On the Number of Rectangular Partitions

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Abstract

How many ways can a rectangle be partitioned into smaller ones? We study two variants of this problem: when the partitions are constrained to lie on n given points (no two of which are corectilinear), and when there are no such constraints and all we require is that the number of (non-intersecting) segments is n. In the first case, when the order (permutation) of the points conforms with a certain property, the number of partitions is the $(n+1)$st Baxter number, $B(n+1)$; the number of permutations conforming with the property is the $(n-1)$st Schröder number; and the number of guillotine partitions is the nth Schröder number. In the second case, it is known [22] that the number of partitions and the number of guillotine partitions correspond to the Baxter and Schröder numbers, respectively. Our contribution is a bijection between permutations and partitions. Our results provide interesting and new geometric interpretations to both Baxter and Schröder numbers and suggest insights regarding the intricacies of the interrelations.

Keywords: Rectangular partitions, guillotine partitions, Baxter permutations, Schröder numbers, quasi-monotone permutations.

1 Introduction

Given a rectangle R, a *Rectangular Partition* (RP) is a subdivision of R into rectangles by non-intersecting axis-parallel segments. We investigate the *number* of different rectangular partitions for two variants of the problem: First, we consider the *point-free* variant, where RP simply comprises n segments and two partitions are considered different if the relative ordering of the segments is not the same. Second, we consider *point-constrained* rectangular partitions, where we are given a set P of n noncorectilinear points inside R and every point in P must lie on (exactly) one segment of RP.

The number of point-free rectangular partitions has been studied elsewhere under a different name: Yao et al. [21, 22] have shown that the number of *mosaic floorplans* containing n blocks is the number of Baxter permutations on n. They have provided two alternative proofs of this fact: the first, by obtaining the same recursive formula used by Chung et al. [4] in their analysis of the number of Baxter permutations; and the second, by showing a bijection between mosaic floorplans and *twin binary trees*, whose number is also known [7] to be the number of Baxter permutations. Our contribution is a simple and *direct* bijection between mosaic floorplans (point-free partitions in our terminology) and Baxter permutations.

For point-constrained rectangular partitions it turns out that the number of rectangular partitions, denoted by $\#RP^{\mathrm{c}}$, depends on the relative order of the points in P. We represent this order by a permutation π on n (reflecting the order of y-coordinates with respect to the x-coordinates), and show that as long as π avoids forbidden subsequences with the same comparisons as 2413 and 3142, $\#RP^{\mathrm{c}}$ is the $(n+1)$st Baxter number. Inspired by the way a permutation of this class can be recursively constructed, we name them *quasi-monotone* permutations; we observe that their number is known to be the $(n-1)$st Schröder number. We also show that when only *guillotine* partitions are allowed, then no matter what the permutation of the points is, the number of different rectangular partitions is the nth Schröder number.

Point-constrained partitions are related to the optimization problem of finding the minimum edge-length rectangular partition of a rectangle with n noncorectilinear points inside it (known as *RGNLP*). It is shown in [2] (as we have observed independently) that an optimal solution of an instance of RGNLP must be composed of exactly n non-intersecting segments, yielding the relation to point-constrained rectangular partitions.

The rest of this paper is organized as follows. In Section 2 we briefly describe some related work on rectangular partitions. In Section 3 we give a short background on Baxter and Schröder numbers. The bijection between point-free partitions and Baxter permutations is described in Section 4. In Section 5 we consider the number of different point-constrained rectangular partitions: we start by describing a method to constructively enumerate them; then we discuss guillotine partitions; next we show that for the identity permutation of n points $\#RP^{\mathrm{c}}$ is the $(n+1)$st Baxter number, and finally we generalize the analysis from identity to quasi-monotone permutations and observe

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that the number of such permutations is the $(n-1)$st Schröder number. In Section 6 we discuss the relation between point-free and point-constrained partitions. We conclude in Section 7.

2 Related Work

Sakanushi and Kajitani [17] were the first to consider the number of distinct (mosaic) floorplans. They found a recursive formula for this number, but did not recognized it to be the same recursive formula suggested by Chung et al. [4] in their analysis of the number of Baxter permutations. Yao et al. [21] were the first to show that the number of distinct floorplans containing n blocks is the number of Baxter permutations on n. They have derived the same recursive formula, however, by using a different analysis. Moreover, in the journal version of their work [22] they have provided an alternative proof: a bijection between floorplans and twin binary trees. The number of twin binary trees is known [7] to be the number of Baxter permutations. Yao et al. have also considered *slicing* floorplans (guillotine partitions in our terminology) and proved that their number is the nth Schröder number.

To the best of our knowledge, enumerative aspects of point-constrained rectangular partitions (as defined above) have never been studied before. However, there is a considerable amount of work that concerns optimization problems related to rectangular partitions, the most general of which is: *Partition a rectilinear (i.e., axis-parallel) polygon which encloses a set of "holes" (non-intersecting rectilinear polygons) into rectangles in a way that minimizes the total length of the edges participating in the partition.* (Other criteria, e.g., obtaining the minimum number of rectangles, have also been considered.) One application of this problem is in the area of integrated-circuit design, e.g., in MIT's "PI" system [16]: Once the electronic modules have been placed on the chip, the routing area is partitioned into rectangles (channels), in order to simplify signal wires routing. This stage is known as "channel definition," and the minimum edge-length criterion was chosen since it results in more "natural-looking" rectangles.

The partitioning problem where the holes are degenerate, i.e., a hole is a point and the bounding rectilinear polygon is a rectangle, is known as *RGP* (partitioning a rectangle with possibly-corectilinear points). This problem has applications to stock (or die) cutting in the presence of material defects. RGP, similarly to most of the polygon partitioning problems mentioned above (but unlike RGNLP), was shown to be NP-hard [13]. Over the years several approximation algorithms for RGP were suggested (see, e.g., [6, 8, 9, 10, 12]), including a polynomial-time approximation scheme [3, 14]. de Meneses and de Souza [5] suggested an integer programming formulation of RGP, and used this formulation and integer programming techniques to find exact solutions for medium sized instances of RGP. In [2], Calheiros, Lucena, and de Souza presented a reduced integer programming formulation for RGNLP instances.

