The maximum number of tangencies among convex regions with a triangle-free intersection graph

Eyal Ackerman*

Abstract

Let $t(\mathcal{C})$ be the number of tangent pairs among a set $\mathcal{C}$ of n Jordan regions in the plane. Pach, Suk, and Treml [6] showed that if $\mathcal{C}$ consists of convex bodies and its *intersection graph* is bipartite then $t(\mathcal{C}) \leq 4n - \Theta(1)$, and, moreover, there are such sets that admit at least $3n - \Theta(\sqrt{n})$ tangencies. We close this gap and generalize their result by proving that the correct bound is $3n - \Theta(1)$, even when the intersection graph of $\mathcal{C}$ is only assumed to be triangle-free.

1 Introduction

Erdős's famous *unit distance* problem [3, 5] asks for the maximum number of pairs of points that are at unit distance from each other among n distinct points in the plane. This is equivalent to asking for the maximum number of *tangency points* among n distinct unit disks in the plane.

Let $\mathcal{C}$ be a family of Jordan regions in the plane. We say that two regions are *tangent* if they intersect at a single point, and denote by $t(\mathcal{C})$ the number of tangent pairs in $\mathcal{C}$. It is easy to see that n (convex) regions might determine $\Theta\left(n^2\right)$ tangency points: e.g., there are n^2 tangency points (*tangencies*) between the $2n$ regions consisting of the n sides of a convex n-gon and a set of n convex n-gons each of which has a vertex on each of the sides of the first polygon. However, more restricted families of regions might determine less tangencies. One example are unit disks (Erdős's unit distance problem): they are are known to admit at most $O\left(n^{4/3}\right)$ tangencies [7], and it is conjectured that the correct bound is Erdős's lower bound of $\Omega\left(ne^{\frac{c \log n}{\log \log n}}\right)$ [5] (see also [3] for the history of this problem and additional references). Another example are regions such that the boundary curves of every pair of them intersect exactly once or twice and no three boundary curves intersect at a point. Such regions admit only $O(n)$ tangencies [1, 2].

For a set of regions $\mathcal{C}$ denote by $\mathcal{I}(\mathcal{C})$ the *intersection graph* of $\mathcal{C}$. That is, the graph whose vertex set is $\mathcal{C}$ and whose edge set consists of all the intersecting pairs. Ben-Dan and Pinchasi [4] observed that if $\mathcal{I}(\mathcal{C})$ is bipartite then $t(\mathcal{C}) = O\left(n^{3/2} \log n\right)$ and suggested that the correct bound is $O(n)$. Pach, Suk, and Treml [6] proved this conjecture for the case of convex regions, and found almost matching lower and upper bounds for the maximum number of tangencies.

Theorem 1 ([6]). *Let $\mathcal{C}$ be a set of n convex bodies in the plane. If $\mathcal{I}(\mathcal{C})$ is bipartite, then $t(\mathcal{C}) \leq 4n - \Theta(1)$. Moreover, for every n there is set $\mathcal{C}$ of n convex bodies in the plane such that $\mathcal{I}(\mathcal{C})$ is bipartite and $t(\mathcal{C}) \geq 3n - \Theta(\sqrt{n})$.*

They also suggested the following conjecture.

*Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. ackerman@sci.haifa.ac.il

Conjecture 2 ([6])**.** *For every fixed integer $k > 2$, if $\mathcal{C}$ is a set of n convex bodies in the plane such that $\mathcal{I}(\mathcal{C})$ is K_k-free, then $t(\mathcal{C}) \le c_k n$, for some constant c_k that depends only on k.*

In this note we close the gap in Theorem 1 and prove Conjecture 2 for $k = 3$.

Theorem 3. *Let $\mathcal{C}$ be a set of n convex bodies in the plane. If $\mathcal{I}(\mathcal{C})$ is triangle-free, then $t(\mathcal{C}) \le 3n - \Theta(1)$. Moreover, for every n there is set $\mathcal{C}$ of n convex bodies in the plane such that $\mathcal{I}(\mathcal{C})$ is bipartite and $t(\mathcal{C}) \ge 3n - \Theta(1)$.*

2 Proof of Theorem 3

Most of the proof is devoted to the upper bound. For the lower bound see Proposition 2.13.

