produce an equivalence for $n = 1$.

Proof of 8.7. It is enough to check the cases in which X is a sphere $S^{i,j}$. The reason for this is that the domain of $\Phi_n(X)$ is equivalent to $\mathrm{Map}^{\mathrm{h}}(X, B_\Lambda(M,n))$ and the range to $\mathrm{Map}^{\mathrm{h}}(\pi_* X, K_\Lambda(M,n))$ (2.7, 2.5); since the functor π_* takes E_2-homotopy pushouts to homotopy pushouts of simplicial Π-algebras, it follows that the domain and range of $\Phi_n(X)$ take homotopy pushouts (in X) to homotopy pullbacks. So if $\Phi_n(X)$ is a weak equivalence for spheres, it is a weak equivalence for any simplicial space Y which can be constructed from spheres by a finite number of homotopy pushouts. To pass to arbitrary X, note that any simplicial space X is up to weak equivalence a filtered colimit of such Y, and that both the domain and range of $\Phi_n(X)$ take filtered colimits in X to homotopy limits of simplicial sets.

So we restrict attention to the bigraded spheres. Each $S^{i,j}$ is a cogroup object in the E_2-homotopy category of simplicial spaces, while $\pi_* S^{i,j}$ is a cogroup object in the category of simplicial Π-algebras. It is easy to check that $\Phi_n(X)$ commutes up to homotopy with the induced multiplications on the spaces involved. This means that in order to prove that $\Phi_n(S^{i,j})$ is a weak equivalence it is enough to show that it induces an isomorphism on ordinary homotopy groups, including π_0; it is not necessary to check all possible basepoints.

By inspection, $\pi_0 \Phi_n(S^{n,j})$ is an isomorphism; both domain and range are isomorphic to M_j. This implies that $\Phi_n(S^{n,j})$ is a weak equivalence, since the higher homotopy groups of the domain (isomorphic to $\pi_{n+k,j} B_\Lambda(M,n)$) and of the range (isomorphic to $(\pi_{n+k} K_\Lambda(M,n))_j$) are trivial. Since $S^{i,j}$ is the E_2-suspension of $S^{i-1,j}$, it follows as above that $\Phi_n(S^{i,j}) \sim \Omega \Phi_n(S^{i-1,j})$. By induction and the fact that the domain and range of $\Phi_n(S^{i,j})$ are connected for $i > 0, i \neq n$, it is easy to conclude that $\Phi_n(S^{i,j})$ is a weak equivalence for $i > 0$, and that $\pi_k \Phi_n(S^{0,j})$ is an isomorphism for $k > 0$. But $\pi_0 \Phi_n(S^{0,j})$ is a map $\Lambda_j \to \Lambda_j$, and it is easy to see by inspection that this is the identity. □

8.9. *Existence of Eilenberg-Mac Lane objects.* The easiest way to do this seems to be with generators and relations. To construct a simplicial space of type B_Λ, start with the wedge $W = \vee_{j\geq 1} \vee_{x\in\Lambda_j} S^{0,j}$; it is clear that $\hat{\pi}_0 W$ is the free Π-algebra on the underlying graded set of Λ. Now attach a one-cell for each relation in some presentation of Λ, and apply the functor $\hat{P}_0$ to obtain an object W' of type B_Λ. Since $\hat{\pi}_0 W' \simeq \Lambda$, there is a map $\alpha : \pi_* W' \to K_\Lambda$ which is an isomorphism on π_0. To construct a map of type $B_\Lambda(M,n)$, $n\geq 1$, start with W' and add on the wedge $\vee_{i\geq 1}\vee_{x\in M_i} S^{n,i}$ to obtain Z, so that $\pi_* Z$ is the coproduct of $\pi_* W'$ with $*_{i\geq 1} *_{x\in M_i} \mathrm{c}S^n \bar{\otimes} \pi_* S^i$. . There is a retraction $Z\to W'$ obtained by mapping the wedge factors trivially; let F be the homotopy fibre. Consider the diagram

$$\begin{array}{ccccc} \pi_* F & \longrightarrow & \pi_* Z & \longrightarrow & \pi_* W' \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow \\ K_0(M,n) & \longrightarrow & K_\Lambda(M,n) & \longrightarrow & K_\Lambda \end{array}$$

in which both rows are fibre sequences; here β is obtained by mapping a factor $\mathrm{c}S^n\bar{\otimes}\pi_*S^i$ of $\pi_* Z$ indexed by $x\in M_i$ so as to represent the element $x\in \pi_n K_0(M,n)\simeq M$. This gives an epimorphism

$$\hat{\pi}_n F \xrightarrow{\simeq} \hat{\epsilon}_n F \to M\,.$$

We now attach $(n+1)$-cells to Z to kill off the kernel of this epimorphism and apply the functor $\hat{P}_n$ to obtain Z'. It is routine to check that $W'\to Z'$ is of type $B_\Lambda(M,n)$.