3 Baxter and Schröder Numbers

3.1 Baxter numbers. The nth Baxter number is the number of Baxter permutations on n. A Baxter permutation on n can be defined as a permutation $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n)$ that satisfies the following two conditions:
For every $1 \leqslant i < j < k < l \leqslant n$:

1. If $\sigma_i + 1 = \sigma_l$ and $\sigma_j > \sigma_l$ then $\sigma_k > \sigma_l$; and
2. If $\sigma_l + 1 = \sigma_i$ and $\sigma_k > \sigma_i$ then $\sigma_j > \sigma_i$.

For example, for $n = 4$, (3,1,4,2) and (2,4,1,3) are the only non-Baxter permutations. This class of permutations was introduced by Baxter [1] in the context of fixed points of the composite of commuting functions. Chung et al. [4] showed that the number of Baxter permutations on n is

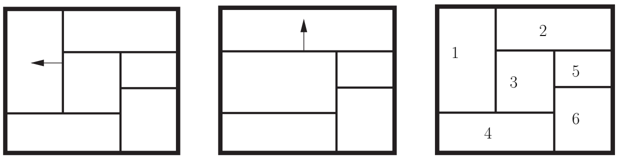
$$\text{(3.1)} \qquad \mathrm{B}(n) = \sum_{r=0}^{n-1} \frac{\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}}{\binom{n+1}{1}\binom{n+1}{2}}$$

Dulucq and Guibert [7] have shown one-to-one correspondences between Baxter permutations, twin binary trees, and three non-intersecting paths on a grid. The first Baxter numbers are $\{0, 1, 2, 6, 22, 92, 422, 2074, \ldots\}$.

3.2 Schröder numbers. The (large) Schröder numbers arise in several enumerative combinatorial problems. One example is the number of paths on a grid from $(0, 0)$ to (n, n), that stay below the line $y = x + 1$ and use only the steps $(1, 0)$, $(0, 1)$, and $(1, 1)$. Other examples can be found in [19]. The first Schröder numbers are $\{1, 2, 6, 22, 90, 394, 1806, \ldots\}$.

4 Point-Free Partitions

A *point-free rectangular partition* of a rectangle R is a partition of R into rectangles by non-intersecting rectilinear segments. Throughout this section, unless stated otherwise, the term 'partitions' refers to point-free rectangular partitions, and n marks the number of rectangles in a partition (it is easy to see that the number of rectangles is the number of segments plus 1). In order to count the number of different partitions (into n rectangles), we must first define equivalence of partitions. We follow the definition of Sakanushi et al. [17]: A *top-, left-, right-,* or *bottom-seg-rect relation* between a segment s and a rectangle r exists if s supports r from the respective direction. Two partitions are *equivalent* if there is a labeling of their rectangles and segments such that they hold the same seg-rect relations.



(a) (b) (c)

Figure 1: Enumerating rectangles according to their deletion order from the top-left corner.

Definition 4.1. *A rectangle* r_1 *is* above *a rectangle* r_2 *if: (1) There is a segment* s *such that there is a bottom-seg-rect relation between* r_1 *and* s *and a top-seg-rect relation between* r_2 *and* s*; or (2) there is a rectangle* r_3 *such that* r_3 *is above* r_2 *and* r_1 *is above* r_3*.*

The relation *left of* is defined similarly.

4.1 A bijection between Baxter permutations and point-free partitions. As we have mentioned in the introduction, there is a bijection between point-free partitions and twin binary trees [22], and a bijection between twin binary trees and Baxter permutations [7]. Here we present a *direct* bijection between Baxter permutations and point-free partitions. We first describe a one-to-one mapping from point-free partitions to Baxter permutations. Then we show a one-to-one mapping in the opposite direction.

4.1.1 A one-to-one mapping from point-free partitions to Baxter permutations. Given a partition, we can delete its top-left rectangle by 'sliding' either the right edge of the rectangle to the left (see Figure 1(a)), or its bottom edge upwards (see Figure 1(b)). This operation is called *block deletion* [11], and it could be defined similarly for every corner of the bounding rectangle. In Figure 1 the rectangles are enumerated according to their deletion order from the top-left corner.
We define a mapping Ψ from partitions to permutations in the following way:

1. Given a partition x we name its rectangles according the order in which they are deleted from the top-left corner.
2. $\Psi(x)$ corresponds to the order in which the rectangles are deleted from the bottom-left corner.

For example, the permutation that corresponds to the partition of Figure 1 is 413652.

Observation 4.1. *Rectangle* r_1 *precedes rectangle* r_2 *according to the top-left corner deletion order and* r_2 *precedes* r_1 *according to the bottom-left corner deletion order iff* r_1 *is above* r_2*. Similarly,* r_1 *precedes* r_2 *according to both orders iff* r_1 *is left of* r_2*.*

Observation 4.2. *If rectangle* r_1 *is next to rectangle* r_2 *according to one of the two orders, then there is a segment that supports both* r_1 *and* r_2*.*

Next, we prove that Ψ is indeed a one-to-one mapping from partitions to Baxter permutations.

Lemma 4.1. *Given a partition* x *with* n *rectangles,* $\Psi(x)$ *is a Baxter permutation on* n*.*

Proof. Suppose $\Psi(x) = \sigma_1\sigma_2\ldots\sigma_n$ is not a Baxter permutation. Then there are four indices $1 \leqslant i < j < k < l \leqslant n$ such that either: (1) $\sigma_k < \sigma_i + 1 = \sigma_l < \sigma_j$; or (2) $\sigma_j < \sigma_l + 1 = \sigma_i < \sigma_k$. Assume the first case holds, and choose w.l.o.g. j and k such that $k = j + 1$. According to observations 4.1 and 4.2 rectangle σ_i is left of rectangle σ_l, and some segment s_1 supports both of them. Similarly, rectangle σ_j is below rectangle σ_k, and some segment s_2 supports both of them. According to Observation 4.1 rectangle σ_k is to the left of rectangle σ_l and above rectangle σ_i. Similarly, rectangle σ_j is to the right of rectangle σ_i and below rectangle σ_l. Thus, s_1 and s_2 must intersect. The proof in the second case is similar and is thus omitted.

