A generalized Grothendieck spectral sequence

David Blanc
Northwestern University

Christopher Stover
University of Chicago

June 5, 1991

Abstract

We construct a generalized Grothendieck spectral sequence for computing the derived functors of a composite functor $T \circ S$, extending the classical version to non-additive functors and non-abelian categories.

1 Introduction

The classical Grothendieck spectral sequence [Gr, Thm 2.4.1] computes the derived functors of a composite functor $T \circ S$ in terms of those of T and S:

That is, suppose $\mathcal{C} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{A}$ are additive functors between abelian categories (with enough projectives), and $T(P)$ is S-acyclic for projective $P \in \mathcal{C}$; then for any $C \in \mathcal{C}$ one has a spectral sequence with

$$E^2_{s,t} \cong (L_sS)(L_tT)C \;\Rightarrow\; (L_{s+t}(S \circ T))C$$

(see [HS, VIII, Thm 9.3]).

The derived functors $L_\star F$ in question are usually defined as the homology groups of certain chain-complexes; however, they may also be defined as the homotopy groups of suitable simplicial objects (see §2.2.4 below), and this more general definition extends to cases where the functors involved are not additive, or the categories are not abelian.

Now if $\mathcal{B}$ is a category of universal algebras – such as groups, rings, or Lie algebras (§2.1) – then the homotopy groups of a simplicial object $X_\bullet$ over $\mathcal{B}$ support an action of the primary homotopy operations (§3.1.2) associated to $\mathcal{B}$: we then say that $\pi_\star X_\bullet$ is a $\mathcal{B}$-Π-*algebra*. In particular, for any functor $F : \mathcal{C} \to \mathcal{B}$, the derived functors $L_\star F = \{L_0F, L_1F, \ldots\}$ together take values in the category of $\mathcal{B}$-Π-algebras. Moreover, if $\mathcal{C}$ is also a category of universal algebras, F induces a functor

$$\bar{F}_* : \mathcal{C}\text{-}\Pi\text{-algebras} \rightarrow \mathcal{B}\text{-}\Pi\text{-algebras}.$$

In this setting we have a generalized Grothendieck spectral sequence:

Theorem 4.4 *Let* $\mathcal{C} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{A}$ *be functors between categories of universal algebras, such that* TF *is* S*-acyclic for every free* $F \in \mathcal{C}$*; then for every* $C \in \mathcal{C}$ *there is a spectral sequence with*

$$E^2_{s,t} \cong (L_s \bar{S}_t)(L_\star T)C \;\Rightarrow\; (L_{s+t}(S \circ T))C.$$

More generally, given a suitable simplicial object $X_\bullet$ over a category $\mathcal{B}$, and a functor $F : \mathcal{B} \rightarrow \mathcal{A}$, one is often interested in determining the homotopy groups of $F(X_\bullet)$ – e.g., in order to calculate derived functors. In particular one may ask how these depend on $\pi_\star X_\bullet$. If $\mathcal{B}$ is a category of universal algebras, one has a spectral sequence converging to $\pi_\star F X_\bullet$ with E^2-term the derived functors of $\bar{F}_*$ applied to $\pi_\star X_\bullet$ (Theorem 4.2).

The generalized Grothendieck spectral sequence is a special case of this; further (degenerate) examples are the Kunneth and Universal Coefficients short exact sequences, and others (§4.2.1).

1.1 notation

For any category $\mathcal{C}$, we denote by $gr\mathcal{C}$ the category of *non-negatively graded objects* over $\mathcal{C}$, and by $s\mathcal{C}$ the category of *simplicial objects* over $\mathcal{C}$. We use the convention that $X_\bullet$ denotes a *simplicial* object (cf. [May, §2]), while $X_* = \{X_i\}_{i=0}^\infty$ denotes a graded object. The category of groups will be denoted $\mathcal{G}p$, that of abelian groups by $\mathcal{A}b\mathcal{G}p$, that of pointed sets by Set_*, and that of pointed simplicial sets by $\mathcal{S}_*$ (rather than $sSet_*$).

We shall assume that all categories are pointed (=have a zero object), and all functors are covariant and pointed (=take zero object to zero object).

1.2 organization

After recalling the needed homotopical algebra in §2, $\mathcal{C}$-Π-algebras and induced functors are defined in §3. The general spectral sequence is set up, and the above theorem is proved, in §4. In §5 we explain why the homotopy group objects over $s\mathcal{C}$ take values in $\mathcal{C}$.

2 universal algebras & homotopical algebra

First, we give some definitions, and remind the reader of some of the (non-abelian) homological algebra needed in our context:

2.1 universal algebras

Recall [McL, I,§7] that a (pointed) *concrete* category $\mathcal{C}$ is one with a faithful functor $U : \mathcal{C} \to Set_*$. In particular, we shall be interested in categories of (possibly graded) *universal algebras* (or varieties of algebras, in the terminology of [McL, V,§6]): that is, categories whose objects, which we shall call simply *algebras*, are (non-negatively graded) sets X ($= \{X_i\}_{i=0}^{\infty}$), together with an action of a fixed set of operators W of the form $\omega : X_{i_1} \times X_{i_2} \times \dots X_{i_n} \longrightarrow X_k$, satisfying a set of identities E.

For simplicity we assume X (or each X_i) has the underlying structure of a group. (This assumption may be relinquished in certain cases – e.g., for $\mathcal{C} = Set_*$ – but our construction will not work in general for arbitrary universal algebras.)

To avoid confusion with the simplicial dimensions needed later, we shall denote the underlying group in degree i by G_iX. Such a *category of universal graded algebras* will be called a *CUGA*. The ungraded version can be thought of as a CUGA with objects concentrated in degree 0.

If each G_iX is abelian we call X *underlying-abelian*; if this is true of all $X \in \mathcal{C}$, we say the category $\mathcal{C}$ is underlying-abelian (of course, $\mathcal{C}$ need not then be an abelian catgeory).

2.1.1 CUGA's

Examples of CUGA's include *regular graded algebras* in the sense of [B2, §2] (all of which are underlying-abelian), such as:

- (i) The category of abelian groups, or more generally of (graded) left R-modules for some ring R.
- (ii) The category of graded commutative algebras over a ring k; similarly the category of associative algebras over k.
- (iii) $\mathcal{K}_p$, the category of $\mathbb{F}_p$-algebras over the mod-p Steenrod algebra.
- (iv) More generally, the category of algebras over E^*E for any ring spectrum E (though these need not be non-negatively graded).
- (v) The category of graded Lie rings; similarly the category $\mathcal{L}_p$ of graded restricted Lie algebras over $\mathbb{F}_p$.

CUGA's which are not underlying-abelian include:

- (vi) The category of (graded) groups.

