We need to have some good control over the homotopy of $MU^{(G)}$. We start with MU. Recall that

$MU \wedge MU = \bigvee_{p \in \mathbb{Z}[b_1, \dots]} MU \wedge S^{|p|}$. We saw that $MU_* MU = MU_*[b_1, \dots]$, so the map from the RHS to the left is the obvious map of MU-modules extending $S^{|p|} \xrightarrow{p} MU \wedge MU$.

There is also algebraic content here: $MU_* MU$ represents 3 pieces of data: 2 formal group laws & an isomorphism between them. The elements b_i are the coefficients of the universal isomorphism: $f(x) = \sum_i b_i x^{i+1}$ $(b_0 = 1)$.

(So how do the data arise? $MU_* \longrightarrow MU_* MU \longrightarrow R$ gives one f.g.l F, The b_i determine an iso f, and $G(x,y) = f^{-1} F(f(x), g(x))$)

All of this can be done remembering the C_2-action. This gives a Real f.g.l and a notion of a Real orientation. We have

$MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \simeq \bigvee_{R} MU_{\mathbb{R}} \wedge S^{\frac{|p|}{2}\rho_2}$, where the indexing set R is the same as before. Therefore, by induction on n,

$$MU_{\mathbb{R}}^{\wedge n} = \bigvee_{R^{\otimes(n-1)}} MU_{\mathbb{R}} \wedge S^{\frac{|p|}{2}\rho_2}.$$

The essential part here is subtle: given any element p in $\pi^u_{2k} MU_{\mathbb{R}}^{\wedge n}$, we have a C_2-equivariant lift $\bar{p}: S^{k\rho_2} \longrightarrow MU_{\mathbb{R}}^{\wedge n}$.

We apply this to $MU^{(4)}$. $L^*_{C_2} MU^{(4)} = MU_{\mathbb{R}}^{\wedge 4}$, and we know $\pi^u_* MU^{(4)} = \mathbb{Z}[r_1, \gamma r_1, \gamma^2 r_1, \gamma^3 r_1, r_2, \dots]$. By the previous remark, we know that all of these lift to equivariant maps $S^{\frac{|p|}{2}\rho_2} \xrightarrow{\bar{p}} L^*_{C_2} MU^{(G)}$.

As an aside, there is a good choice for the r_i. $MU_{\mathbb{R}}^{\wedge 4}$ classifies 4 Real formal group laws and isomorphisms:

$$\begin{tikzcd} F_1 \arrow[r, bend left, "\gamma"] & F_2 \arrow[d, bend left, "\gamma"] \\ F_4 \arrow[u, bend left, "\gamma"] & F_3 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

The elements r_i are the coefficients of the isomorphism between F_1 & the 2-typification (reduction to BP) of γF_1. It's a bit formal group intensive, but this choice also plays nicely w/ geometric fixed points.

So we have generators $\bar{r}_i$. We want to filter $MU^{(4)}$ by the ideal generated by the $\bar{r}_i$. Of course, they are just C_2-equivariant, not C_8-equivariant.

The underlying goal is to show an analogue of the classical result: $MU/(x_1, \dots) = H\mathbb{Z}$, and then use this to better compute.

We'll handle this by doing more algebra in spectra:

(Random Empty Space)

Remember we had maps $S^{i\rho_2} \xrightarrow{\bar{r}_i} i^*_{C_2} MU^{((G))}$. Now let's bootstrap up to something that sees the homotopy generators.

$\mathbb{P}(S^V) = \bigvee_{i \geq 0} S^{iV}$. This has a multiplication (its the obvious one). This is associative but not commutative.

= free associative ring on S^V. So for $V = j\rho_2$, have a map $\mathbb{P}(S^{j\rho_2}) \to i^*_H MU^{((G))}$. Smash all of these together giving

$$\underset{\displaystyle\bigvee_{\substack{\text{monomials}\\ \text{in } \mathbb{Z}[r_1, \dots]}} S^{\frac{|p|}{2}\rho_2}}{\underset{\shortparallel}{\bigwedge \mathbb{P}[S^{j\rho_2}]}} \longrightarrow \bigwedge i^*_H MU^{((G))} \to i^*_H MU^{((G))}.$$

Now norm up.

$$A = N_{C_2}^G\left(\bigvee S^{\frac{|p|}{2}\rho_2}\right) \longrightarrow \underbrace{N_{C_2}^G i^*_H MU^{G}}_{\text{counit}} \longrightarrow N_{C_2}^G MU^G$$

Indexing set is "$\overset{\text{monic}}{\text{monomials}}$ in $N_{C_2}^G \mathbb{Z}[r_1, \dots] = \mathbb{Z}[r_1, \gamma r_1, \dots, r_2, \dots]$" To a monomial p, have a group: $C_2 \subseteq \underset{H_p}{\underset{\shortparallel}{\mathrm{Stab}}}(p \bmod 2) \subseteq G$. Then the orbit for p corresponds to the wedge summand

$$G_+ \underset{H_p}{\wedge} S^{\frac{|p|}{|H_p|}\rho_{H_p}}.$$

<u>Thm</u> ① $MU^{((G))} \underset{A}{\wedge} S^0 = H\underline{\mathbb{Z}}$.

② The "degree" filtration gives a filtration on $MU^{((G))}$ with $Gr(MU^{((G))}) = H\underline{\mathbb{Z}} \wedge A$.

Now the "Gap theorem": $\Omega_{\mathbb{O}} = D^{-1} MU^{((G))}$ with $D : S^{?\rho} \to MU^{((G))}$ (the actual element doesn't matter).

This colimit arises from $\Sigma^{-?m\rho} MU^{((G))}$. Smashing our tower with $S^{-?m\rho}$ has the same form:

$$\left(\bigvee G_+ \underset{H}{\wedge} S^{j\rho_H}\right) \wedge H\underline{\mathbb{Z}}.$$

<u>Thm</u> For any $\{e\} \subsetneq H \subseteq G$, $\pi_{-2}\left(\left(G_+ \underset{H}{\wedge} S^{j\rho_H} \wedge H\underline{\mathbb{Z}}\right)^G\right) = 0$.

<u>Cor</u> If $\bar{D} : S^{k\rho_G} \to MU^{((G))}$, then $\pi_{-2}\left(\bar{D}^{-1} MU^{((G))}\right)^G = 0$.

Pf of theorem:

It suffices to show $\underset{\underset{\displaystyle\pi^H_{-2}\left(S^{j\rho_H} \wedge H\underline{\mathbb{Z}}\right)}{\wr\shortparallel}}{\pi^G_{-2}\left(G_+ \underset{H}{\wedge} S^{j\rho_H} \wedge H\underline{\mathbb{Z}}\right)} = 0$ for $|H| \neq 1$.

If $j \geq 0$, connectivity gives this, and if $j < -2$, co-connectivity does so. For $j = 1, 2$, the complex computing the relevant groups is dual to $\overset{j}{\mathbb{Z}} \xleftarrow{\nabla} \overset{j+1}{\mathbb{Z}^2}$.