Lemma 4.2. *Given two partitions* x_1 *and* x_2*, if* $\Psi(x_1) = \Psi(x_2)$ *then* x_1 *and* x_2 *are equivalent.*

Proof. By induction on the number of rectangles n. Let x'_1 (resp., x'_2) be the partition we get by deleting the top-left rectangle of x_1 (resp., x_2), and let s_1 (resp., s_2) be the segment we 'slide' in order to delete the rectangle. Then s_1 and s_2 must have the same orientation, otherwise the numbers 1 and 2 will have different orders in $\Psi(x_1)$ and $\Psi(x_2)$. Every pair of rectangles in x_1 (resp., x_2) hold the same relation (above or left of) before and after the deletion, thus, $\Psi(x'_1) = \Psi(x'_2)$. It follows that $x'_1 = x'_2$ and since s_1 and s_2 have the same orientation, $x_1 = x_2$.

4.1.2 A one-to-one mapping from Baxter permutations to point-free partitions. Algorithm `BP2PFP` (see Figure 2) constructs a partition of n rectangles given a Baxter permutation on n, by inserting rectangles from the top-right corner and setting their boundaries according to the given permutation. See Figure 3 for an example.

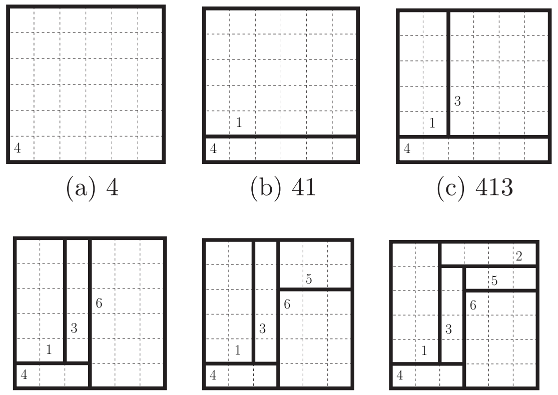
Lemma 4.3. *There is a one-to-one mapping from Baxter permutations on* n *elements to partitions containing* n *rectangles.*

Proof. Let π be a Baxter permutation on n, and let x be the output of Algorithm `BP2PFP` when applied to π. Clearly x is a valid partition containing n rectangles. We will show that $\Psi(x) = \pi$. It is easy to see that during the computation of $\Psi(x)$, the rectangles are deleted

Input: A Baxter permutation $\pi = \sigma_1\sigma_2 \dots \sigma_n$
Output: A point-free partition

```
Draw a rectangle σ1.
Construct an n × n grid within σ1.
for i ← 2 to n do
   if σi < σi−1 then
      Slice the top-right rectangle by a horizontal
      segment at the (i − 1)st level of the grid.
      Name the newly-created rectangle σi.
      while the rectangle to the left of σi has a label
      greater than σi do
         Extend rectangle σi to the left.
   else
      Slice the top-right rectangle by a vertical seg-
      ment at the (i − 1)st level of the grid.
      Name the newly-created rectangle σi.
      while the rectangle below σi has a label smaller
      than σi do
         Extend rectangle σi downwards.
```

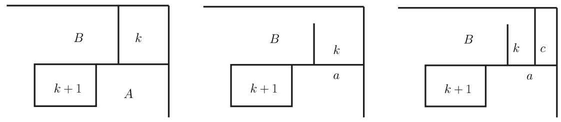
Figure 2: Algorithm BP2PFP



(a) 4 (b) 41 (c) 413

(d) 4136 (e) 41365 (f) 413652

Figure 3: Algorithm BP2PFP applied to 413652



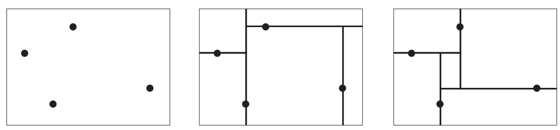
(a) (b) (c)

Figure 4: Illustrations for the proof of Lemma 4.3

from the bottom-left corner of x in the same order they were added to x in the course of Algorithm BP2PFP. Therefore, it is sufficient to prove that the order in which the rectangles of x are deleted from the top-left corner is $1, 2, \dots, n$. It is clear that the rectangle labeled 1 is the first to be removed. Assume that for every $1 \leqslant i \leqslant k$ the rectangle labeled i is the ith rectangle which is removed from the top-left corner. We show next that the next rectangle to be deleted is the rectangle labeled $k + 1$. Suppose for contradiction that $k + 1$ precedes k in π: $\pi = \dots, (k + 1), A, B, k, \dots,$ where A is a (possibly empty) sequence of integers greater than $k + 1$ and B is a (possibly empty) sequence of integers smaller than k (there are no other options since π is a Baxter permutation). Figure 4(a) shows the partition after k was added. According to the induction hypothesis all the rectangles in B are removed before rectangle k is removed, so when k is removed the segment supporting its left edge also supports the left edge of $k + 1$. The bottom-right corner of k is either a '⊥'-junction or a '⊣'-junction. In the second case $k + 1$ is the next rectangle to be deleted. A '⊥'-junction can only be formed when the first rectangle greater than k and to the right of it in π (denote it by c) is smaller than the rectangle below k and sharing the same segment as a right edge (denote this rectangle by a). Figures 4(b,c) describe the situation before and after c is added. Note that a is the last integer in A and $k < c < a$. If A is empty then $a = k + 1$; thus, there cannot be such a rectangle c. Otherwise, there must be an integer i such that $k + 1 \leqslant i \leqslant c - 1$, i is to the left of a in π, and $i + 1$ is either c or to the right of c. Therefore $i, a, k, i + 1$ is a forbidden subsequence in π. The proof for the second case (k precedes $k+1$ in π) is similar and is thus omitted.