(vii) the category of Π-algebras: recall ([B1, §3] or [St, §4]) that a Π-*algebra* is a graded group $X = \{X_i\}_{i=0}^{\infty}$ together with an action of the primary homotopy operations − Whitehead products, compositions, and action of the fundamental group − which satisfies all the universal relations on such operations (cf. [W, XI, §1]). These are modeled on $\pi_\star X$, where X is a pointed space.

2.1.2 free algebras and underlying sets

Each CUGA $\mathcal{C}$ is equipped with a pair of adjoint functors $\mathcal{C} \underset{F}{\overset{U}{\rightleftharpoons}} grSet_*$ to the category of graded pointed sets, with $U(A)$ the *underlying* graded set of $A \in \mathcal{C}$, and $F(X)$ the *free* algebra on a graded set of *generators* $X = \{X_i\}_{i=1}^{\infty}$ (where in each degree the base point $*$ is identified with the group identity element e). Thus every CUGA $\mathcal{C}$ has enough projectives. It can also be shown to have all limits and colimits, as in [McL, V,§1 & IX,§1].

2.2 homotopical algebra

When $\mathcal{C}$ is a CUGA, Quillen shows in [Q1, II,§4] that $s\mathcal{C}$ can be given a *closed (simplicial) model category* structure as follows:

2.2.1 the model category $s\mathcal{C}$

A map $f : X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ is called a *fibration* if it is a surjection on the basepoint components of the underlying simplicial groups; it is called a *weak equivalence* (w.e.) if it induces an isomorphism on homotopy groups (§3.1.1). A map $i : A_\bullet \to B_\bullet$ is called a *cofibration* if it has the *left lifting property* (LLP) with respect to trivial fibrations − that is, if the dotted arrow exists in the following commutative diagram whenever f is both a fibration and a weak equivalence:

$$\begin{array}{ccccc} & A_\bullet & \longrightarrow & X_\bullet & \\ \text{cof.} & i \downarrow & \nearrow & \downarrow f & \text{fib. w.e.} \\ & B_\bullet & \longrightarrow & Y_\bullet & \end{array}$$

These three classes of maps satisfy certain axioms (cf. [BF, §1]), which allow one to 'do homotopy theory' in $s\mathcal{C}$.

2.2.2 cofibrant and free objects

An object $A_\bullet$ is called *cofibrant* if $* \to A_\bullet$ is a cofibration; the full subcategory of such objects is denoted $s\mathcal{C}_c$. Examples are the *free objects*

$C_\bullet$, where for each $n \geq 0$ there is a (graded) set $T^n \subseteq C_n$ such that $C_n \in \mathcal{C}$ is the free object generated by T^n, and each degeneracy map $s_j : C_n \to C_{n+1}$ takes T^n to T^{n+1}.

2.2.3 resolutions

The *homotopy category* $Ho\mathcal{X}$ of any model category $\mathcal{X}$ is obtained from it by localizing with respect to the weak equivalences, with $\gamma : \mathcal{X} \to Ho\mathcal{X}$ the localization functor. In our case, because all objects in $s\mathcal{C}$ are fibrant, $Ho(s\mathcal{C})$ is equivalent to the category $\pi(s\mathcal{C}_c)$, whose objects are the cofibrant ones of $s\mathcal{C}$, and whose morphisms are homotopy classes of maps (cf. [Q1, I, §1]). Here homotopies between maps in $s\mathcal{C}$ are defined simplicially, as in [Q1, II, §2]; the set of homotopy classes of maps $A_\bullet \to B_\bullet$ is denoted $[A_\bullet, B_\bullet]_{s\mathcal{C}}$.

Under this equivalence of $Ho(s\mathcal{C})$ and $\pi(s\mathcal{C}_c)$, the localization functor is determined by the choice, for each object $X_\bullet \in s\mathcal{C}$, of a cofibrant object $A_\bullet$ with a weak equivalence $A_\bullet \to X_\bullet$. This is called a *resolution* of $X_\bullet$, and all such are homotopy equivalent. We use the embedding $c(-)_\bullet : \mathcal{C} \to s\mathcal{C}$ (which sends $X \in \mathcal{C}$ to the constant simplicial object $c(X)_\bullet$) to define resolutions of objects in $\mathcal{C}$.

2.2.4 derived functors

If $H : \mathcal{X} \to \mathcal{Y}$ is a functor between model categories which preserves weak equivalences between cofibrant objects, the *total left derived functor* of H is the functor $\mathbf{L}H = \tilde{H} \circ \gamma : \mathcal{X} \to Ho(\mathcal{Y})$, where $\tilde{H} : Ho\mathcal{X} \to Ho\mathcal{Y}$ is induced by H on $\mathcal{X}_c$.

Any functor $T : \mathcal{C} \to \mathcal{C}'$ may be *prolonged* to a functor $sT : s\mathcal{C} \to s\mathcal{C}'$, by applying it dimensionwise (by abuse of notation we shall often denote sT simply by T.) In particular, if $T : \mathcal{C} \to \mathcal{C}'$ is a functor from a CUGA into a concrete category, the usual n-th *derived functor* of F, denoted L_nT, assigns to an object $X \in \mathcal{C}$ the object $(L_nT)X = \pi_n((\mathbf{L}sT)c(X)_\bullet) = \pi_n TA_\bullet$, where $A_\bullet \to X$ is any resolution, and $\pi_n(-)$ is defined in §3.1.1 below. The derived functors $L_\star F = \{L_0F, L_1F, \ldots\}$ together take values in the category of $\mathcal{C}'$-Π-algebras (see §3.2).

Note that T need only be given on the full subcategory of free objects of $\mathcal{C}$ in order to define L_nT on $\mathcal{C}$. In this case L_0T is called the *extension* of T to all of $\mathcal{C}$; it agrees with T on the subcategory of free objects, and often agrees with T on all of $\mathcal{C}$ (e.g., if T preserves colimits).

3 homotopy operations over $\mathcal{C}$

Simplicial object $X_\bullet$ over a CUGA $\mathcal{C}$ have *homotopy group objects* $\pi_n X_\bullet \in \mathcal{C}$. These are represented in the homotopy category by certain models:

3.1 models in sC

Let $\mathbf{\Delta}[n]_\bullet$ denote the standard simplicial n-simplex, $\dot{\mathbf{\Delta}}[n]_\bullet = \mathbf{\Delta}[n]_\bullet^{(n-1)}$ its $(n-1)$-skeleton, and $\mathbf{S}^n_\bullet = \mathbf{\Delta}[n]_\bullet / \mathbf{\Delta}[n]_\bullet^{(n-1)}$ the simplicial n-sphere. We denote by $\mathbf{S}^n(k)_\bullet$ the graded simplicial set with $\mathbf{S}^n_\bullet$ in degree k. The simplicial objects $F(\mathbf{S}^n(k)_\bullet) \in s\mathcal{C}$ $(n, k \geq 0)$ should be thought of as the "spheres" of $s\mathcal{C}$. The free simplicial algebras (§2.2.1) which are weakly equivalent to "wedges of spheres" $F(\bigvee_{i\in I} \mathbf{S}^{n_i}(k_i)_\bullet) \cong \coprod_{i\in I} F(\mathbf{S}^{n_i}(k_i)_\bullet)$ will be called *models* for $s\mathcal{C}$; the full subcategory of models in $s\mathcal{C}$ is denoted by $\mathcal{M}$.