Let $\mathcal{C}$ be a set of $n \ge 4$ convex bodies in the plane. We prove by induction on n that if $\mathcal{I}(\mathcal{C})$ is triangle-free, then $t(\mathcal{C}) \le 3n - 6$. The first steps of the proof follow the proof of [6, Theorem 7]. Since $t(\mathcal{C}) \le \binom{n}{2}$, for $n = 4$ there are at most $\binom{4}{2} = 6 \le 3n - 6$ tangencies. Suppose now that $n \ge 5$ and that the theorem holds for every $\mathcal{C}'$ as above, with $4 \le |\mathcal{C}'| < n$. Let $\mathcal{C}$ be a set of n convex bodies in the plane, such that $\mathcal{I}(\mathcal{C})$ is triangle-free, that is, $\mathcal{C}$ does not contain three pairwise intersecting bodies. We may assume that every body in $\mathcal{C}$ is tangent to at least 4 other bodies, for otherwise we can conclude by induction. We begin by replacing every convex body $C \in \mathcal{C}$ by a convex polygon whose vertices are the tangency points along the boundary of C. Henceforth, $\mathcal{C}$ denotes the set of convex polygons.

Proposition 2.1. *There are no two polygons $P, Q \in \mathcal{C}$ such that a vertex of P is inside Q.*

Proof. Suppose there is such a vertex v. Then P touches another polygon $R \ne P, Q$ at v. Therefore, P, Q, R are pairwise intersecting. □

Denote by T the set of tangency points, and let $m = |T|$. Next, we define a planar graph $G = (V, E)$ by placing a vertex at every point in T and at every intersection point between sides of two polygons. Note that this graph is 4-regular. Denote by F the set of faces of G, and by $|f|$ the size of a face $f \in F$. A *k-face* is a face of size k. We write $f \subseteq P$ when a face f is contained in a polygon P. Note that each face $f \in F$ is contained in exactly 0, 1, or 2 polygons, since $\mathcal{I}(\mathcal{C})$ is triangle-free. Denote by F_i the set of faces that are contained in exactly i polygons, for $i = 0, 1, 2$. We proceed with a few observations on G, most of them already appear in [6].

Proposition 2.2. *Let v be a vertex of a face $f \in F_1$. If $v \notin T$ then one of its neighbors in f is also a crossing point.*

Proof. Suppose that $v \notin T$ is a vertex of $f \in F_1$ and let $P \in \mathcal{C}$ be the polygon that contains f. Then v is the intersection point of a side of P and a side of another polygon Q. This side must intersect P at another point u, since by Proposition 2.1, P does not contain a vertex of Q. The segment vu cannot be crossed by a side of another polygon, since there are no three pairwise intersecting polygons. Therefore, uv is an edge of f, and $u \notin T$ is a neighbor of v in f. □

Proposition 2.3. *If $f \in F_0 \cup F_1$ is a 3-face, then $f \in F_1$ and f has exactly one vertex from T.*

Proof. The three edges of f must be contained in sides of two polygons. Indeed, if all of them are contained in sides of one polygon, then this polygon is a triangle, however, we assumed that any polygon has at least 4 vertices. Otherwise, if each edge of f is contained in a side of a different polygon, then we have three pairwise intersecting polygons.

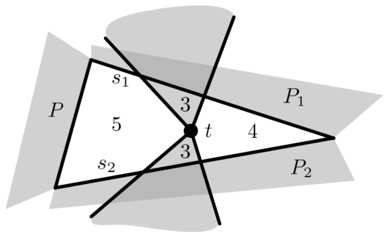


Figure 1: $F(t) = (3, 3, 4, 5)$ yields three pairwise intersecting polygons.

Therefore, f has two edges that are contained in sides of the same polygon, hence they intersect at vertex of this polygon and $f \in F_1$. The third edge of f must belong to a side of another polygon, thus, the remaining vertices of f are crossing points. □

Every tangency point $t \in T$ is adjacent to two faces from F_1 and to two faces from F_0. Define $F(t) \stackrel{\text{def}}{=} (|f_1|, |f_2|, |f_3|, |f_4|)$, where the faces f_i, $1 \le i \le 4$, are the four faces adjacent to t, such that $f_1, f_2 \in F_1$, $|f_1| \le |f_2|$, and $f_3, f_4 \in F_0$, $|f_3| \le |f_4|$. We may assume that all the faces adjacent to a tangency point are distinct, for otherwise G has a cut vertex and we can conclude by induction. Call a vertex *bad* if it is adjacent to at least one 3-face, and *double* bad if it is adjacent to two 3-faces. The next observation follows from Proposition 2.3.