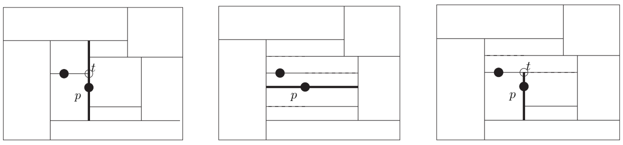
5 Point-Constrained Partitions

Given a rectangle R containing a set P of n noncorectilinear points, a *point-constrained rectangular partition* of R is a partition of R into rectangles, by n rectilinear segments, such that every point in P lies on exactly one segment (see Figure 5 for examples). Throughout this section, unless stated otherwise, the term 'partitions' refers to point-constrained rectangular partitions. In the first part of this section we present a mechanism for exploring the space of partitions. Next, we define *guillotine* partitions and show that the number of guillotine partitions is the nth Schröder number. Then, we argue that when both guillotine and nonguillotine partitions are considered $\#RP^{\mathrm{C}}$ depends only on the *permutation* of the points in P, and show that for the identity permu-



(a) (R, P) (b) A guillotine partition (c) A nonguillotine partition

Figure 5: Point-constrained rectangular partitions



(a) x (b) $\texttt{Flip}(x, p)$ (c) $\texttt{Rotate}(x, t)$

Figure 6: Applying the `Flip` and `Rotate` operators

tation $\#RP^{\text{C}}$ is the $(n+1)$st Baxter number. Finally, we define a new class of permutations, so-called *quasi-monotone* permutations, and extend the previous result for this type of permutations.

5.1 Generating point-constrained partitions. In this section we define two operators that enable us to explore the space of all the partitions. Given a partition x we can get new partitions by applying each of the following operators on x.

Definition 5.1. *Let p be a point in P, and suppose the segment s that passes through p is horizontal. The operator* $\texttt{Flip}(x, p)$ *changes the orientation of s to vertical, while extending the segments supported by s (that is, one of whose endpoints is contained by s) until each one of these segments reaches a horizontal segment or the bounding rectangle. Flipping a vertical segment is defined in a similar way.*

Definition 5.2. *Suppose t is an endpoint of a segment s_1, that lies on another segment s_2. The operator* $\texttt{Rotate}(x, t)$ *extends s_1 beyond t until it reaches another segment (or the boundary), and shrinks s_2 to t (deleting the portion of s_2 that does not contain the point of P) while extending the segments that were supported by the shrunk part (in order to maintain a valid partition).*

See Figure 6 for examples of the `Flip` and `Rotate` operators.

Given a rectangle R which encloses a set of points P, we denote by $G(R, P) = (V, E)$ the directed graph of partitions of (R, P), where $V = \{x : x \text{ is a partition of } (R, P)\}$ and $E = \{(x_1, x_2) : x_2 \text{ is reachable from } x_1 \text{ by a single } \texttt{Flip} \text{ or } \texttt{Rotate} \text{ operation}\}$.

Lemma 5.1. *$G(R, P)$ is connected.*

Proof. Let x_1 and x_2 be two distinct partitions, and let x_v be the partition in which all the segments are vertical. The partition x_v can be reached from both x_1 and x_2 by a series of at most n `Flip` operations, where n is the size of P. Note that every `Flip` operation is reversible by a single `Flip` operation on the same point, and a series of `Rotate` operations. Thus, there is a path from x_1 to x_2 (through x_v) in $G(R, P)$.

It is thus possible to generate and iterate over all the partitions of (R, P) by traversing $G(R, P)$ by, say, a standard depth- (or breadth-) first search.

5.2 Guillotine partitions.

Definition 5.3. *In a* guillotine *partition the segments can be ordered so that when the partition is executed according to that order, the current segment always partitions a rectangle into two rectangles.*

See Figure 5 for examples of guillotine and non-guillotine partitions. In this section we consider the number of guillotine partitions. It is easy to see that this number depends only on the number of points in P. Let $\text{GP}(n)$ be the number of guillotine partitions when $|P| = n$. We derive a recursive formula for $\text{GP}(n)$ by assuming that the leftmost vertical segment that cuts R goes through the kth point (left to right) in P, then summing up over all possible values of k, and finally by multiplying by 2 (for the symmetric partitions in which R is cut by at least one horizontal segment):

$$(5.2)\quad \text{GP}(n) = 2\Big(\text{GP}(n-1) + \sum_{k=2}^{n}\Big(\frac{1}{2}\text{GP}(k-1)\Big)\text{GP}(n-k)\Big),$$

where $\text{GP}(0) = 1$. This formula is equivalent to a recursive formula of the nth Schröder number:

$$(5.3)\quad S(n) = S(n-1) + \sum_{k=0}^{n-1} S(k)S(n-1-k), \quad S(0) = 1$$

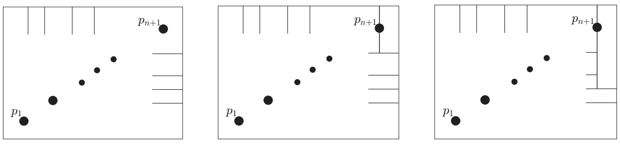
Thus we have:

Theorem 5.1. *Given a rectangle R which encloses a set P of n noncorectilinear points, the number of guillotine partitions of (R, P) is the nth Schröder number.*

5.3 Partitions and permutations.

Definition 5.4. *Given a set P of noncorectilinear points, we refer to the relative order of the points in P as the* permutation *of P and denote it by $\pi(P)$.*

Representing the relative order of the points by a permutation $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n)$ is feasible since the points are noncorectilinear. By $\sigma_i = j$ we mean that the ith point along the x-axis is the jth point along the y-axis. It is easy to see that given two pairs of a rectangle and a set of points, (R_1, P_1) and (R_2, P_2), such that $|P_1| = |P_2|$ and $\pi(P_1) = \pi(P_2)$, we always have $\#RP^{\text{C}}(R_1, P_1) = \#RP^{\text{C}}(R_2, P_2)$. In other words, the number of partitions depends only on the permutation of points and neither on the bounding rectangle nor on



(a) An additional point (b) Set a vertical segment (c) Extend the vertical segment

Figure 7: From $T_n(i, j+k)$ to $T_{n+1}(i+1, j+1)$

the actual point coordinates. Therefore, we will also use the notation $\#RP^{\mathrm{C}}(\pi)$. However, experiments we have performed showed that when $\pi(P_1) \neq \pi(P_2)$ it is possible to have $\#RP^{\mathrm{C}}(P_1) \neq \#RP^{\mathrm{C}}(P_2)$. For example, $\#RP^{\mathrm{C}}(1,2,3,4) = 92$ while $\#RP^{\mathrm{C}}(3,1,4,2) = 93$.