3.1.1 homotopy groups

If $\mathcal{C}$ is ungraded, the homotopy groups $\pi_n X_\bullet$ $(n \geq 0)$ of any $X_\bullet \in s\mathcal{C}$ are defined to be those of its underlying simplicial group (§5.2). When $\mathcal{C}$ is graded, $\pi_n X_\bullet$ is the graded group with $G_k(\pi_n X_\bullet) = \pi_n(G_k X_\bullet)$.

The models of $s\mathcal{C}$ then represent the homotopy groups over $s\mathcal{C}$, in the sense that $[F(\mathbf{S}^n(k)_\bullet), X_\bullet]_{s\mathcal{C}} \cong G_k \pi_n X_\bullet$, by the adjointness of U and F (§2.1.2; and compare [Q1, I,§4]).

In fact, $\pi_n X_\bullet$ takes values in $\mathcal{C}$, with the $\mathcal{C}$-structure induced by maps in

$$Hom_{\mathcal{C}}(F(\mathbf{S}^n(k)_n),\ F(\bigvee_{i=1}^{N} \mathbf{S}^n(k_i)_n)\ ) \ \cong\ G_k F\{x_1, \ldots x_N\}, \quad \text{where } |x_i| = k_i\ ,$$

which correspond to N-ary operations in W of $\mathcal{C}$. For $n \geq 1$ it turns out that $\pi_n X_\bullet$ is actually an *abelian* object in $\mathcal{C}$ (§5.1.3) – as in the case $\mathcal{C} = \mathcal{G}p$ – so that many of these operations will be trivial (see Lemma 5.2.1).

3.1.2 homotopy operations

In addition, $\pi_\star X_\bullet \in gr\mathcal{C}$ has an action of the (primary) $\mathcal{C}$-homotopy operations, which are described as usual by the universal examples – homotopy classes of maps between models. Any such class

$$\alpha \in [F(\mathbf{S}^n(k)_\bullet), F(\bigvee_{i=1}^{N} \mathbf{S}^{n_i}(k_i)_\bullet)]_{s\mathcal{C}} \tag{1}$$

induces a $\mathcal{C}$*-homotopy operation* (natural in $X_\bullet$)

$$\alpha^\# \ :\ \pi_{n_1} G_{k_1}(X_\bullet) \times \pi_{n_2} G_{k_2}(X_\bullet) \times \ldots \pi_{n_N} G_{k_N}(X_\bullet) \ \longrightarrow\ \pi_n G_k(X_\bullet)\ .$$

3.2 $\mathcal{C}$-Π-algebras

We define a $\mathcal{C}$-Π-*algebra* A_* to be a sequence $A_0, A_1, \dots$ of objects in $\mathcal{C}$ (abelian, if $n \geq 1$), together with an action of the $\mathcal{C}$-homotopy operations, subject to the universal relations coming from (1). These form a category, denoted $\mathcal{C}$-Π-Alg, which is itself a CUGA (now *bigraded*, if $\mathcal{C}$ is graded). The *free* $\mathcal{C}$-Π-algebras are those that are isomorphic to $\pi_\star(F(\bigvee_i \mathbf{S}^{n_i}(k_i)_\bullet))$ for various n_i, k_i.

Note that the category $s(\mathcal{C}$-Π-$Alg)$ of simplical $\mathcal{C}$-Π-algebras thus has a model category structure, as in §2.2.1.

3.2.1 examples of $\mathcal{C}$-Π-algebras

For some specific CUGA's $\mathcal{C}$, the category of $\mathcal{C}$-Π-algebras has a familiar description:

(I) If the CUGA $\mathcal{C}$ is an abelian category, $s\mathcal{C}$ is equivalent to the category $c\mathcal{C}$ of chain complexes over $\mathcal{C}$ (as in [Do, §1]), with a model $F(\mathbf{S}^n(k)_\bullet)$ in $s\mathcal{C}$ corresponding to a minimal chain complex $\Sigma^n F[x_k]$. Then

$$[F(\mathbf{S}^n(k)_\bullet), F(\bigvee_{i=1}^{N} \mathbf{S}^{n_i}(k_i)_\bullet)]_{s\mathcal{C}} \cong \bigoplus_{n_i=n} G_k F(\mathbf{S}^n(k_i)_\bullet),$$

so there are no homotopy operations in $s\mathcal{C}$, except for the internal ones of $\mathcal{C}$ – and thus the category $\mathcal{C}$-Π-Alg is equivalent to $gr\mathcal{C}$.

(II) If $\mathcal{C} = \mathcal{G}p$, the category of groups, then the homotopy category of $s\mathcal{G}p$ is equivalent to that of connected toplogical spaces (cf. [Kan, §9,11]), so a $\mathcal{G}p$-Π-algebra is just an ordinary Π-algebra (see §2.1.1(vii)), with a shift in dimension.

(III) The homotopy operations for $\mathcal{L}_p$, the category of ungraded Lie algebras over $\mathbb{F}_p$, include a graded Lie bracket which satisfies a Hilton-Milnor theorem (see [Sch], where this is shown to hold over $\mathbb{Z}$). The homotopy groups of the individual models:

$$\pi_\star F(\mathbf{S}^n_\bullet) \;\cong\; \Lambda(n)$$

are just the "Λ-algebra spheres" $\pi_\star LAS_n$ of [6A, 5.4 & 5.4'].

(IV) Similarly, the homotopy operations for commutative algebras over $\mathbb{F}_2$ have been calculated by Bousfield and Dwyer (in [Bo, Dw]): They form a commutative algebra on certain unary "divided power" operations δ_i, of degree i, for each $2 \leq i$, subject to the relations of [Dw, Thm. 2.1].

3.2.2 other categories

We may extend the concept of a $\mathcal{C}$-Π-algebra to categories $\mathcal{C}$ which are not CUGA's, by the following convention:

a) If $\mathcal{C}$ is *any* abelian category, we let $\mathcal{C}\text{-}\Pi\text{-}Alg = gr\mathcal{C}$, as in §3.2.1(I).

b) If $\mathcal{C} = Set_*$, a Set_*-Π-algebra $\{X_i\}_{i=0}^{\infty}$ is an ordinary Π-algebra (§2.1.1(vii)) $\{X_i\}_{i=1}^{\infty}$ in positive degrees, together with a pointed set X_0 in degree 0.

c) If $\mathcal{C}$ is any concrete category which is neither abelian nor a CUGA, we set $\mathcal{C}\text{-}\Pi\text{-}Alg = Set_*\text{-}\Pi\text{-}Alg$, and for each $X_\bullet \in s\mathcal{C}$ we let $\pi_\star X_\bullet \in \mathcal{C}\text{-}\Pi\text{-}Alg$ denote $\pi_\star(UX_\bullet)$, where $U : \mathcal{C} \to Set_*$ is the faithful "underlying set" functor.