8.10. *Uniqueness of Eilenberg-Mac Lane objects.* Recall from above that if f is of type $B_\Lambda(M,n)$ then $\Delta_{n-1}(\pi_* f)$ is of type $K_\Lambda(M,n)$.

8.11. **Proposition.** *Let Λ be a Π-algebra, M a Λ-module, and $n\geq 1$ an integer. Let $\mathcal{M}_n$ denote the moduli space of all maps of type $B_\Lambda(M,n)$. Then the functor $\Delta_{n-1}(\pi_*)$ induces a weak equivalence*

$$\mathcal{M}_n \to \mathcal{M}(K_\Lambda \looparrowright K_\Lambda(M,n))\,.$$

8.12. *Remark.* By 6.5, the moduli space on the right is equivalent to $\bar{W}\mathrm{Aut}(\Lambda, M)$. In particular, the moduli space is connected.

Proof. We first handle the case $M=0$; it is easy to see that this amounts to showing that the functor $P_0\pi_*$ induces a weak equivalence from the moduli space of all objects of type B_Λ to $\mathcal{M}(K_\Lambda)$. In view of 2.2, it is enough to show that B_Λ is unique up to weak equivalence, that $\mathrm{Aut}^{\mathrm{h}}(B_\Lambda)$ is homotopically discrete, and that the map $\pi_0 \mathrm{Aut}^{\mathrm{h}}(B_\Lambda)\to \mathrm{Aut}(\Lambda)$ obtained by recording the effect of a self-map on π_0 is an isomorphism.

Suppose that X is a fibrant object of type B_Λ and let W be as in 8.9. By the construction of W it is possible to obtain a map $W \to X$ which is an isomorphism on π_0; this will induce equivalences $W' = \hat{P}_0 W \to \hat{P}_0 X \leftarrow X$. This shows that there is only one such X up to weak equivalence. The same kind of argument shows that $\pi_0 \operatorname{Aut}^{\mathrm{h}}(X)$ maps surjectively to $\operatorname{Aut}(\Lambda)$. Pick such an X which is fibrant and cofibrant, and in particular constructed by cell attachment. Attaching a cell $(D^{i+1,j}, S^{i,j})$ to an object Y to get Y' gives a homotopy fibre sequence

$$\operatorname{Map}^{\mathrm{h}}(Y', X) \to \operatorname{Map}^{\mathrm{h}}(Y, X) \to \operatorname{Map}^{\mathrm{h}}(S^{i,j}, X)$$

in which the base space is contractible for $i > 0$ and homotopically discrete for $i = 0$ (its homotopy groups are $\pi_{i+*,j}X$). Moreover, the map from $[S^{0,j}, X]$ to the set of Π-algebra maps $\hat{\pi}_0 S^{0,j} \to \hat{\pi}_0 X$ is an isomorphism. A formal inductive argument now shows that for any Y, the space $\operatorname{Map}^{\mathrm{h}}(Y, X)$ is homotopically discrete and the map $[Y, X] \to \operatorname{Hom}(\hat{\pi}_0 Y, \hat{\pi}_0 X)$ is injective. The case $Y = X$ of this is what we are looking for.

Now we consider the case of a general M. For any simplicial model category $\mathbf{C}$, there is an induced simplicial model category structure on the category of arrows in $\mathbf{C}$, in which a morphism