5.4 The number of partitions of identity permutations.

Lemma 5.2. *Let $\mathcal{I}_n$ be the identity permutation on n. $\#RP^{\mathrm{C}}(\mathcal{I}_n) = B(n+1)$.*

Proof. Given a partition x we denote by *bottom(x)* (resp., *top(x)*) the set of vertical segments touching the bottom (resp., top) edge of the bounding rectangle R. Similarly, *left(x)* (resp., *right(x)*) denote the set of horizontal segments touching the respective edges of R. Let $T_n(i,j)$ be the number of different partitions of n points with the identity permutation, such that for every partition x, $|\mathrm{top}(x)| = i$ and $|\mathrm{right}(x)| = j$. Then we can write the following recurrence relation for $n \geqslant 0$:

(5.4)
$$T_{n+1}(i+1, j+1) = \sum_{k=1}^{\infty} \Big(T_n(i, j+k) + T_n(i+k, j) \Big),$$

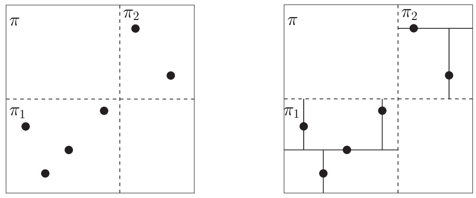
where $T_0(0,0) = 1$ and $T_n(i,j) = 0$ for $n < 0$. To understand why this relation holds, note that we can create a partition x of $n+1$ points such that $|\mathrm{top}(x)| = i+1$ and $|\mathrm{right}(x)| = j+1$ from a partition x' of n points, such that $|\mathrm{top}(x')| = i$ and $|\mathrm{right}(x')| = j+k$ (for $k \geqslant 1$), by:

1. Adding an additional point p_{n+1} to the right and above all the points of x';
2. Setting a vertical segment s_{n+1} through p_{n+1}; and
3. Extending s_{n+1} downwards using `Rotate` operations until $k-1$ segments are removed from $\mathrm{right}(x)$.

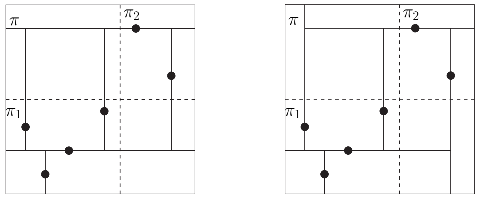
Figure 7 shows these steps. We can create in a similar way a partition x of $n+1$ points, for which $|\mathrm{top}(x)| = i+1$ and $|\mathrm{right}(x)| = j+1$, from a partition x' of n points, such that $|\mathrm{top}(x')| = i+k$ (for $k \geqslant 1$) and $|\mathrm{right}(x')| = j$, by passing a horizontal segment through a new point p_{n+1}. Clearly, every partition x of $n+1$ points can be created from a partition x' of n points as described above, and there are no two different partitions x'_1, x'_2 of n points, that lead to the same partition of $n+1$ points. Therefore,

(5.5)
$$\#RP^{\mathrm{C}}(\mathcal{I}_n) = \sum_{i,j \geqslant 0} T_n(i,j),$$

which is exactly $B(n+1)$ [4].



(a) π_2 concatenated above π_1 (b) Partitions of π_1 and π_2



(c) One combination of the partitions of π_1 and π_2 (d) Another combination of the partitions of π_1 and π_2

Figure 8: Partitions of a quasi-monotone permutation

5.5 Quasi-monotone permutations and their number of partitions.

In this section we first define quasi-monotone permutations and explore some of their properties. Then we show that the number of partitions for a quasi-monotone permutation is $B(n+1)$.

5.5.1 Quasi-monotone permutations.

Let $\pi_1 = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ and $\pi_2 = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ be two permutations on n and m, respectively. We say that $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n+m})$ is the result of concatenating π_2 *above* π_1 if $\sigma_i = \alpha_i$ for $1 \leqslant i \leqslant n$ and $\sigma_{n+i} = n + \beta_i$ for $1 \leqslant i \leqslant m$ (see Figure 8(a)). Likewise, we say that $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n+m})$ is the result of concatenating π_2 *below* π_1 if $\sigma_i = m + \alpha_i$ for $1 \leqslant i \leqslant n$ and $\sigma_{n+i} = \beta_i$ for $1 \leqslant i \leqslant m$.

Definition 5.5. *π is a* quasi-monotone *permutation if*
1. *$\pi = (1)$; or*
2. *There are two quasi-monotone permutations π_1 and π_2 such that π is the result of* concatenating *π_2 above or below π_1.*

A similar definition was suggested by Shapiro and Stephens [18] in their analysis of matrices that eventually fill up under bootstrap percolation. The following follows from their results:

OBSERVATION 5.1. *The number of quasi-monotone permutations of length n is the $(n-1)$st Schröder number.*

They also showed that the portion of quasi-monotone permutations (out of all permutations) approaches zero as n tends to infinity.

Another characterization of quasi-monotone permutations is in terms of *forbidden subsequences.* A permutation $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n) \in S_n$ *avoids* a certain subpermutation $\tau \in S_k$ (for $k \leqslant n$) if it does not contain a subsequence $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k})$ with the same pairwise comparisons as τ. The set of permutations of length n avoiding τ is denoted by $S_n(\tau)$. It can be shown that the set of quasi-monotone permutations is equal to $S_n(3142, 2413)$, suggesting an alternative proof [20] that their number is the $(n-1)$st Schröder number.

5.5.2 The number of partitions for quasi-monotone permutations.

In this section we prove that the number of partitions when the points are in a quasi-monotone permutation is $B(n+1)$.

Let x be a partition. The *interface* of x, denoted by $\mathcal{F}(x)$, is an ordered quadruple (l, t, r, b), such that $l = |\text{left}(x)|$, $t = |\text{top}(x)|$, $r = |\text{right}(x)|$, and $b = |\text{bottom}(x)|$. We denote by $\#RP^{\text{c}}(\pi, \mathcal{F})$ the number of partitions with permutation π and interface $\mathcal{F}$.