With this convention, for any concrete category $\mathcal{C}$ the functor $\pi_\star$, and thus the derived functors of any $T : \mathcal{C}' \to \mathcal{C}$, take values in $\mathcal{C}\text{-}\Pi\text{-}Alg$.

3.2.3 Proposition. *Any covariant functor $T : \mathcal{C} \to \mathcal{B}$ from a CUGA $\mathcal{C}$ into a concrete category $\mathcal{B}$ induces a functor $\bar{T}_* : \mathcal{C}\text{-}\Pi\text{-}Alg \to \mathcal{B}\text{-}\Pi\text{-}Alg$, which is the extension* (§2.2.4) *of the functor on free $\mathcal{C}$-Π-algebras defined by $\bar{T}_t(\pi_\star A_\bullet) = \pi_t(TA_\bullet)$ for $A_\bullet \in \mathcal{M}$.*

Proof: If $\mathcal{C}$ is a CUGA, it is evident that for any two models $A_\bullet, B_\bullet \in \mathcal{M}$ there is a natural bijection

$$\pi_\star(-) : Hom_{Ho(s\mathcal{C})}(A_\bullet, B_\bullet) \to Hom_{\mathcal{C}\text{-}\Pi\text{-}Alg}(\pi_\star A_\bullet, \pi_\star B_\bullet).$$

In particular, this means that if $\pi_\star A_\bullet \cong \pi_\star B_\bullet$ *as $\mathcal{C}$-Π-algebras*, then $A_\bullet$ and $B_\bullet$ are actually homotopy equivalent, so $\pi_t(TA_\bullet) \cong \pi_t(TB_\bullet)$. The naturality of the bijection implies that $\bar{T}_t$ is in fact a functor. □

Note that $\bar{T}_*(\pi_\star X_\bullet)$ is usually *not* the same as $\pi_\star TX_\bullet$ for $X_\bullet \notin \mathcal{M}$ – e.g., for $T = - \otimes \mathbb{Z}/p : \mathcal{A}b\mathcal{G}p \to \mathcal{A}b\mathcal{G}p$.

3.2.4 examples in the abelian case

If $\mathcal{C}$ and $\mathcal{B}$ are abelian categories, then $\mathcal{C}\text{-}\Pi\text{-}Alg$, $\mathcal{B}\text{-}\Pi\text{-}Alg$ are equivalent to $gr\mathcal{C}$, $gr\mathcal{B}$ respectively (§3.2.1(I)), and any $A_\bullet \in \mathcal{M}$ is equivalent to a minimal chain complex $\hat{A}_*$.

a) When T is additive, $\pi_n TA_\bullet \cong T\hat{A}_n = T\pi_n A_\bullet$ and $\bar{T}_*$ is, in each degree, just the 0-th derived functor of T (§2.2.4).

b) When T is not additive, this need not be the case: For example, if $T : \mathcal{A}b\mathcal{G}p \times \mathcal{A}b\mathcal{G}p \to \mathcal{A}b\mathcal{G}p$ is $T(X,Y) = X \otimes Y$, its prolongation to $s\mathcal{A}b\mathcal{G}p \times s\mathcal{A}b\mathcal{G}p$ corresponds (under the equivalence $c\mathcal{A}b\mathcal{G}p \to s\mathcal{A}b\mathcal{G}p$ of §3.2.1(I)) to the chain-complex tensor product, so for any $X_\bullet$, $Y_\bullet$

$$\bar{T}_t(\pi_\star X_\bullet, \pi_\star Y_\bullet) = \bigoplus_{i+j=t} \pi_i X_\bullet \otimes \pi_j Y_\bullet$$

(compare [Do, §6]).

4 bisimplicial objects

We now consider the category $ss\mathcal{C}$ of *bisimplicial* objects over a CUGA $\mathcal{C}$. We think of an object $A_{\bullet\bullet} \in ss\mathcal{C}$ as having *internal* and *external* simplicial structures, with corresponding homotopy group objects $\pi^i_t A_{\bullet\bullet}$ and $\pi^e_s A_{\bullet\bullet}$ (each in $s\mathcal{C}$). There are two embeddings $c^e, c^i : s\mathcal{C} \hookrightarrow ss\mathcal{C}$, with $c^e(X_\bullet)_{t,s} = X_t$, and $c^i(X_\bullet)_{t,s} = X_s$. We use the convention that for $A_{t,s}$, t is the internal dimension and s is the external one.

The category $ss\mathcal{C}$ can be given a number of different closed model category structures (e.g. [BF, Thm B.6]). We shall need the one defined by Dwyer, Kan and Stover in [DKS]:

4.1 the model category ssC

One can use the models for $s\mathcal{C}$ to provide $ss\mathcal{C}$ with a closed simplicial model category structure, in which a map $f : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ is a weak equivalence if for each $s, t \geq 0$, $f_\star : \pi_s \pi^i_t X_{\bullet\bullet} \to \pi_s \pi^i_t Y_{\bullet\bullet}$ is an isomorphism. Fibrations and cofibrations are defined as in [DKS, §5].

4.1.1 $\mathcal{M}$-free objects

We say that $A_{\bullet\bullet} \in ss\mathcal{C}$ is $\mathcal{M}$-*free* if for each $m \geq 0$ there are (graded) simplicial sets $X[m]_\bullet \simeq \bigvee_i \mathbf{S}^{n_i}(k_i)_\bullet$ such that $A_{\bullet,m} \cong F(X[m]_\bullet)$, and the external degeneracies of $A_{\bullet\bullet}$ are induced under F by maps $X[m]_\bullet \to X[m+1]_\bullet$ which are, up to homotopy, the inclusion of wedge summands. Such an object is cofibrant, and any $X_\bullet \in s\mathcal{C}$ may be resolved (§2.2.3) by such an $\mathcal{M}$-free $A_{\bullet\bullet}$, as follows:

The construction is that of [St, §2]. Let $C\mathbf{S}^n(k)_\bullet$ be the cone on the (graded) simplicial set $\mathbf{S}^n(k)_\bullet$. We obtain the simplicial algebra $W(X_\bullet) \in s\mathcal{C}$ from

$$\coprod_{n,k \geq 0} \Big(\coprod_{f : F(\mathbf{S}^n(k)_\bullet) \to X_\bullet} F(\mathbf{S}^n(k)_\bullet) \ \amalg \coprod_{H : F(C\mathbf{S}^n(k)_\bullet) \to X_\bullet} F(C\mathbf{S}^n(k)_\bullet) \ \Big) \ ,$$

by identifying the natural sub-object $F(\mathbf{S}^n(k)_\bullet) \stackrel{i}{\hookrightarrow} F(C\mathbf{S}^n(k)_\bullet)$ of the copy of $F(C\mathbf{S}^n(k)_\bullet)$ indexed by H in the second sum with the copy of $F(\mathbf{S}^n(k)_\bullet)$ indexed by $f = H \circ i$ in the first sum.