Observation 2.4. *If $t \in T$ is a bad vertex, then in each of the two faces from F_0 that are adjacent to t, at least one neighbor of t is a crossing point. If t is double bad, then all of its neighbors are crossing points.*

Proposition 2.5. *If $t \in T$ is a double bad vertex, then $F(t) \in \{(3, 3, 4, \ge 6), (3, 3, \ge 5, \ge 5)\}$.*

Proof. Let $F(t) = (|f_1|, |f_2|, |f_3|, |f_4|)$, such that $|f_1| = |f_2| = 3$. It follows from Proposition 2.3 that $|f_3| \ge 4$. Since $|f_3| \le |f_4|$ then clearly if $|f_3| \ge 5$ then $|f_4| \ge 5$. Suppose that $|f_3| = 4$. For $i = 1, 2$, let e_i be the opposite edge to t in f_i, and let s_i be the side of the polygon P_i that contains e_i. Since $|f_3| = 4$, the vertex opposite to t in f_3 must be an intersection point of s_1 and s_2. Clearly s_1 and s_2 intersect once, so $|f_4| \ge 5$. However, if $|f_4| = 5$ then there must be a side of a polygon $P \ne P_1, P_2$ that intersects s_1 and s_2, hence we have three pairwise intersecting polygons (see Figure 1). Therefore, $|f_4| \ge 6$ in this case. □

Proposition 2.6. *Any face $f \in F_1$ has at most $|f| - 2$ vertices $t \in T$ on its boundary such that $F(t) = (3, |f|, 4, 4)$.*

Proof. Let f be a face in F_1, and suppose there are two neighboring vertices on its boundary u, v such that $F(u) = F(v) = (3, |f|, 4, 4)$. Then there is one polygon side that supports the 3-faces adjacent to u or v, and the 4-faces from F_0 that are adjacent to u or v (see Figure 2). Therefore, if f has $|f|$ $(3, |f|, 4, 4)$-vertices on its boundary, then there is a polygon side that surrounds f. If f has exactly $|f| - 1$ such vertices, then the remaining vertex must be a concave vertex of a polygon (see Figure 2). □

We proceed by assigning every face $f \in F$ a *weight* $w(f) = |f| - 4$. Let W be the total weight we assigned, and observe that by Euler's polyhedral formula we have:

$$W = \sum_{f \in F} (|f| - 4) = \sum_{f \in F} (|f| - 4) + \sum_{v \in V} (\deg(v) - 4) = 4(|E| - |F| - |V|) = -8. \quad (1)$$

Proposition 2.7. *For every polygon $P \in \mathcal{C}$ it holds that $w(P) \stackrel{\text{def}}{=} \sum_{f \subseteq P} w(f) = |P| - 4$.*

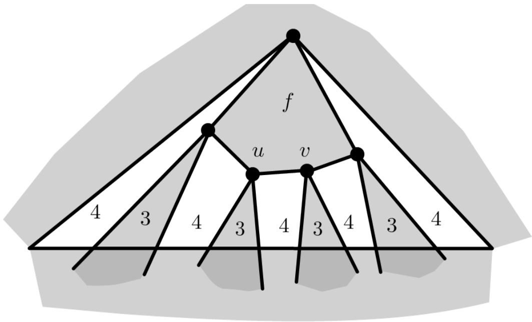


Figure 2: If f has $|f|-1$ $(3,|f|,4,4)$-vertices on its boundary, then the remaining vertex is concave.

Proof. Since there are no three pairwise intersecting polygons, the interior of P is divided into faces by disjoint segments connecting pairs of interior points on the sides of P. Assume that we add these segments one by one, while keeping track of $w(P)$. Every new segment we add increases the number of faces by one (thus, contributing -4 to $w(P)$), and increases by 4 the number of sides of faces in P. Therefore, $w(P)$ maintains its initial value, which is $|P|-4$. □

Every tangency point is a vertex of exactly two polygons. Therefore,

$$\sum_{P\in\mathcal{C}}(|P|-4) = 2m-4n. \tag{2}$$

Combining (1) and (2) we get:

$$\begin{aligned} -8 &= W = \sum_{f\in F_1} w(f) + \sum_{f\in F_2} w(f) + \sum_{f\in F_0} w(f) \\ &= \frac{1}{2}\sum_{P\in\mathcal{C}} w(P) + \frac{1}{2}\sum_{f\in F_1} w(f) + \sum_{f\in F_0} w(f) \\ &= m-2n+\frac{1}{2}\sum_{f\in F_1} w(f) + \sum_{f\in F_0} w(f). \end{aligned} \tag{3}$$

Pach et al. [6] showed that if $\mathcal{I}(\mathcal{C})$ is bipartite then

$$\frac{1}{2}\sum_{f\in F_1} w(f) + \sum_{f\in F_0} w(f) \geq -m/2, \tag{4}$$

which, when plugged into (3), gives $m \leq 4n-16$. We use the *discharging method* to refine their analysis and replace the right hand side of (4) by $-m/3$, and obtain $m \leq 3n-12 \leq 3n-6$. Namely, we prove:

Lemma 2.8. $\frac{2}{3}m + \sum_{f\in F_1}(|f|-4) + 2\sum_{f\in F_0}(|f|-4) \geq 0$.