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ C & \xrightarrow{v} & D \end{array}$$

from u to v is a weak equivalence (resp. fibration) if α and β are weak equivalences (resp. fibrations) in $\mathbf{C}$, and a cofibration if α is a cofibration in $\mathbf{C}$ and the natural map $C \coprod_A B \to D$ is a cofibration in $\mathbf{C}$. We use this when $\mathbf{C}$ is the E_2 model category structure on simplicial spaces in order to have an explicit way (2.2) to identify the moduli space of a map. Let f be a map of type $B_\Lambda(M, n)$. What we have to prove is that f is unique up to weak equivalence, that $\operatorname{Aut}^{\mathrm{h}}(f)$ is homotopically discrete, and that the natural map $\alpha : \pi_0 \operatorname{Aut}^{\mathrm{h}}(f) \to \operatorname{Aut}(\Lambda, M)$ is an isomorphism. Uniqueness of f and surjectivity of α are proved as above using the explicit models from 8.9. Write $f : B_\Lambda \to X$. We can assume that f is obtained by starting with the identity map $B_\Lambda \to B_\Lambda$ and attaching cells to the target of dimension n and higher. An inductive argument, exactly the same as above, shows that if $g : B_\Lambda \to Y$ is a map obtained in this way, then $\operatorname{Map}^{\mathrm{h}}(g, f)$ is homotopically discrete, and the natural map $[g, f] \to \operatorname{Hom}(\Lambda, \Lambda) \times \operatorname{Hom}(\hat{\pi}_n Y, \hat{\pi}_n X)$ is injective. Applying this in the case $Y = X$ finishes the proof. □

For convenience, we will denote Eilenberg-Mac Lane objects by B_Λ and $B_\Lambda(M, n)$.

Classification of Postnikov stages. Suppose that X is a simplicial space with $X \sim \hat{P}_{n-1}X$ and that M is a module over $\hat{\pi}_0 X$. If Y is a simplicial space, we write $Y \sim X + (M, n)$ if $\hat{P}_n Y \sim Y$, $\hat{P}_{n-1}Y \sim X$, and $\hat{\pi}_n Y$ is isomorphic to M as a module over $\hat{\pi}_0 X$, where the isomorphism is realized with respect to some isomorphism $\hat{\pi}_0 Y \simeq \hat{\pi}_0 X$. We write $\mathcal{M}(X + (M, n))$ for the moduli space of all simplicial spaces of type $X + (M, n)$. The following result is proved in the same way as 6.8.

8.13. **Theorem.** *Suppose that* X *is a simplicial space with* $X \sim \hat{P}_{n-1}X$, $n \geq 1$. *Let* $\Lambda = \hat{\pi}_0 X$, *and let* M *be a module over* Λ. *Then there is an natural weak equivalence*

$$\mathcal{M}(X + (M, n)) \sim \mathcal{M}(X \looparrowright B_\Lambda(M, n+1) \looparrowleft B_\Lambda)\ .$$

8.14. *Remark.* The arrows $\looparrowright$ on the right indicate maps which induce isomorphisms on $\hat{\pi}_i$ for appropriate i (2.3); in this case it is just isomorphisms on $\hat{\pi}_0$. Again, the remarks at the beginning of 3.9 could be repeated here with some slight modifications.

The fundamental homotopy fibre square. The following theorem is at the basis of our classification result.

8.15. **Theorem.** *Suppose that* X *is a simplicial space,* Λ *is a* Π*-algebra, and* M *is a* Λ*-module. Then for any* $n \geq 2$ *there is a natural homotopy fibre square*

$$\begin{array}{ccc}
\mathcal{M}(X \rightsquigarrow B_\Lambda(M, n) \looparrowleft B_\Lambda) & \longrightarrow & \mathcal{M}(\pi_* X \rightsquigarrow K_\Lambda(M, n) \looparrowleft K_\Lambda) \\
\downarrow & & \downarrow \\
\mathcal{M}(X) & \longrightarrow & \mathcal{M}(\pi_* X)
\end{array}$$

8.16. *Remark.* The moduli spaces on the left here involve simplicial spaces, and the ones on the right simplicial Π-algebras. The vertical arrows are induced by the obvious functors which take a diagram and select the first component; the lower horizontal arrow is induced by the functor π_*. The upper horizontal arrow is induced (as in 8.5) by the functor which takes a diagram $U \to V \xleftarrow{f} W$ to the diagram

$$\pi_* U \to \Delta^{\mathrm{t}}_{n-1}(\pi_* f) \leftarrow \Delta^{\mathrm{s}}_{n-1}(\pi_* f)\ .$$