This defines a cotriple $W : s\mathcal{C} \to s\mathcal{C}$, with the obvious counit ("evaluation") and comultiplication, and $A_{\bullet\bullet}$ is defined to be the simplicial object over $s\mathcal{C}$ induced by this cotriple (cf. [Go, App., §3]). Clearly each $A_{\bullet,n} = W^{n+1}X_\bullet \in \mathcal{M}$; moreover, the natural augmentation $A_{\bullet\bullet} \to c^e(X_\bullet)_\bullet$ is a weak equivalence in $ss\mathcal{C}$ (see [St, Prop. 2.6]).

4.2 Theorem.

Let $S : \mathcal{B} \to \mathcal{A}$ be a functor from a CUGA to any concrete category; then for any $X_\bullet \in s\mathcal{B}$ there is a first quadrant spectral sequence with

$$E^2_{s,t} \cong (L_s\bar{S}_t)\pi_\star X_\bullet \;\Rightarrow\; (L_{s+t}S)X_\bullet\ .$$

Here $\bar{S}_*$ and $L_s\bar{S}_*$ have values in $\mathcal{A}$-Π-Alg (see §3.2.2 and Proposition 3.2.3). Note that if $X_\bullet$ is cofibrant, $(L_nS)X_\bullet = \pi_n(SX_\bullet)$.

Proof: For any bisimplicial set $Y_{\bullet\bullet}$ there is a spectral sequence with

$$E^2_{s,t} \cong \pi_s\pi^i_t Y_{\bullet\bullet} \;\Rightarrow\; \pi_{s+t}\Delta Y_{\bullet\bullet}$$

where the diagonal $\Delta Y_{\bullet\bullet} \in \mathcal{S}_*$ is defined by $(\Delta Y_{\bullet\bullet})_n = Y_{n,n}$, with $d_j = d^e_j d^i_j$, $s_j = s^e_j s^i_j$ (cf. [BF, Thm B.5], [Q3]).

Given $X_\bullet \in s\mathcal{B}$, construct an $\mathcal{M}$-free resolution $B_{\bullet\bullet} \to X_\bullet$ as in §4.1.1. Note that $\pi * B_{\bullet\bullet} \to \pi_\star X_\bullet$ is a free simplicial resolution in the category of $\mathcal{C}$-Π-algebras, by definition of the weak equivalences for $ss\mathcal{B}$. Therefore, by Proposition 4.2 $\pi^i_t SB_{\bullet\bullet} = \bar{S}_t\pi^i_* B_{\bullet\bullet}$ (since each $B_{\bullet,n} \in \mathcal{M}$), and in the spectral sequence for $SB_{\bullet\bullet} \in ss\mathcal{A}$ we have

$$E^2_{s,t} \cong \pi_s\pi^i_t SB_{\bullet\bullet} \cong \pi_s\bar{S}_t\pi^i_* B_{\bullet\bullet} \cong (L_s\bar{S}_t)\pi_\star X_\bullet \qquad (2)$$

Since $B_{\bullet\bullet}$ is weakly equivalent to $c^e(X_\bullet)_\bullet$, by definition 4.1 $\pi_s\pi^i_t B_{\bullet\bullet} \cong \pi_s\pi^i_t c^e(X_\bullet)_\bullet$, so the spectral sequence for $B_{\bullet\bullet}$ collapses and $\pi_\star(\Delta B_{\bullet\bullet}) \cong \pi_\star X_\bullet$. Moreover, by the construction of §4.1.1 $\Delta B_{\bullet\bullet}$ is a free object in $s\mathcal{B}$, so in fact $\Delta B_{\bullet\bullet} \to X_\bullet$ is a resolution in $s\mathcal{C}$, and thus $\Delta SB_{\bullet\bullet} = S\Delta B_{\bullet\bullet} \cong (\mathbf{L}S)X_\bullet$ and (2) converges to $(L_{s+t}S)X_\bullet$. □

4.2.1 examples

(i) If $\mathcal{B} = \mathcal{A} = R$-$Mod$, M is an R-module, and $S = Hom_{R-Mod}(-, M)$ or $S = - \otimes M$, the spectral sequence reduces to the Universal Coefficients short exact sequences for (co)homology.

(ii) For $S = \otimes : R\text{-}Mod \times R\text{-}Mod \to R\text{-}Mod$ (§3.2.4) one similarly obtains the Kunneth short exact sequence for homology.

(iii) If $\mathcal{B} = \mathcal{G}p$, $\mathcal{A} = R\text{-}Mod$, and $S = M \otimes_{\mathbb{Z}} Ab(-)$ for some $M \in \mathcal{A}$, one obtains the Hurewicz spectral sequence of [B1, §2]. This generalizes to the homology functor in any CUGA $\mathcal{C}$ (in particular, the André-Quillen homology of a supplemented algebra over a ground ring k – see [Q2, §1]), by taking $S = Ab : \mathcal{C} \to \mathcal{C}_{ab}$ to be the abelianization functor (cf. §5.1.4).

4.3 Theorem. *Let $\mathcal{Y}$ be an arbitrary model category, and $H : \mathcal{Y} \to s\mathcal{B}$ a functor which preserves cofibrancy and weak equivalences between cofibrants. Let $S : \mathcal{B} \to \mathcal{A}$ be a functor from a CUGA to any concrete category. For every $Y \in \mathcal{Y}$ there is a* Grothendieck spectral sequence *with*

$$E^2_{s,t} \cong (L_s\bar{S}_t)(L_\star H)Y \;\Rightarrow\; (L_{s+t}(S \circ H))Y.$$

Proof: We may assume $Y \in \mathcal{Y}$ is cofibrant, and let $W_\bullet = HY \in s\mathcal{B}$, so $\pi_t W_\bullet = (L_t H)Y$ and the spectral sequence of Theorem 4.2 has $E^2_{s,t} \cong (L_s\bar{S}_t)(L_* H)Y$. For $B_{\bullet\bullet}$ as in the Theorem, we have $\Delta B_{\bullet\bullet} \simeq W_\bullet = HY$, and since H preserves cofibrancy, $\Delta SB_{\bullet\bullet} \cong S\Delta B_{\bullet\bullet} \simeq SW_\bullet \simeq (S \circ H)Y$ so the spectral sequence converges to $(L_{s+t}(S \circ H))Y$. □

It should be pointed out that we have different assumptions on the three categories in question: $\mathcal{Y}$ may be any model category – e.g., $s\mathcal{C}$, for a wide range of allowable $\mathcal{C}$'s (cf. [Q1, II,§4]; $\mathcal{B}$ must actually be a CUGA, as defined in §2.1; while $\mathcal{A}$ may be any concrete category.