Proof. We assign initial *charges* as follows.

- For every face $f \in F_1$ we set $ch_0(f) = w(f) = |f|-4$.
- For every face $f \in F_0$ we set $ch_0(f) = 2w(f) = 2|f|-8$.
- For every tangency point $t \in T$ we set $ch_0(t) = \frac{2}{3}$.

It remains to show that the total initial charge is non-negative. We do that by redistributing the charges (*discharging*) in several rounds, such that the total charge remains the same, and eventually, every element has a non-negative final charge. We denote by $ch_i(x)$ the charge of an element x after the ith discharging round. Note that the only elements with a negative initial charge are 3-faces, whose charge is -1. A vertex is *good* if it has a positive charge.

Round 1: A face $f \in F_0$ such that $|f| \in \{6,7\}$ sends $\frac{4}{3}$ units of charge (henceforth, "units") to each double bad vertex on its boundary, and distributes the rest of its charge evenly to every other tangency point on its boundary. Any other face $f' \in F_0 \cup F_1$ sends $ch_0(f')/k$ units to each of the k tangency points on its boundary.

Proposition 2.9. *Let $f \in F_0$ be a face and let t be a tangency point on its boundary. Then the following holds in Round 1:*

(i) if t is double bad and $|f| = 5$, then f sends at least $\frac{2}{3}$ units to t;

(ii) if t is double bad and $|f| > 5$, then f sends at least $\frac{4}{3}$ units to t;

(iii) if $|f| = 5$, then f sends at least $\frac{2}{5}$ units to t, and at least $\frac{2}{4}$ units if t is also adjacent to a 3-face;

(iv) if $|f| \geq 6$, then f sends at least $\frac{2}{3}$ units to t.

Proof. The claims follow from the definition of Round 1, and from Observation 2.4. □

Proposition 2.10. *After Round 1 the following holds:*

(i) every face in $F_0 \cup F_1$ has a non-negative charge;

(ii) for every vertex $t \in T$ if $ch_1(t) < 0$, then $ch_1(t) = -\frac{1}{3}$ or $ch_1(t) = -\frac{2}{15}$. In the first case $F(t) = (3,4,4,4)$, while in the second case $F(t) = (3,5,4,4)$.

Proof. Observe that by Proposition 2.3 every 3-face has one vertex from T to which it sends its negative charge and ends up with charge zero. Every face f has at most $\lfloor |f|/2 \rfloor$ double bad vertices on its boundary, therefore 6- and 7-faces from F_0 remain with a non-negative charge. Any other face clearly remains with a non-negative charge, therefore (i) holds.

For the second claim, note that if $ch_1(t) < 0$ then t must be a bad vertex. If t is double bad, then it follows from Proposition 2.5 that $F(t) = (3,3,4,\geq 6)$ or $F(t) = (3,3,\geq 5,\geq 5)$. By Proposition 2.9, in the first case $ch_1(t) \geq \frac{2}{3} + 2 \cdot (-1) + \frac{4}{3} \geq 0$, while in the second case $ch_1(t) \geq \frac{2}{3} + 2 \cdot (-1) + 2 \cdot \frac{2}{3} \geq 0$.

Finally, suppose that t is adjacent to exactly one 3-face. If t is adjacent to a face f such that f is a 6-face or $|f| = 5$ and $f \in F_0$, then t receives from f at least $\frac{1}{3}$ units and ends up with a non-negative charge. The other cases are listed (note that by Proposition 2.2, a 5-face in F_1 sends either $\frac{1}{5}$ or at least $\frac{1}{3}$ units to each tangency point on its boundary). □

Round 2: Let $t \in T$ be a tangency point with $ch_1(t) < 0$. Then t is adjacent to a 3-face and therefore has at most two good neighbors. If t has exactly one good neighbor or $ch_1(t) > -\frac{2}{3}$, then t asks for $-ch_1(t)$ units from one of its good neighbors. If t has two good neighbors and $ch_1(t) = -\frac{2}{3}$, then t asks for $\frac{1}{6}$ units from each of its good neighbors. If a good vertex $q \in T$ is being asked for ε units by a vertex t, then q accepts t's request and send it ε units if and only if $\varepsilon \leq ch_1(q)/j$, where j is the number of requests q got.

Round 3: Repeat Round 2 with respect to $ch_2(\cdot)$.