PROPOSITION 5.1. *For every n, l, t, r, b, $\#RP^{c}(\mathcal{I}_n, (l, t, r, b)) = \#RP^{c}(\mathcal{I}_n, (l, b, r, t))$.*

Note that this property is not intuitive and does not follow from simple symmetry arguments. In this short version of the paper we only provide a brief sketch of the proof.

Proof. We prove this proposition by showing a one-to-one mapping, ψ, such that for every partition x of n points in the identity permutation and with the interface (l, t, r, b), $\psi(x)$ is a partition of n points in the identity permutation, and $\mathcal{F}(\psi(x)) = (l, b, r, t)$. The mapping ψ is defined recursively: If x contains a guillotine cut, then $\psi(x)$ is the result of applying ψ on the partitions induced by that cut (if the cut is horizontal, then reflection about the ascending diagonal is required before applying ψ on the subproblems, and reflection about the descending diagonal is required afterwards). Otherwise, we find two segments (after reflection if needed), $s_1 \in \text{top}(x)$ and $s_2 \in \text{bottom}(x)$, such that s_1 is to the left of s_2. We use these segments as 'pseudo' guillotine cuts and apply ψ on the subproblems they induce.

COROLLARY 5.1. *Let $\overline{\mathcal{I}}_n$ be the reverse identity permutation on n $(n, n-1, \ldots, 3, 2, 1)$, then for every n, l, t, r, b, $\#RP^{c}(\mathcal{I}_n, (l, t, r, b)) = \#RP^{c}(\overline{\mathcal{I}}_n, (l, t, r, b))$.*

Proof. Let a be a partition of n points in the identity permutation, such that $\mathcal{F}(a) = (l, t, r, b)$. When we reflect a with respect to the x-axis we get a partition a' of n points in the reverse identity permutation, such that $\mathcal{F}(a') = (l, b, r, t)$. The corollary follows directly from this fact and from Proposition 5.1.

LEMMA 5.3. *Let π be a quasi-monotone permutation of n points. For every interface $\mathcal{F}$, $\#RP^{\text{c}}(\pi, \mathcal{F}) = \#RP^{\text{c}}(\mathcal{I}_n, \mathcal{F})$.*

Proof. By induction on n. For $n = 1$ a permutation of one point is both the identity permutation and quasi-monotone permutation. Assume the claim is true for every quasi-monotone permutation of $n' < n$ points, and let π be a quasi-monotone permutation of n points. The permutation π may be a concatenation-above or a concatenation-below of two quasi-monotone permutations.

Suppose that π is the result of concatenating a quasi-monotone permutation $\pi_2 \in S_{n-k}$ above another quasi-monotone permutation $\pi_1 \in S_k$. Then all the partitions of π can be created by considering every pair of a partition of π_1 and a partition of π_2, and by "combining" every such pair in all the possible combinations (see Figure 8). Note that given x_1 and x_2, partitions of π_1 and π_2, respectively, the number of partitions of π that are created by combining x_1 and x_2 in all the possible combinations depends only on $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$. Moreover, the interface of every such combined partition also depends only on $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ and the way they were combined.

According to the induction hypothesis, for every pair of interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ we have $\#RP^{\text{c}}(\pi_1, \mathcal{F}_1) = \#RP^{\text{c}}(\mathcal{I}_k, \mathcal{F}_1)$ and $\#RP^{\text{c}}(\pi_2, \mathcal{F}_2) = \#RP^{\text{c}}(\mathcal{I}_{n-k}, \mathcal{F}_2)$. All the partitions of $\mathcal{I}_n$ can be created by combining all the pairs of a partition of $\mathcal{I}_k$ and a partition of $\mathcal{I}_{n-k}$ in all possible combinations. Again, the number of combinations and the interface of every such combined partition depends only on the interfaces of the partitions of $\mathcal{I}_k$ and $\mathcal{I}_{n-k}$, and on the way they were combined. Thus, for every concatenation-above quasi-monotone permutation π and interface $\mathcal{F}$, $\#RP^{\text{c}}(\pi, \mathcal{F}) = \#RP^{\text{c}}(\mathcal{I}_n, \mathcal{F})$.

Suppose now that π is the result of concatenating a quasi-monotone permutation $\pi_2 \in S_{n-k}$ below another quasi-monotone permutation $\pi_1 \in S_k$. It follows from Corollary 5.1 that for every pair of interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$, $\#RP^{\text{c}}(\mathcal{I}_k, \mathcal{F}_1) = \#RP^{\text{c}}(\overline{\mathcal{I}}_k, \mathcal{F}_1)$ and $\#RP^{\text{c}}(\mathcal{I}_{n-k}, \mathcal{F}_2) = \#RP^{\text{c}}(\overline{\mathcal{I}}_{n-k}, \mathcal{F}_2)$. Using the induction hypothesis we conclude that for every pair of two interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$, $\#RP^{\text{c}}(\pi_1, \mathcal{F}_1) = \#RP^{\text{c}}(\overline{\mathcal{I}}_k, \mathcal{F}_1)$ and $\#RP^{\text{c}}(\pi_2, \mathcal{F}_2) = \#RP^{\text{c}}(\overline{\mathcal{I}}_{n-k}, \mathcal{F}_2)$. Then, according to the combination arguments given above and by using Corollary 5.1, for every concatenation-below quasi-monotone permutation π and interface $\mathcal{F}$, $\#RP^{\text{c}}(\pi, \mathcal{F}) = \#RP^{\text{c}}(\overline{\mathcal{I}}_n, \mathcal{F}) = \#RP^{\text{c}}(\mathcal{I}_n, \mathcal{F})$.

In conclusion, the claim holds for all quasi-monotone permutations.

THEOREM 5.2. *Given a rectangle R which encloses a set P of n noncorectilinear points, such that $\pi(P)$ is a quasi-monotone permutation on n, $\#RP^C(R,P) = B(n+1)$.*

Proof. The claim follows from lemmata 5.2 and 5.3.