If in fact $\mathcal{Y} = s\mathcal{C}$ for some CUGA $\mathcal{C}$, and H is the prolongation of $T : \mathcal{C} \to \mathcal{B}$, then $H = sT$ will preserve cofibrancy if T takes free objects in $\mathcal{C}$ to free objects in $\mathcal{B}$. However, in this case the requirement may be weakened using the following

Definition: An object $B \in \mathcal{B}$ is called *S-acyclic* if $(L_n S)B = 0$ for $n > 0$, and $(L_0 S)B \cong SB$ (with the obvious map).

4.4 Theorem. *Let $\mathcal{C} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{A}$ be covariant functors, $\mathcal{C}$, $\mathcal{B}$ CUGA's, and $\mathcal{A}$ any concrete category. Suppose that TF is S-acyclic for every* free *$F \in \mathcal{C}$. Then for every $C \in \mathcal{C}$ there is a Grothendieck spectral sequence with*

$$E^2_{s,t} \cong (L_s\bar{S}_t)(L_* T)C \;\Rightarrow\; (L_{s+t}(S \circ T))C.$$

Proof: In fact we show that the theorem holds for any $C_\bullet \in s\mathcal{C}$ (rather than just $C_\bullet = c(C)_\bullet$). We may assume $C_\bullet \in s\mathcal{C}$ is free; then it suffices to produce an object $W_\bullet$ and a map $f : W_\bullet \to TC_\bullet$ in $s\mathcal{B}$ such that

(a) $W_\bullet$ is cofibrant,

(b) $f : W_\bullet \to TC_\bullet$ is a weak equivalence,

(c) $Sf : SW_\bullet \to (ST)C_\bullet$ is also a weak equivalence,

since then we can proceed as in the proof of Theorem 4.3. Such a $W_\bullet$ may be obtained by a diagonal construction, as follows.

(I) Recall that if $f : X_\bullet \to Y_\bullet$ is a map of simplicial sets such that for each $l \geq 0$, the inclusion $\overset{\bullet}{\mathbf{\Delta}}[l]_\bullet \overset{i_l}{\hookrightarrow} \mathbf{\Delta}[l]_\bullet$ has the LLP (see §2.2.1) with respect to f, then f is a weak equivalence (in fact, a trivial fibration – cf. [Q1, II,§1]).

(II) Let $\mathbf{\Delta}[m]_\bullet \tilde{\times} \mathbf{\Delta}[l]_\bullet$ denote the bisimplicial set with $(\mathbf{\Delta}[m]_\bullet \tilde{\times} \mathbf{\Delta}[l]_\bullet)_{s,t} = \mathbf{\Delta}[m]_s \times \mathbf{\Delta}[l]_t$, and similarly $(\mathbf{\Delta}[m]_\bullet \tilde{\times} \overset{\bullet}{\mathbf{\Delta}}[l]_\bullet)_{s,t} = \mathbf{\Delta}[m]_s \times \overset{\bullet}{\mathbf{\Delta}}[l]_t$. Assume that a map $f : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ of bisimplicial sets has the property that, for all $m, l \geq 0$, the map $i_{m,l} : \mathbf{\Delta}[m]_\bullet \tilde{\times} \overset{\bullet}{\mathbf{\Delta}}[l]_\bullet \hookrightarrow \mathbf{\Delta}[m]_\bullet \tilde{\times} \mathbf{\Delta}[l]_\bullet$ has the LLP with respect to f. Then $f_{m,\bullet} : X_{m,\bullet} \to Y_{m,\bullet}$ is a weak equivalence (of simplicial sets) for each $m \geq 0$, by (I).

(III) Now for any $Y_{\bullet\bullet} \in ss\mathcal{B}$ we may use the "small object" construction of [Q1, II,§3] to obtain an object $Z_{\bullet\bullet}$ with a map $f : Z_{\bullet\bullet} \to Y_{\bullet\bullet}$ as follows:

Define $Z_{\bullet\bullet}$ to be the direct limit of a sequence $0 = Z^0_{\bullet\bullet} \hookrightarrow \dots Z^{n-1}_{\bullet\bullet} \hookrightarrow Z^n_{\bullet\bullet} \dots$ of cofibrant objects in $ss\mathcal{B}$ with compatible maps $f^n : Z^n_{\bullet\bullet} \to Y_{\bullet\bullet}$, which are defined inductively by the pushout diagrams

$$\begin{array}{ccc} \coprod_{\mathcal{D}} F(\mathbf{\Delta}[m]_\bullet \tilde{\times} \overset{\bullet}{\mathbf{\Delta}}[l]_\bullet) & \xrightarrow{\coprod_{\mathcal{D}} g_d} & Z^{n-1}_{\bullet\bullet} \\ {\scriptstyle \coprod_{\mathcal{D}} F(i_{m,l})} \downarrow & \boxed{\text{PO}} & \downarrow \\ \coprod_{\mathcal{D}} F(\mathbf{\Delta}[m]_\bullet \tilde{\times} \mathbf{\Delta}[l]_\bullet) & \longrightarrow & Z^n_{\bullet\bullet} \end{array}$$

where the coproducts are taken over the set $\mathcal{D} = \mathcal{D}_n$ of all commutative diagrams of the form:

$$\begin{array}{ccc} F(\mathbf{\Delta}[m]_\bullet \tilde{\times} \overset{\bullet}{\mathbf{\Delta}}[l]_\bullet) & \xrightarrow{g_d} & Z^{n-1}_{\bullet\bullet} \\ {\scriptstyle F(i_{m,l})} \downarrow & & \downarrow {\scriptstyle f^{n-1}} \\ F(\mathbf{\Delta}[m]_\bullet \tilde{\times} \mathbf{\Delta}[l]_\bullet) & \xrightarrow{h_d} & Y_{\bullet\bullet} \end{array} \qquad \text{(d)}$$

for all m, l (in all degrees k, in the graded case).

Using the adjointness of F and U (§2.1.2), one sees that each $i_{m,l}$, for all m, l (and k), has the LLP with respect to the underlying map of $f : Z_{\bullet\bullet} \to Y_{\bullet\bullet}$, so that $Z_{m,\bullet} \overset{w.e.}{\simeq} Y_{m,\bullet}$ for each $m \geq 0$, by (II).