Round 4: Suppose that $F(t) = (3,5,4,4)$ and let f be the 5-face that is adjacent to t. If all the vertices of f are from T and $ch_3(t) = -\frac{2}{15}$, then t asks for $\frac{1}{15}$ units from each of the

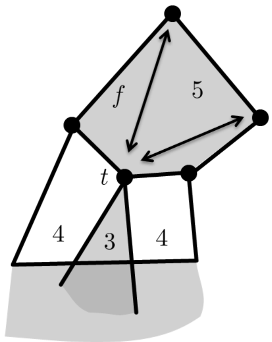


Figure 3: Round 4: if $F(t) = (3, 5, 4, 4)$ and $ch_3(t) = -\frac{2}{15}$, then t ask for $\frac{1}{15}$ unit from each of its non-neighbors in f.

two vertices of f that are not its neighbors (see Figure 2). They accept t's requests if they can afford it (like in Round 2).

Clearly, for any $t \in T$ and $i \geq 1$, if $ch_i(t) \geq 0$ then $ch_{i+1}(t) \geq 0$. Thus, by Proposition 2.10 it remains to verify that $ch_4(t) \geq 0$, for $t \in T$ such that $F(t) \in \{(3, 4, 4, 4), (3, 5, 4, 4)\}$.

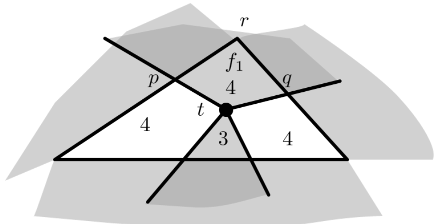
Proposition 2.11. *If* $F(t) = (3, 4, 4, 4)$ *then* $ch_3(t) \geq 0$.

Proof. Let f_1 be the 4-face from F_1 that is adjacent to t, let p and q be the vertices adjacent to t in f_1, and let r be the opposite vertex to t in f_1. If neither p nor q are in T, then the local neighborhood of t looks like Figure 4(a). If r is a crossing point, then we have three pairwise intersecting polygons. Otherwise, if $r \in T$, then it is a concave vertex of a polygon. Therefore, we may proceed by considering the case in which $p, q \in T$ and the case that exactly one of p and q is in T. In the latter case we assume, w.l.o.g., that $p \in T$ and $q \notin T$.

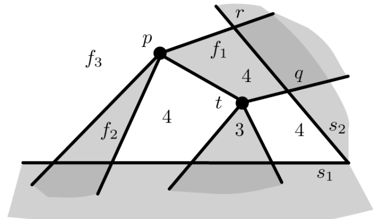
Case 1: $p \in T$ and $q \notin T$. Since p is the only good neighbor of t, t asks $\frac{1}{3}$ units from p in Round 2 (and again in the next round if its first request is denied). Note that in this case $r \notin T$ by Proposition 2.2, and therefore t is the only bad neighbor of p. Let f_2 be the other face from F_1 that is adjacent to p. We consider four subcases, according to whether $|f_2| = 3, 4, 5$ or $|f_2| \geq 6$.

Subcase 1.a: $|f_2| = 3$. Refer to Figure 4(b) and consider $|f_3|$. If $|f_3| = 4$ then the segments s_1 and s_2 must intersect twice. If $|f_3| = 5$ then there are three pairwise intersecting polygons. Therefore $|f_3| \geq 6$ and (by Proposition 2.9) it sends at least $\frac{2}{3}$ units to p in Round 1. Thus, $ch_1(p) \geq \frac{2}{3} - 1 + \frac{2}{3} = \frac{1}{3}$. Since t is the only bad neighbor of p, p accepts t's request in Round 2, and $ch_2(t) = 0$.

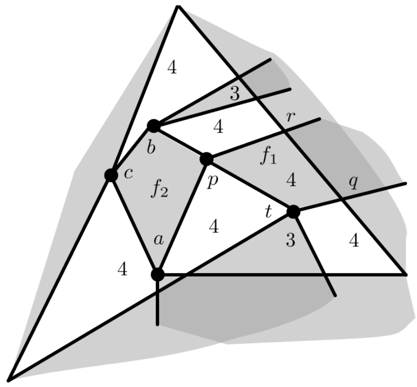
Subcase 1.b: $|f_2| = 4$. Since $F(p) = (4, 4, 4, \geq 4)$, we have $ch_1(p) \geq \frac{2}{3}$. If p has at most one other neighbor but t that asks for charge in Round 2, then p can accept t's request in this round and $ch_2(t) = 0$. Otherwise, p has exactly three neighbors with a negative charge after Round 1, refer to Figure 4(c). Since both a and b are adjacent to f_2, they must each be adjacent to a 3-face, and to two 4-faces from F_0. That is, $F(a) = F(b) = (3, 4, 4, 4)$. Observe that $F(c) \neq (3, 4, 4, 4)$, by Proposition 2.6. Therefore, $ch_1(c) \geq \frac{2}{3}$, and in Round 2 each of a and b asks (and receives) only $\frac{1}{6}$ units from each of p and c. Thus, $ch_2(a), ch_2(b) = 0$ and in the next round p can accept t's request.