Proof of 8.15. Consider the commutative square

$$\begin{array}{ccc}
\mathcal{M}(X \rightsquigarrow B_\Lambda(M, n) \looparrowleft B_\Lambda) & \longrightarrow & \mathcal{M}(\pi_* X \rightsquigarrow K_\Lambda(M, n) \looparrowleft K_\Lambda) \\
\downarrow & & \downarrow \\
\mathcal{M}(X) \times \mathcal{M}(B_\Lambda(M, n) \looparrowleft B_\Lambda) & \longrightarrow & \mathcal{M}(\pi_* X) \times \mathcal{M}(K_\Lambda(M, n) \looparrowleft K_\Lambda)
\end{array}$$

in which the second factor of the lower horizontal arrow is induced by the difference construction (8.5). The lower spaces are connected, and by 2.11, 2.7, and 8.7 the induced map on vertical fibres is a weak equivalence. Note in this connection that with the help of functorial factorization it is easy to replace the upper left hand moduli space by an equivalent moduli space of diagrams $U \to V \leftarrow W$ in which the simplicial space V equivalent to $B_\Lambda(M,n)$ is fibrant. The proof is finished by observing that the map

$$\mathcal{M}(B_\Lambda(M,n) \hookleftarrow B_\Lambda) \to \mathcal{M}(K_\Lambda(M,n) \hookleftarrow K_\Lambda)$$

is a weak equivalence (8.11). □

9. The main theorem

Recall that if A is a Π-algebra, the moduli space $\mathcal{TM}(A)$ of realizations of A is defined by

$$\mathcal{TM}(A) = \coprod_{\langle X \rangle} \mathcal{M}(X) \ ,$$

where X ranges over weak equivalence classes of (pointed) topological spaces with $\pi_* X \simeq A$. In this section we give the main structure theorems for this moduli space.

9.1. **Definition.** Suppose that X is a simplicial space. We say that X is *a potential n-stage for the Π-algebra A* if the following three conditions are satisfied:

- $\hat{\pi}_0(X)$ is isomorphic to A as a Π-algebra,
- $\hat{\pi}_i(X) \simeq 0$ for $i > n$, and
- $\hat{\epsilon}_i(X) \simeq 0$ for $1 < i \leq n+1$.

The *partial moduli space* or *partial realization space* $\mathcal{TM}_n(A)$ is defined to be the moduli space of all simplicial spaces which are potential n-stages for A.

9.2. *Remark.* It follows from the spiral exact sequence that a potential n-stage X for A has $\hat{\pi}_i X \simeq \Omega^i A$ for $0 \leq i \leq n$, $\hat{\pi}_i X = 0$ for $i > n$, $\hat{\epsilon}_i X \simeq 0$ for $i \neq 0, n+2$, $\hat{\epsilon}_0 X \simeq A$, and $\hat{\epsilon}_{n+2} X \simeq \Omega^{n+1} A$.

The above definition makes sense for $n = \infty$ (the simplicial space X involved would have $\hat{\pi}_0 X \simeq A$ and $\hat{\epsilon}_i X \simeq 0$ for $i > 0$). Our first theorem says that the potential ∞-stages for A are essentially the same as realizations of A.

9.3. **Theorem.** *The geometric realization functor induces a weak equivalence $\mathcal{TM}_\infty(A) \to \mathcal{TM}(A)$.*

Proof. Let F be the functor which assigns to a potential ∞-stage Y for A the geometric realization $|Y^c|$, where Y^c is some functorial cofibrant approximation to Y; by inspection of the homotopy spectral sequence of a realization (7.9) [11, 8.3], $F(Y)$ is a topological realization of A. Let G be the functor which assigns to such a topological realization X the constant simplicial space given by X; it is easy to see directly that $G(X)$ is a potential ∞-stage for A. The two composites GF and FG are connected to the respective identity functors by chains of natural transformations which are weak equivalences, and so induce weak equivalences of the moduli spaces. □

It is easy to see from 7.11 that if X is a potential n-stage for A and $m < n$, then the horizontal Postnikov section $\hat{P}_m X$ is a potential m-stage for A, In particular the functor $\hat{P}_{n-1}$ induces a map $\mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A)$. Our next theorem gives an expression for $\mathcal{TM}_\infty(A)$ in terms of these maps. Let $\mathrm{holim}^{\mathbf{R}}$ denote the derived homotopy limit functor for diagrams of simplicial sets; this is the functor obtained by replacing the diagram in some functorial way by a diagram of fibrant simplicial sets, and applying the ordinary homotopy limit functor of [3].