Applying the construction of (III) to $Y_{\bullet\bullet} = c^e(TC_\bullet)_\bullet \in s\mathcal{B}$, we obtain such a $Z_{\bullet\bullet}$. We claim that the diagonal $W_\bullet \overset{Def}{=} \Delta Z_{\bullet\bullet}$ has properties (a)-(c) above:

First, note that $W_\bullet = \Delta Z_{\bullet\bullet}$ is free (and thus cofibrant) in $s\mathcal{B}$, by construction – so it satisfies (a). Next, we have $Z_{m,\bullet} \overset{w.e.}{\simeq} Y_{m,\bullet} = c(TC_m)_\bullet$ for each $m \geq 0$, and thus

$$\pi_s \pi_t^e Z_{\bullet\bullet} \cong \pi_s \pi_t^e Y_{\bullet\bullet} \quad = \begin{cases} \pi_s TC_\bullet & \text{if } t = 0 \\ 0 & \text{if } \ t > 0 \end{cases} ,$$

so the Quillen spectral sequence (cf. [Q3]) shows $W_\bullet = \Delta Z_{\bullet\bullet} \overset{w.e.}{\simeq} \Delta Y_{\bullet\bullet} = TC_\bullet$ – i.e., (b) holds.

Moreover, each $Z_{m,\bullet} \in s\mathcal{B}$ is also free, by construction, so it is a resolution of TC_m, and thus (§2.2.4) $\pi_s S Z_{m,\bullet} = (L_s S)(TC_m)$ ($= 0$, for $s > 0$ by the assumption of S-acyclicity). Thus the Quillen spectral sequence for $SZ_{\bullet\bullet} \in ss\mathcal{A}$ shows that $SW_\bullet = S\Delta Z_{\bullet\bullet} \cong \Delta S Z_{\bullet\bullet} \overset{w.e.}{\simeq} STC_\bullet$ – so (c) holds, too. Therefore, $W_\bullet \to TC_\bullet$ has the required properties, which completes the proof of the Theorem. □

4.4.1 remark

When B and $\mathcal{A}$ are abelian and S is additive, then (§3.2.4) $\bar{S}_*$ may be identified with S (applied dimensionwise), and one has the classical Grothendieck spectral sequence of [Gr, Thm. 2.4.1].

Note however that while $\mathcal{C}$ can be any abelian category with enough projectives (in which case we require that TP be S-acyclic for any projective $P \in \mathcal{C}$), $\mathcal{B}$ must actually be a CUGA, since the construction of §4.1.1 depends on the existence of free objects. Of course, if $\mathcal{B}$ is any *abelian* category with enough projectives, the standard construction of the resolution of $TC_\bullet$ in $cc\mathcal{B} \simeq ss\mathcal{B}$ (cf. [HS, VIII, §9]) can be used instead.

5 appendix: homotopy group objects for $s\mathcal{C}$

To illustrate some of the structure of $\mathcal{C}$-Π-algebras, we here show that the homotopy groups of a simplicial object $X_\bullet \in s\mathcal{C}$ actually take value in $\mathcal{C}$, and are abelian in dimensions≥ 1.

5.1 abelian objects in $\mathcal{C}$

In the case of CUGA's, abelian objects have a convenient explicit description:

5.1.1 operations

For any CUGA $\mathcal{C}$, we fix a subset Ω of the semi-groupoid of all operations W, not including the group operation of the G_k's, which (together with these group operations) generates W. Let Ω' denote the subset of W consisting of the n-ary operations $\omega(x,y,\ldots) \in \Omega$ with $n \geq 2$, together with the "operation commutators" $[\omega,i](x_1,\ldots x_{i-1},x,y,x_{i+1},\ldots,x_n) \stackrel{Def}{=}$

$$\omega(x_1,...,x_{i-1},x,x_{i+1},...,x_n)\omega(x_1,...,x_{i-1},y,x_{i+1},...,x_n)\omega(x_1,...,x_{i-1},xy,x_{i+1},...,x_n)^{-1}$$

for all $\omega \in \Omega$, $1 \leq i \leq n$.

(For $\omega(x) = x^{-1}$, the inverse operation of the underlying group structure, this is just the group commutator: $[\omega,1](x,y) = x^{-1}y^{-1}xy$).

We shall assume that the operations in Ω are *pointed* in the sense that they vanish if at least one operand is 0. This is then true of Ω', too.

5.1.2 ideals

In this situation we call a sub-algebra $X \subseteq Y$ an *ideal* of Y if for each $\theta \in \Omega'$ and $y_1,\ldots,y_n \in Y$, we have $\theta(y_1,\ldots,y_n) \in X$ if $y_j \in X$ for some $1 \leq j \leq n$. Thus in particular, $G_kX \lhd G_kY$ for every $k \geq 0$, and the quotient graded group Y/X is an object of $\mathcal{C}$. For example, $Ker(f)$ is an ideal of X for any $f: X \to Y$.

5.1.3 abelian objects

An object A in a CUGA $\mathcal{C}$ is called *abelian* if $Hom_{\mathcal{C}}(X,A)$ has a *natural* abelian group structure for any $X \in \mathcal{C}$. We denote by $\mathcal{C}_{ab}$ the full subcategory of abelian objects in $\mathcal{C}$. Equivalently, A is abelian iff there are in $\mathcal{C}$ "abelian group object structure maps": $\mu: A \times A \to A$ (*group operation*), $\nu: A \to A$ (*inverse*), and $\eta: \star \to A$ (*identity*), fitting into the obvious commutative diagrams.

Since $\mathcal{C}$ is a CUGA, *any* $A \in \mathcal{C}$ has an underlying group structure, which takes the form of "group structure maps" on the underlying (graded) set of A, viz.: $\bar{\mu}: A \times A \to A$, $\bar{\nu}: A \to A$, and $\bar{\eta}: \star \to A$ − again fitting into suitable commutative diagrams.

It is straightforward to verify that if A has "abelian group object structure maps" μ, ν, and η as above, then $\bar{\mu}$, $\bar{\nu}$, and $\bar{\eta}$ must equal μ, ν, and η respectively, as maps of graded sets. Conversely, if the given group structure maps $\bar{\mu}$, $\bar{\nu}$, and $\bar{\eta}$ can be lifted to $\mathcal{C}$ (i.e., are in the image of the faithful functor U of §2.1.2), then the liftings will be "abelian group object structure maps".

It is also readily verified that if the operations Ω' act trivially on an object A in $\mathcal{C}$, then $\bar{\mu}$, $\bar{\nu}$, and $\bar{\eta}$ in fact lift to $\mathcal{C}$, so A is abelian – and conversely, Ω' must act trivially on any abelian $A \in \mathcal{C}$.

5.1.4 abelianization

For any $X \in \mathcal{C}$, let $P(X) \subset X$ be the sub-algebra generated by the image of X under Ω'. This an ideal of X, and the graded group $Ab(X) = X/P(X)$ thus lies in $\mathcal{C}$.

Note that $Ab(X)$ lies in $\mathcal{C}_{ab}$, since Ω' acts trivially on it. In fact, $Ab(X)$ is the *abelianization* of X – i.e., any map from X into $B \in \mathcal{C}_{ab}$ factors uniquely through the natural map $X \to Ab(X)$.

5.1.5 examples

In most of the examples of §2.1.1, $\mathcal{C}_{ab}$ is just a category of graded R-modules, for suitable rings R, with other operations trivial.