(a) if $p, q \notin T$ then r is a concave vertex or there are three pairwise intersecting polygons.



(b) Subcase 1.a. If $|f_3| = 4$ then s_1 and s_2 intersect twice. If $|f_3| = 5$ then there are three pairwise intersecting polygons.



(c) Subcase 1.b. p has three bad neighbors.

Figure 4: Case 1 in the proof of Proposition 2.11.

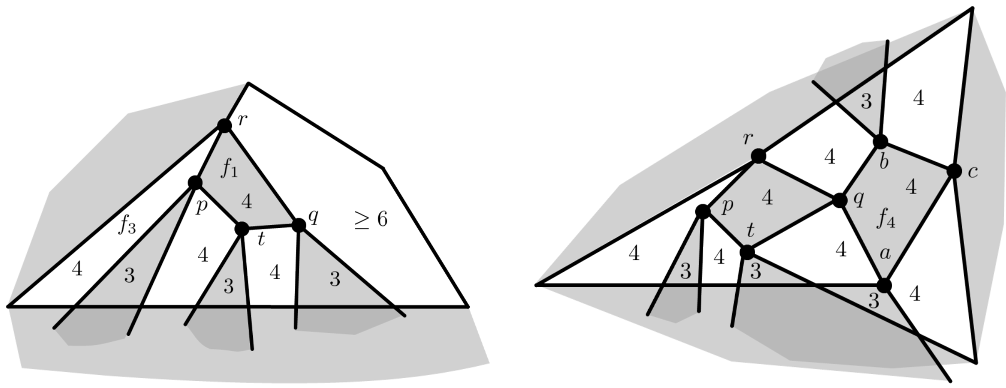
Subcase 1.c: $|f_2| = 5$. In this case $ch_1(p) \geq \frac{13}{15}$. If p has at most one other neighbor but t that requests charge in Round 2, then p can accept t's request in this round and $ch_2(t) = 0$. Otherwise, p has exactly three bad neighbors asking for charge in Round 2. Denote by a and b the other two, and observe that both of them are adjacent to f_2. Therefore, $F(a) = F(b) = (3, 5, 4, 4)$, and $ch_1(a) = ch_1(b) = -\frac{2}{15}$. If follows that in Round 2 p sends at most $\frac{2}{15}$ units to each of them and $\frac{1}{3}$ units to t in the next round.

Subcase 1.d: $|f_2| \geq 6$. In this case t is the only neighbor of p with a negative charge at the beginning of Round 2, and so p can accept t's request.

Case 2: $p \in T$ and $q \in T$. Note that in this case $r \in T$ as well. The first subcase that we consider is when $ch_1(p) < 0$. By symmetry the same arguments apply when $ch_1(q) < 0$, so the second subcase we need to consider is $ch_1(p), ch_1(q) \geq 0$.

Subcase 2.a: $ch_1(p) < 0$, that is, $F(p) = (3, 4, 4, 4)$. Since $ch_1(p) < 0$, in Round 2 (and perhaps also in Round 3) t might only ask $\frac{1}{3}$ units from q (if q has a positive charge). Observe that $ch_1(q), ch_1(r) \geq 0$. Indeed, if $ch_1(q) < 0$, then $F(q) = (3, 4, 4, 4)$. But then f_1 has three $(3, 4, 4, 4)$-vertices, which contradicts Proposition 2.6. For the same reason $ch_1(r) \geq 0$.

Observe that $F(q) \neq (3, 4, 4, 5)$ for otherwise there must be three pairwise intersecting polygons (refer to Figure 5(a)). If q is not adjacent to a 3-face or $F(q) = (3, 4, 4, > 6)$ then $ch_1(q) \geq \frac{2}{3}$. Therefore, if $ch_1(q) < \frac{2}{3}$ then $F(q) = (3, 4, 4, 6)$, in which case $ch_1(q) \geq \frac{1}{3}$. We now consider both possibilities.



(a) $F(q) = (3,4,4,\geq 6)$, since $F(q) = (3,4,4,4)$ contradicts Proposition 2.6 and $F(q) = (3,4,4,5)$ implies three pairwise intersecting polygons.

(b) $F(q) = (4,4,4,4)$ and $|f_4| = 4$.