9.4. **Theorem.** *There is a natural weak equivalence of simplicial sets*

$$\mathcal{TM}_\infty(A) \sim \mathrm{holim}^{\mathbf{R}}_n \mathcal{TM}_n(A)\ .$$

Proof. This follows from [7]; the main result there is stated for simplicial sets, but the arguments apply to any cofibrantly generated simplicial model category with arbitrary small limits and colimits. The main result of [7] is applied in exactly the same as in [7, 4.6]. □

This reduces the study of $\mathcal{TM}_\infty(A)$ to the study of the individual spaces $\mathcal{TM}_n(A)$, together with the maps between them. We begin with $\mathcal{TM}_0(A)$. The following is clear from 6.5, since $\mathcal{TM}_0(A)$ is the moduli space of all simplicial spaces of type B_A.

9.5. **Theorem.** *The space $\mathcal{TM}_0(A)$ is naturally weakly equivalent to* $\mathrm{BAut}(A)$.

In this statement, $\mathrm{Aut}(A)$ denotes the discrete group of Π-algebra automorphisms of A; in particular, the theorem states that $\mathcal{TM}_0(A)$ is an Eilenberg-Mac Lane space of type $K(\pi, 1)$ for $\pi = \mathrm{Aut}(A)$.

The next theorem analyzes the difference between $\mathcal{TM}_n(A)$ and $\mathcal{TM}_{n-1}(A)$.

9.6. **Theorem.** *Suppose that $n \geq 1$. Then there is a natural homotopy fibre square*

$$\begin{array}{ccc}
\mathcal{TM}_n(A) & \longrightarrow & \mathcal{M}(A \looparrowright K_A(\Omega^n A, n+2)) \\
{\scriptstyle \hat{P}_{n-1}}\downarrow & & \downarrow \\
\mathcal{TM}_{n-1}(A) & \longrightarrow & \mathcal{M}(A \looparrowright K_A(\Omega^n A, n+2) \looparrowleft A)
\end{array}$$

The vertical map on the right is induced by the functor which takes a map $U \to V$ and repeats it to obtain $U \to V \leftarrow U$. The other two maps in the square are constructed below.

9.7. *Interpretation.* According to 2.11 and 6.5, the space $Z = \mathcal{M}(A \looparrowright K_A(\Omega^n A, n+2) \looparrowleft A)$ fibres over $\bar{W}\mathrm{Aut}(A) \times \bar{W}\mathrm{Aut}(A, \Omega^n A)$ with fibre

$$\coprod_f \mathcal{H}_A^{n+1}(A; \Omega^n A) \ , \tag{9.8}$$

where the coproduct is taken over the set of all isomorphisms $A \to \pi_0 K_A(\Omega^n A, n+2)$. It is clear that $\mathrm{Aut}(A)$ acts simply transitively on this set, and it follows that Z fibres over $\bar{W}\mathrm{Aut}(A, \Omega^n A)$ with fibre $\mathcal{H}_A^{n+1}(A; \Omega^n A)$. In this way each potential $(n-1)$-stage Y for A, i.e., each vertex of $\mathcal{TM}_{n-1}(A)$, determines an element o_Y in $H_A^{n+2}(A; \Omega^n A)$ modulo the action of $\mathrm{Aut}(A, \Omega^n A)$. This element (which can be identified with the k-invariant (6.8) of the simplicial Π-algebra $\pi_* Y$) is the obstruction to lifting Y to a potential n-stage. Let $\mathcal{TM}_n(A)_Y$ denote the moduli space of all potential n-stages X for A with $\hat{P}_{n-1} X \sim Y$. If o_Y is nontrivial, then $\mathcal{TM}_n(A)_Y$ is empty, otherwise (given that $\Omega \mathcal{H}_A^{n+1}(A; \Omega^n A) \sim \mathcal{H}_A^n(A; \Omega^n A)$), there is a fibration sequence

$$\mathcal{H}_A^n(A; \Omega^n A) \to \mathcal{TM}_n(A)_Y \to \mathcal{M}(Y) \ .$$

On the level of π_0 this can be interpreted as saying that weak equivalence classes of lifts of Y to a potential n-stage for A correspond to trivializations of o_Y; of course the sequence also indicates how the space of such trivializations contributes to the spaces of self-equivalences of these lifts.