When $\mathcal{C} = \mathcal{K}_p$, then $\mathcal{C}_{ab}$ is a suitable category of unstable modules over the mod-p Steenrod algebra $\mathcal{A}_p$, with trivial multiplication: For $p = 2$, $(\mathcal{K}_2)_{ab} = \Sigma\mathcal{U}$ = the category of $\mathcal{A}_2$-modules with $Sq^i x = 0$ for $|x| \leq i$. For $p > 2$, $(\mathcal{K}_p)_{ab} = \mathcal{V}$ = the category of $\mathcal{A}_p$-modules with $\mathcal{P}^i x = 0$ for $|x| \leq 2i$ (cf. [Mi, §1]).

Abelian Π-algebras, and the category $(\Pi\text{-}Alg)_{ab}$, are discussed in [B3].

5.2 homotopy group objects over $\mathcal{C}$

A simplicial object $X_\bullet$ over any concrete category $\mathcal{C}$ has *homotopy groups* $\pi_\star X_\bullet$ – namely, those of the underlying simplicial set. In our case, since $X_\bullet$ has the underlying structure of a simplical group, these are defined to be the homology of the (not necessarily abelian) chain complex $\{N_\star X_\bullet, \partial\}$ (cf. [May, §17]), where

$$N_n X_\bullet = \bigcap_{1 \leq j \leq n} ker\{d_j : X_n \to X_{n-1}\} \subset X_n, \text{ and } \partial_n = d_0|_{N_n X_\bullet} \text{ for each } n \geq 0.$$

5.2.1 Lemma. *For any CUGA $\mathcal{C}$ and $X_\bullet \in s\mathcal{C}$, $\pi_k X_\bullet \in \mathcal{C}_{ab} \subset \mathcal{C}$ for all $k \geq 1$, and $\pi_0 X_\bullet \in \mathcal{C}$.*

Proof: The image of $d_0 : N_{k+1}X_\bullet \to N_k X_\bullet$ is an ideal (§5.1.2) in $Z_k X_\bullet = Ker(\partial_n)$, since if $x = d_0 x'$ for some $x' \in N_{k+1}X_\bullet$, then for any $\omega \in \Omega$ and $x, y, \ldots \in Z_k X_\bullet$ we have $\omega(x, y, \ldots) = d_0(\omega(x', s_0 y, s_0 \ldots))$ and

$d_i(\omega(x', s_0 y, s_0 \ldots)) = 0$ for $i \geq 1$ (because $\mathcal{C}$ is pointed). Thus $\pi_k X_\bullet \in \mathcal{C}$ for all $k \geq 0$.

Now let $k \geq 1$: for any operation $\omega \in \Omega'$ and $x, y, \ldots \in Z_k X_\bullet$, let $c = \omega(s_0 x, s_0 y, s_0 \ldots) \cdot \omega(s_0 x, s_1 y, s_0 \ldots)^{-1}$; then $d_0 c = \omega(x, y, \ldots)$, and $d_j c = 0$ for $j \geq 1$. Thus any element of $P(Z_k X_\bullet)$ is a boundary in $N_k X_\bullet$, so $\pi_k X_\bullet = Ab(\pi_k X_\bullet)$ is an abelian object in $\mathcal{C}$ (§5.1.3). □

5.2.2 Corollary. *For any functor $T : \mathcal{C}' \to \mathcal{C}$, the 0-th derived functor of T takes values in $\mathcal{C}$, and higher derived functors take values in $\mathcal{C}_{ab}$.*

References

[B1] D. Blanc, "A Hurewicz spectral sequence for homology", *Trans. AMS* **318** (1990) No. 1, pp. 335-354.

[B2] D. Blanc, "Derived functors of graded algebras", *J. Pure Appl. Alg* **64** (1990) No. 3, pp. 239-262.

[B3] D. Blanc, "Abelian Π-algebras and their projective dimension", *Preprint* 1991.

[Bo] A.K. Bousfield, "Operations on Derived Functors of Non-additive Functors", (Brandeis preprint, 1967).

[6A] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, & J.W. Schlesinger, "The mod-p lower central series and the Adams spectral sequence", *Topology* **5** (1966), pp. 331-342.

[BF] A.K. Bousfield & E.M. Friedlander, "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", in *Geometric Applications of Homotopy Theory, II*, ed. M.G. Barratt & M.E. Mahowald, Springer-Verlag *Lec. Notes Math.* **658**, Berlin-New York 1978, pp. 80-130.

[Do] A. Dold, "Homology of symmetric products and other functors of complexes", *Ann. of Math.* **68** (1958) No. 1, pp. 54-80.

[Dw] W.G. Dwyer, "Homotopy operations for simplicial commutative algebras", *Trans. AMS* **260** (1980) No. 2, pp. 421-435.

[DKS] W.G. Dwyer, D.M. Kan, & C.R. Stover, "An E^2 model category structure for pointed simplicial spaces", *J. Pure & Appl. Alg.*, (to appear).

[Go] R. Godement, *Topologie algébrique et théorie des faisceaux*, Act. Sci. & Ind. No. 1252, Publ. Inst. Math. Univ. Strasbourg **XIII**, Hermann, Paris 1964.

[Gr] A. Grothendieck, "Sur quelques points d'algèbre homologique", *Tôhoku Math. J.* (Ser. 2) **9** (1957) Nos. 2-3, pp. 119-221.

[HS] P.J. Hilton & U. Stammbach, *A Course in Homological Algebra* Springer-Verlag, Berlin-New York 1971.

[Kan] D.M. Kan, "On homotopy theory and c.s.s. groups", *Ann. of Math.*(2) **68** (1958), pp. 38-53.

[McL] S. Mac Lane, *Categories for the working mathematician*, Grad. Texts in Math. No. 5, Springer-Verlag, Berlin-New York 1971.

[May] J.P. May, *Simplicial Objects in Algebraic Topology*, Univ. of Chicago Press, Chicago-London 1967.

[Mi] H. Miller, "Correction to 'The Sullivan conjecture on maps from classifying spaces' " *Ann. of Math.* **121** (1985), pp. 605-609.

[Q1] D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* 20, Berlin-New York 1963.

[Q2] D.G. Quillen, "On the (co-)homology of commutative rings", in: *Applications of categorical algebra*, Proc. Symp. Pure Math. **XVII** American Mathematical Society, Providence, RI 1970 pp. 65-87.

[Q3] D.G. Quillen, "Spectral sequences of a double semi-simplicial group", *Topology* **5** (1966), pp. 155-156.

[Sch] J.W. Schlesinger, "The semi-simplicial free Lie ring", *Trans. AMS* **122** (1966) No. 2, pp. 436-442.

[St] C.R. Stover, "A Van Kampen spectral sequence for higher homotopy groups", *Topology*, **29** (1990) No. 1, pp. 9-26.

[W] G.W. Whitehead, *Elements of homotopy theory*, Grad. Texts in Math. No. 61, Springer-Verlag, Berlin-New York 1971.