Figure 5: Subcase 2.a: $p, q \in T$, $F(t) = F(p) = (3,4,4,4)$.

Subcase 2.a.i: $F(q) = (3,4,4,6)$. Refer to Figure 5(a). Since q is adjacent to a 3-face, its only neighbors from T are t and r. Because $ch_1(r) \geq 0$, t in the only neighbor of q that requests charge at Round 2, so q accepts t's request and $ch_2(t) = 0$.

Subcase 2.a.ii: $ch_1(q) \geq \frac{2}{3}$. If q has at most two neighbors asking charge in Round 2, then q can send $\frac{1}{3}$ units to t in Round 2. Otherwise, let a and b be the other two neighbors of q that ask charge in Round 2, and let f_4 be the face that is adjacent to a,b, and q (see Figure 5(b)). Since $ch_1(a), ch_1(b) < 0$ either $|f_4| = 4$ or $|f_4| = 5$. We consider these two possibilities.

- Suppose that $|f_4| = 4$ and let c denote its remaining vertex. Then $c \in T$ and observe that $ch_1(c) > 0$. Indeed, if $ch_1(c) < 0$ then $F(c) = (3,4,4,4)$, which contradicts Proposition 2.6. Therefore, each of a and b ask only $\frac{1}{6}$ units from q at Round 2, and q is able to accept t's request at the next round.
- Suppose that $|f_4| = 5$. In this case we have $ch_1(q) \geq \frac{13}{15}$ and $ch_1(a), ch_1(b) \geq -\frac{2}{15}$. Therefore q accepts t's request in Round 3.

Subcase 2.b: $ch_1(p), ch_1(q) \geq 0$. Observe that in this case $ch_1(p), ch_1(q) \geq \frac{1}{6}$. Indeed, if p is not adjacent to a 3-face then $ch_1(p) \geq \frac{2}{3}$. Otherwise, $F(p) = (3,4,4,\geq 5)$ and therefore $ch_1(p) \geq \frac{2}{3} - 1 + \frac{2}{4} = \frac{1}{6}$. For the same reason $ch_1(q) \geq \frac{1}{6}$. Therefore, t asks in Round 2 for $\frac{1}{6}$ units from each of p and q. Note that t's requests are accepted, since, as we already observed, if p is not adjacent to a 3-face, then $ch_1(p) \geq \frac{2}{3}$ and, thus, it can accept t's request since it has at most four asking neighbors. Otherwise, $ch_1(p) \geq \frac{1}{6}$ and t is the only bad neighbor of p (since r is adjacent to a face from F_0 of size at least 5). Therefore, both p and q accept t's request in Round 2.

This concludes the proof of Proposition 2.11. □

Proposition 2.12. *If $F(t) = (3,5,4,4)$ then $ch_4(t) \geq 0$.*

Proof. Let $f_1 \in F_1$ be the 5-face adjacent to t. If f_1 has at most three tangency points, then each of them gets at least $\frac{1}{3}$ units in Round 1, and so $ch_2(t) \geq 0$. It remains to consider the case where all the vertices of f_1 are from T, i.e., f_1 is a polygon from $\mathcal{C}$. (By Proposition 2.2

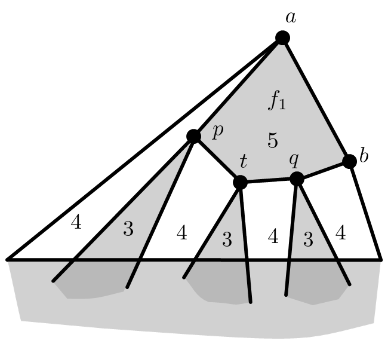


Figure 6: $F(t) = F(p) = F(q) = (3, 5, 4, 4)$.

f_1 cannot have exactly one vertex which is a crossing point.) Let p and q be the neighbors of t in f_1. If at least one of them has a positive charge after Round 1, then t asks one of them for $\frac{2}{15}$ units in Round 2. The other case to consider is when $ch_1(p), ch_1(q) < 0$.

Case 1: $ch_1(p) > 0$ or $ch_1(q) > 0$. Assume, w.l.o.g., that $ch_1(p) > 0$ and t asks p for $\frac{2}{15}$ units in Round 2. If p is not adjacent to a 3-face, then $ch_1(p) \geq \frac{13}{15}$. In this case, in Round 2 p accepts t's request, since p may send at least $\frac{13}{15}/4 > \frac{2}{15}$ to each of its at most four bad neighbors. Otherwise, since p receives $\frac{1}{5}$ units from f_1 in Round 1 and $ch_1(p) > 0$, it follows that p is adjacent to a face of size at least 5 from F_0. That is, $F(p) = (3, 5, 4, \geq 5)$. By proposition 2.9 this face sends at least $\frac{2}{4}$ units to p in Round 1, therefore, $ch_1(p) \geq \frac{2}{3} - 1 + \frac{1}{5} + \frac{2}{4} = \frac{11}{30}$. Note that t is the only bad neighbor of p, thus, p accepts t's request in Round 2.