6 Point-Free and Point-Constrained Partitions

In this section we establish a relation between point-free partitions and point-constrained partitions. First, we present a new representation of a point-free partition—the *partition graph*. Then, we use this representation to describe the relation between point-free and point-constrained partitions.

6.1 The partition graph

DEFINITION 6.1. *Given a (point-free) partition we define two partial orders of the segments in the partition. We say that segment s_1 is* left *of segment s_2, denoted $s_1 \prec_\ell s_2$, if:*

1. *s_1 is vertical, s_2 is horizontal, and s_1 contains the left endpoint of s_2; or*
2. *s_1 is horizontal, s_2 is vertical, and s_2 contains the right endpoint of s_1; or*
3. *s_1 and s_2 are vertical, ℓ_1 is the line supporting s_1, s_2 is to the right of ℓ_1, and the projection of s_2 on ℓ_1 contains more than one point of s_1; or*
4. *There is a segment $s \notin \{s_1, s_2\}$ such that $s_1 \prec_\ell s$ and $s \prec_\ell s_2$.*

In a similar manner we define the relation *below* between two segments s_1 and s_2, and denote it by $s_1 \prec_b s_2$.

We show below that the union of $\prec_\ell$ and $\prec_b$ is a total relation on any set of segments defining a valid partition.

Let s be a horizontal segment. We denote by s^l and s^r the vertical segments that bound s from left and from right, respectively. Similarly, for a vertical segment s, we denote by s^b and s^a the horizontal segments that bound s from below and from above, respectively. The next observation follows from Definition 6.1 and will be useful later on:

OBSERVATION 6.1. *Let s be a horizontal segment and let $x \neq s$ be another segment. If $s \prec_\ell x$ then either $x = s^r$ or $s^r \prec_\ell x$. If $x \prec_\ell s$ then either $x = s^l$ or $x \prec_\ell s^l$. We can derive a similar observation about vertical segments and the 'below' relation.*

Using the 'left' and 'below' relations, we can now define the "partition graph."

DEFINITION 6.2. *Given a partition r of a rectangle by n segments $s_1, s_2, \ldots, s_n$, the* partition graph *of r is a directed graph $G(r) = (V = \{v_1, v_2, \ldots, v_n\}, E)$, such that: (1) Every vertex $v_i \in V$ has a label 'vertical' or 'horizontal,' according to the orientation of s_i; and (2) There is an edge from v_i to v_j labeled 'left' (resp., 'below') if $s_i \prec_\ell s_j$ (resp., $s_i \prec_b s_j$).*

Using the concept of a partition graph it is possible to define two point-free partitions as equivalent if their partition graphs are isomorphic. We proceed by mentioning some properties of partition graphs.

PROPERTY 6.1. *Given a partition r, $G(r)$ is a tournament.*

Proof. (sketch) This property can be proven by induction on n and considering the partition we get by removing either the lowest segment in left(r) or the leftmost segment in bottom(r) (the one whose endpoint lies on the other is removed).

Note that since a partition graph is also transitive (this follows from Definition 6.1(4)), it must be acyclic (even when the labels on the edges are ignored). Thus, any partition graph defines a total order of its vertices. The next observation follows from this fact and from Observation 6.1.

OBSERVATION 6.2. *Given a partition r, let s_i be a vertical (resp., horizontal) segment, and let $v_1 < v_2 < \cdots < v_n$ be the total order of the vertices of $G(r)$. ($v_i < v_j$ if there is an edge $v_i \to v_j$.) Let $v_j \equiv v_i^b$ and $v_k \equiv v_i^a$ (resp., $v_j \equiv v_i^l$ and $v_k \equiv v_i^r$) be the vertices that correspond to s_i^b and s_i^a (resp., s_i^l and s_i^r). Then j is the maximal index such that there is an edge $v_j \stackrel{below}{\to} v_i$ (resp., $v_j \stackrel{left}{\to} v_i$), and k is the minimal index such that there is an edge $v_i \stackrel{below}{\to} v_k$ (resp., $v_i \stackrel{left}{\to} v_k$).*

LEMMA 6.1. *Suppose $G(V,E)$ is a partition graph, and let $v_1 < v_2 < \cdots < v_n$ be the total order of V. Then the graph induced by $v_2, v_3, \ldots, v_n$ is also a partition graph.*

Proof. Let r be a partition whose partition graph is G, and let s_1 be the segment in r that corresponds to v_1. s_1 must be the lowest segment in left(r) or the leftmost segment in bottom(r). Removing s_1 from r while extending the segments that have their bottom endpoint (in the first case) or their left endpoint (in the second case) on s_1, towards the bounding rectangle, will result in a valid partition whose corresponding graph is the subgraph of G induced by $v_2, v_3, \ldots, v_n$.

We can argue the same about the subgraph induced by $v_1, v_2, \ldots, v_{n-1}$. Thus:

COROLLARY 6.1. *Suppose $G(V,E)$ is a partition graph, let $v_1 < v_2 < \cdots < v_n$ be the total order of V, and $1 \leqslant i < j \leqslant n$. Then the subgraph induced by $v_i, v_{i+1}, \ldots, v_j$ is also a partition graph.*

6.2 A relation between point-free and point-constrained partitions.

DEFINITION 6.3. *Given a partition graph $G(V, E)$, $|V| = n$, and a set P of n points whose relative order is given by a permutation $\pi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, we say that a one-to-one mapping $\phi : V \to \{1, 2, \ldots, n\}$* embeds G *in P if for every edge $v_i \overset{left}{\to} v_j$ (resp., $v_i \overset{below}{\to} v_j$), $\phi(v_i) < \phi(v_j)$ (resp., $\sigma_{\phi(v_i)} < \sigma_{\phi(v_j)}$). We denote by* embeddings(G, P) *the set of different mappings that embed G in P.*

Intuitively, an embedding describes a way to "place" the segments of a point-free partition on a set of points, in order to obtain a valid point-constrained partition.