9.9. *Potential n-stages.* Suppose that Y is a potential n-stage for A; we can assume that Y is cofibrant as a simplicial space. According to 9.2, the homotopy spectral sequence for $\pi_*|Y|$ (7.9) has only two nontrivial columns at the E_2-page: $\hat{\epsilon}_0 Y \simeq A$ in column $E^2_{0,*}$ and $\hat{\epsilon}_{n+2} Y \simeq \Omega^{n+1} A$ in column $E^2_{n+2,*}$. It follows from the description of the spectral sequence in [11, 8.3] that the differential d_{n+2} maps column $n+2$ as much as possible isomorphically to column 0. Consequently, $\pi_i|Y|$ is trivial for $i \geq n+2$, and $\pi_i|Y| \simeq A_i$ for $i \leq n+1$. But more is true. Let $P^m Y$ be the simplicial space obtained by applying the $(m-1)$-connective cover functor degreewise to Y. The spectral sequence of P^mY can be computed by a naturality argument, and it follows that $\pi_i|P^mY|$ is trivial for $i \geq n+m+1$ or for $i < m$, and that $\pi_i|P^mY| \simeq A_i$ for the remaining values of i. In particular, the algebraic constituents of A are knitted together by Y in a way which is much more comprehensive than is reflected by the single ordinary Postnikov stage $|Y|$.

The rest of this section is taken up with the proof of 9.6.

The first step is to analyze the difference between potential n-stages for A and potential $(n-1)$-stages. Suppose that X is a potential n-stage for A. According to 9.1 and the spiral exact sequence, $\hat{\pi}_n X \simeq \Omega^n A$. Let $Y = \hat{P}_{n-1}X$. Then Y is a potential $(n-1)$-stage for A, and according to 8.3, after adjusting X and Y up to weak equivalence there is a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & B_A \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow \\ Y & \xrightarrow{f} & B_A(\Omega^n A, n+1) \end{array} \tag{9.10}$$

in which the maps f and v give isomorphisms on $\hat{\pi}_0$. We now determine how to reverse this construction.

9.11. **Proposition.** *Suppose that Y is a potential $(n-1)$-stage for A $(n \geq 1)$ and that X lies in a homotopy fibre square of the form 9.10. Then X is a potential n-stage for A if and only if the map $g : \pi_* Y \to K_A(\Omega^n A, n+1)$ corresponding (8.6) to f is a weak equivalence of simplicial Π-algebras.*

Proof. The main thing to prove in showing that X is a potential n-stage for A is that $\hat{\epsilon}_i X$ vanishes for $i = n, n+1$; the other conditions are simple to check. The homotopy fibre F of v is of type $B_0(\Omega^n A, n)$. Consequently, $\hat{\epsilon}_i F$ vanishes unless i is n or $n+2$, and the long exact $\hat{\epsilon}_*$-homotopy sequence of u (7.7) degenerates around dimension n into the exact sequence

$$0 \to \hat{\epsilon}_{n+1} X \to \hat{\epsilon}_{n+1} Y \to \hat{\epsilon}_n F \to \hat{\epsilon}_n X \to 0\,.$$

Thus X is a potential n-stage if and only if the connecting homomorphism $\hat{\epsilon}_{n+1} Y \to \hat{\epsilon}_n F \simeq \Omega^n A$ is an isomorphism. A naturality argument identifies this connecting homomorphism with the map $\pi_{n+1}\pi_* Y \to \Omega^n A$ induced by g. Since $\pi_0(g)$ is an isomorphism by assumption, and both domain and range of g have trivial homotopy except in dimensions 0 and $n+1$, the result follows. □

Suppose that Y is a potential $(n-1)$-stage for A. We write $X \sim Y \oplus (\Omega^n A, n)$ if X is a potential n-stage for Y and $\hat{P}_{n-1}X \sim Y$. The space $\mathcal{M}(Y \oplus (\Omega^n A, n))$ is the moduli space of all such X.