Case 2: $ch_1(p), ch_1(q) < 0$. Then by Proposition 2.10 $F(p) = F(q) = (3, 5, 4, 4)$. Let $a, b \in T$ be the other vertices of f_1 (see Figure 6). It follows from Proposition 2.6 that $F(a), F(b) \neq (3, 5, 4, 4)$, therefore, $ch_1(a), ch_1(b) \geq 0$. Moreover, we claim that $ch_2(a) \geq \frac{k}{15}$, where $k \in \{1, 2\}$ is the number of 5-faces from F_1 that are adjacent to a. Indeed, if $F(a) = (3, 5, 4, \geq 5)$ then $ch_1(a) \geq \frac{11}{30}$, and only p requests charge ($\frac{2}{15}$ units) from a in Round 2. Hence, $ch_2(a) \geq \frac{7}{30} > \frac{1}{15}$. Otherwise, if a is not adjacent to a 3-face, then $ch_1(a) \geq \frac{2}{3} + \frac{k}{5}$ and $ch_2(a) \geq \frac{2}{3} + \frac{k}{5} - 2 \cdot \frac{1}{3} - \frac{2}{15} = \frac{3k-2}{15} \geq \frac{k}{15}$.

Similarly, we have that $ch_2(b) \geq \frac{l}{15}$, where $l \in \{1, 2\}$ is the number of 5-faces from F_1 that are adjacent to b. It follows that if $ch_2(t) = -\frac{2}{15}$ then in Round 4 a and b can each send $\frac{1}{15}$ units to t and so $ch_4(t) = 0$. $\square$

Lemma 2.8 now follows from Propositions 2.11 and 2.12. $\square$

The upper bound in Theorem 3 follows from (3) and Lemma 2.8.

Proposition 2.13. *For any n there is a set $\mathcal{C}$ of n convex regions in the plane, such that $\mathcal{I}(\mathcal{C})$ is bipartite and $t(\mathcal{C}) \geq 3n - \Theta(1)$.*

Proof. We use the same construction of Pach et al. [6] (see Figure 7) which yields a set $\mathcal{C}$ of n convex hexagons with $t(\mathcal{C}) \geq 3n - \Theta(\sqrt{n})$. However, we observe that one can take this double hexagonal grid with a constant number of rows, 'wrap' it around a cylinder, and then project it back to the plane. Observe that in such a construction all but a constant number of hexagons touch exactly 6 other hexagons. $\square$

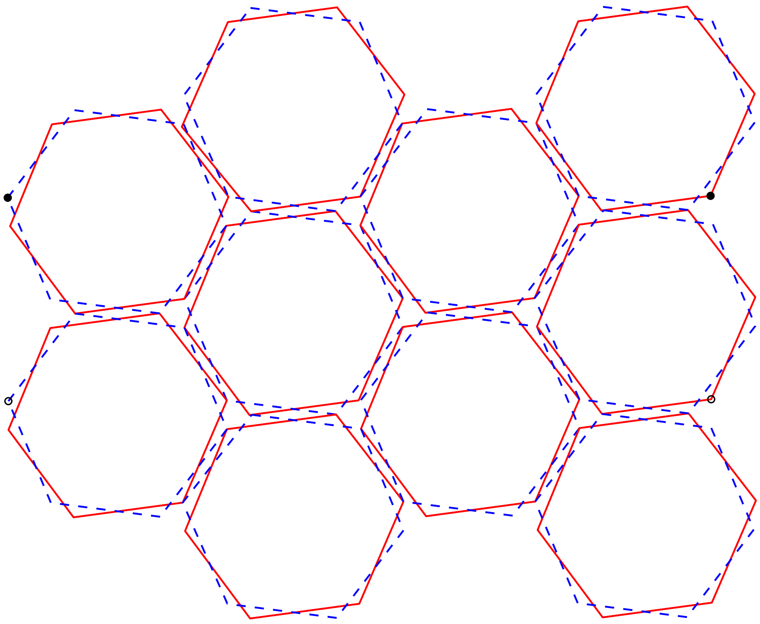


Figure 7: A set of n convex hexagons with a bipartite intersection graph and $3n - \Theta(1)$ tangencies. This double grid is 'wrapped' around a cylinder such that the pairs of black and hollow points coincide, then projected back to the plane.

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