LEMMA 6.2. *Given a rectangle R which encloses a set of points P, a partition graph $G(V, E)$, and an embedding ϕ of G in P, there is exactly one partition r of (R, P) whose partition graph is equivalent to G and in which for every $1 \leqslant i \leqslant n$ the segment in r that corresponds to v_i contains $p_{\phi(v_i)}$ (the $(\phi(v_i))$th left-to-right point in P).*

Proof. (sketch) We construct a partition r of (R, P) in the following way. Let v_i be a vertex in V labeled 'horizontal,' and let v_i^l and v_i^r be the vertices as defined in Observation 6.2. Draw a horizontal segment s_i through $p_{\phi(v_i)}$ and set its left and right endpoints to the x-coordinates of the points $p_{\phi(v_i^l)}$ and $p_{\phi(v_i^r)}$, respectively. If v_i^l (resp., v_i^r) is not defined, set the left (resp., right) endpoint at the left (resp., right) edge of R. Construct vertical segments in a similar way.

A simple case analysis shows that the endpoints of every segment in r lie on different segments or on the boundary, and that there are no intersecting segments, thus r is a valid partition.

We prove that the segments in r have the same relations as the vertices of G by considering all the possible ways that an edge $v_i \overset{left}{\to} v_j$ (resp., $v_i \overset{below}{\to} v_j$) was created and by showing that in all those cases $v_i \overset{left}{\to} v_j$ implies $s_i \prec_\ell s_j$ (resp., $v_i \overset{below}{\to} v_j$ implies $s_i \prec_b s_j$). Thus, the partition graph of r is isomorphic to G.

Finally, suppose that r' is another partition of (R, P) whose partition graph is equivalent to G and in which for every index i the segment that corresponds to v_i contains $p_{\phi(v_i)}$. It follows from the construction of r and from Observation 6.2 that for every i the segment through $p_{\phi(v_i)}$ has the same orientation and endpoints in r and in r', thus r and r' are equivalent.

LEMMA 6.3. *Let $G(V, E)$ be a partition graph, $|V| = n$, and let P be a set of n points such that $\pi(P) = \mathcal{I}_n$, then $|embeddings(G, P)| = 1$.*

Proof. Let $v_1 < v_2 < \cdots < v_n$ be the total order of V, and let $\phi(v_i) = i$. Then ϕ is the only embedding of G in P.

COROLLARY 6.2. *There is a bijection between point-free partitions and point-constrained partitions of a set of points arranged in the identity permutation.*

Proof. Follows immediately from Lemma 6.3 and the fact that every point-constrained partition can be turned into a point-free partition by removing the point constraints. Note that we do not use here the fact that the number of distinct partitions is the Baxter number in both cases.

In fact, Lemma 6.3 and Corollary 6.2 can be generalized for any quasi-monotone permutation.

LEMMA 6.4. *Let $G(V, E)$ be a partition graph, $|V| = n$, and let P be a set of n points in a quasi-monotone permutation. Then $|embeddings(G, P)| = 1$.*

Proof. By induction on n. For $n = 1$ the only permutation is the identity permutation which is also quasi-monotone. Let $\pi \in S_n$ be the quasi-monotone permutation of P. Suppose π is the result of concatenating a quasi-monotone permutation $\pi_2 \in S_{n-k}$ above another quasi-monotone permutation $\pi_1 \in S_k$. Assume $v_1 < v_2 < \cdots < v_n$ is the total order of V. Let G_1 and G_2 be the graphs induced by $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$, respectively, and let $P_1 = \{p_1, p_2, \ldots, p_k\}$ and $P_2 = \{p_{k+1}, p_{k+2}, \ldots, p_n\}$. It follows from Corollary 6.1 that G_1 and G_2 are partition graphs. Thus, according to the induction hypothesis $|\text{embeddings}(G_1, P_1)| = 1$ and $|\text{embeddings}(G_2, P_2)| = 1$. Let ϕ be the union of these embeddings. For each pair of vertices $v_i \in V_1$ and $v_j \in V_2$, the edge connecting them in G is directed from v_i to v_j since v_i precedes v_j in the order. Recall that P_1 is below and to the left of P_2, thus ϕ is an embedding of G in P.

Suppose there was another embedding ϕ' of G in P. Let P_1' and P_2' be the sets of points to which V_1 and V_2 are mapped under ϕ'. If $P_1' = P_1$ and $P_2' = P_2$ then by the induction hypothesis $\phi' = \phi$. Otherwise, there must be two vertices $v_i \in V_1$ and $v_j \in V_2$ such that $p_{\phi'(v_i)} \in P_2$ and $p_{\phi'(v_j)} \in P_1$, thus ϕ' is not a valid embedding.

The proof for the case in which π is the result of concatenating a quasi-monotone permutation π_2 below another quasi-monotone permutation π_1 is quite similar. The main difference is that instead of using the total order of V, we use the order of V, in which for every two vertices v_i and v_j, if $v_i \overset{left}{\to} v_j$ then v_i precedes v_j, and if $v_i \overset{below}{\to} v_j$ then v_j proceeds v_i. Since this order is equivalent to a total order of the graph obtained from G by reversing the edges labeled 'below,' it is unique.

Corollary 6.3. *There is a bijection between point-free partitions and point-constrained partitions of a set of points arranged in a quasi-monotone permutation.*

7 Conclusion

In this paper we discuss the number of distinct rectangular partitions for point-free and point-constrained partitions. For the point-free variant we present a direct bijection between partitions and Baxter permutations. It is worth mentioning that this bijection also maps quasi-monotone permutations to guillotine partitions. The number of partitions with point constraints depends on the permutation of points. We show that if the permutation of the points is quasi-monotone, then the number of partitions is also the number of Baxter permutations. Finally, we mention a few open questions related to the problems discussed in this paper:

1. How many point-constrained partitions are there for non-quasi-monotone permutations?
2. The original minimum edge-length partitioning problem (RGNLP). Furthermore, what is the computational complexity of RGNLP when it is restricted to (quasi-)monotone permutations?
3. What is the number of point-free (or point-constrained) partitions when the problem is generalized to higher dimensions?

Note added in proof: After completion of this manuscript, we discovered the existence of a paper by Murata et al. [15], in which a mapping from *sequence-pairs* to rectangular dissections is presented. From this mapping it is not hard to induce a bijection between Baxter permutations and point-free partitions. However, we feel that the bijection we describe in Section 4.1 is much simpler.

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