9.12**. Proposition.** *Suppose that Y is a potential $(n-1)$-stage for A $(n \geq 1)$. Then there is a natural homotopy fibre square*

$$\begin{array}{ccc} \mathcal{M}(Y \oplus (\Omega^n A, n)) & \longrightarrow & \mathcal{M}(\pi_* Y \looparrowright K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\ {\scriptstyle \hat{P}_{n-1}}\downarrow & & \downarrow \\ \mathcal{M}(Y) & \xrightarrow{\pi_*} & \mathcal{M}(\pi_* Y) \end{array} .$$

9.13. *Remark.* As usual, $\looparrowright$ signifies maps which induce isomorphisms on appropriate homotopy groups; in the case $\pi_* Y \looparrowright K_A(\Omega^n A, n+1)$ these isomorphisms are such that the map is an equivalence. The right vertical arrow in the square is induced by the functor which takes a diagram $U \to V \leftarrow W$ of simplicial Π-algebras and selects the first component. As would be revealed by unraveling the proof, the upper horizontal arrow is induced by two applications of the difference construction, one in the category of simplicial spaces (8.4) to obtain $Y \to B_A(\Omega^n A, n+1)$, and the second in the category of simplicial Π-algebras (8.5) to obtain $\pi_* Y \to K_A(\Omega^n A, n+1)$.

Proof of 9.12. We let $M = \Omega^n A$ and $m = n+1$. There is a square

$$\begin{array}{ccc} \mathcal{M}(Y \oplus (M, n)) & \longrightarrow & \mathcal{M}(Y \xrightarrow{\oplus} B_A(M, m) \hookleftarrow B_A) \\ {\scriptstyle \hat{P}_{n-1}}\downarrow & & \downarrow \\ \mathcal{M}(Y) & \xrightarrow{=} & \mathcal{M}(Y) \end{array}$$

whose upper arrow is a weak equivalence obtained by using 9.11 to select appropriate components of the weak equivalence from 8.13. Here $\xrightarrow{\oplus}$ denotes maps which correspond via 8.5 to weak equivalences $\pi_* Y \to K_A(M, m)$. Passing to appropriate components with 8.15 gives a homotopy fibre square

$$\begin{array}{ccc} \mathcal{M}(Y \xrightarrow{\oplus} B_A(M, m) \hookleftarrow B_A) & \longrightarrow & \mathcal{M}(\pi_* Y \looparrowright K_A(M, m) \hookleftarrow K_A) \\ \downarrow & & \downarrow \\ \mathcal{M}(Y) & \xrightarrow{\pi_*} & \mathcal{M}(\pi_* Y) \end{array} .$$

Combining these squares finishes the proof. □

Proof of 9.6. For any Π-algebra Λ, Λ-module M, and $m \geq 1$ there is a commutative diagram

(9.14)

$$\begin{array}{ccc}
\mathcal{M}(K_\Lambda(M,m) \hookleftarrow K_\Lambda) & \xrightarrow{\sim} & \mathcal{M}(K_\Lambda(M,m+1) \hookleftarrow K_\Lambda) \\
\downarrow & & \downarrow \\
\mathcal{M}(K_\Lambda + (M,m)) & \xrightarrow{\sim} & \mathcal{M}(K_\Lambda \hookrightarrow K_\Lambda(M,m+1) \hookleftarrow K_\Lambda)
\end{array}$$

in which the horizontal arrows are equivalences obtained with the difference construction; see the proof of 6.5 for the upper arrow and 6.8 for the lower one. Clearly, this is a homotopy fibre square. Suppose that Y is a potential $(n-1)$-stage for A. Let $\Lambda = A$, $M = \Omega^n A$, and $m = n+1$. Then $\mathcal{M}(\pi_* Y)$ is one component of $\mathcal{M}(K_\Lambda + (M,m))$. Moreover, the map $\mathcal{M}(K_\Lambda(M,n)) \to \mathcal{M}(\pi_* Y \hookrightarrow K_A(\Omega^n A, n+1))$ obtained by sending a map $U \leftarrow V$ to $U \xrightarrow{=} U \leftarrow V$ is a weak equivalence (a homotopy inverse is given by the functor sending $U \to V \leftarrow W$ to $V \leftarrow W$). Combining this observation with 9.12 and 9.14 then gives a homotopy fibre square

$$\begin{array}{ccc}
\coprod_{\langle Y \rangle} \mathcal{M}(Y \oplus (\Omega^n A, n)) & \longrightarrow & \mathcal{M}(K_\Lambda(M,m+1) \hookleftarrow K_\Lambda) \\
\downarrow & & \downarrow \\
\coprod_{\langle Y \rangle} \mathcal{M}(Y) & \longrightarrow & \mathcal{M}(K_\Lambda \hookrightarrow K_\Lambda(M,m+1) \hookleftarrow K_\Lambda)
\end{array}$$

which is the one we are looking for, since the left vertical arrow is $\mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A)$. □

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