On the Number of Rectangulations of a Planar Point Set

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Abstract

We investigate the number of different ways in which a rectangle containing a set of n noncorectilinear points can be partitioned into smaller rectangles by n (non-intersecting) segments, such that every point lies on a segment.

We show that when the relative order of the points forms a *separable* permutation, the number of *rectangulations* is exactly the $(n+1)$st Baxter number. We also show that no matter what the order of the points is, the number of *guillotine* rectangulations is always the nth Schröder number, and the total number of rectangulations is $O(20^n/n^4)$.

1 Introduction

Given a set P of n points within a rectangle R, a *rectangulation* (or *rectangular partition*) of (R, P) is a subdivision of R into rectangles by non-intersecting axis-parallel segments, such that every point in P lies on a segment.

The problem of finding a rectangulation that minimizes the sum of lengths of the segments (known as RGP [11], or RPP [5]) has attracted some attention. First, it was introduced by Lingas et al. [16] as a special case of partitioning a rectilinear polygon containing rectilinear holes into rectangles. The motivation for this partitioning problem comes from integrated circuits design. This problem as well as RGP were shown to be NP-hard [16]. Later, several approximation algorithms for RGP were suggested (see, e.g., [9, 11, 12, 13, 15]), including a polynomial-time approximation scheme [6, 17]. RGP has applications to stock (or die) cutting in the presence of material defects.

When the points in P are in general position in the sense that no two points have the same x or y coordinate, i.e., the points are *noncorectilinear*, then the complexity class of the minimization problem (known as RGNLP [11], or NCRPP [5]) is still unknown. However, Calheiros et al. [5] have shown that an optimal solution must comprise exactly n non-intersecting segments.

In this paper we consider the *number* of such rectangulations, namely:

> Given a set P of n noncorectilinear points in the plane within a rectangle R, how many different ways are there to divide R into smaller rectangles by n (non-intersecting) segments such that every point in P lies on a segment?

See Figure 1 for examples of such rectangulations. We denote the number of rectangula-

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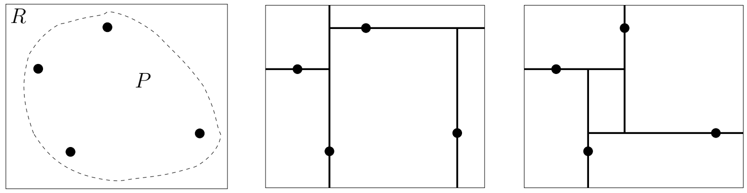


(a) (b) (c)

Figure 1: Rectangulations of (R, P)

tions of a set of points P within a rectangle R by $\Xi(P)$, since, clearly, the dimensions of the bounding rectangle do not affect the number of rectangulations. Moreover, we observe that $\Xi(P)$ depends only on the relative order of the points in P. We represent this order by a permutation π on $[n]$ (reflecting the order of y-coordinates with respect to the x-coordinates when listing the points in P from left to right), and show that if π is a *separable* permutation [4], then the number of rectangulations is the $(n+1)$st Baxter number, $\mathrm{B}(n+1) = \sum_{r=0}^{n} \frac{\binom{n+2}{r}\binom{n+2}{r+1}\binom{n+2}{r+2}}{\binom{n+1}{1}\binom{n+1}{2}} = \Theta(8^n/n^4)$. A separable permutation can be characterized by the recursive process in which it is constructed, or by the absence of subsequences with the same comparisons as 2413 or 3142.

When the permutation of the points is arbitrary, we use a novel technique of Santos and Seidel [20] to show an upper bound of $O(20^n/n^4)$ for the number of rectangulations. We also show that the number of *guillotine* rectangulations (see Definition 4.1) in this case is the nth Schröder number.

Previous work has considered the number of different *point-free* rectangulations, i.e., the number of different ways to divide a rectangle R into $n+1$ smaller rectangles by n non-intersecting segments. Point-free rectangulations have applications in integrated circuits design: During the physical design of a chip, the shape, size, and position on chip of every module are determined. The shape of the chip and the modules (blocks) is usually a rectangle. A *floorplan* describes the topological structure of the blocks, thus, it is often represented by a partition (dissection) of a rectangle into m rectangles (rooms) such that there is a one-to-one mapping from the n $(n \leq m)$ blocks to the rooms. In a *mosaic* floorplan [14] there are no empty rooms: every room contains exactly one block. Thus, a mosaic floorplan is equivalent to a point-free rectangulation.

Sakanushi and Kajitani [19] were the first to consider the number of distinct mosaic floorplans. They found a recursive formula for this number, but did not recognize it to be the same formula suggested by Chung et al. [7] in their analysis of the number of Baxter permutations. Yao et al. [26] showed a bijection between mosaic floorplans and *binary twin trees* whose number is known [10] to be the number of Baxter permutations. They have also considered *slicing* floorplans and proved that their number is the nth Schröder number.

In this work we show that given a set P of n points whose permutation is separable, and a mosaic floorplan f with n segments, f can be drawn such that every point in P is on exactly one segment of f. From this result we conclude a stronger version of a result of de Fraysseix, de Mendez and Pach [8] about the embedding of bipartite planar graphs as contact graphs of vertical and horizontal segments in the plane.

The rest of this paper is organized as follows. In Section 2 we describe two methods to enumerate rectangulations. Next, we show the upper bound for the number of rectangulations. In Section 4 we discuss guillotine rectangulations. The heart of the paper (Section 5) is an analysis of the exact number of rectangulations. We start by observing that this number depends only on the permutation of the points in P, then we show that for points arranged in an identity permutation the number of rectangulations is $\mathrm{B}(n+1)$. Next we define separable permutations and generalize this result for them. In Section 6 we discuss the relation between rectangulations and floorplans, and finally, we conclude in Section 7. For clarity, implementation issues related to the two enumeration methods suggested in Section 2 and a proof of one of the lemmata in Section 5 appear in an appendix.

2 Enumerating Rectangulations

In this section we present two methods of computing the number of rectangulations. The first generates all the rectangulations using two simple operators; the second method counts the number of rectangulations without actually generating them, and is thus more efficient.

2.1 Enumeration by Generating All the Rectangulations

Following we define two operators that enable us to explore the space of all the rectangulations of a given point set P (within a rectangle[1] R). Given a rectangulation x we can obtain new rectangulations by applying each of the following operators on x.

Definition 2.1 (Flip) *Let p be a point in P such that the segment s containing p does not contain any endpoints of other segments. The operator* $\texttt{Flip}(x, p)$ *changes the orientation of s from vertical to horizontal or vice-versa.*

Definition 2.2 (Rotate) *Let s_1 be a segment that contains one or more endpoints of other segments, and let t be such an endpoint which is extreme on s_1 (closest to one of its endpoints). Denote by s_2 the segment terminated at t. The operator* $\texttt{Rotate}(x, t)$ *extends s_2 beyond t until it reaches another segment (or the boundary) and shortens s_1 to t.*

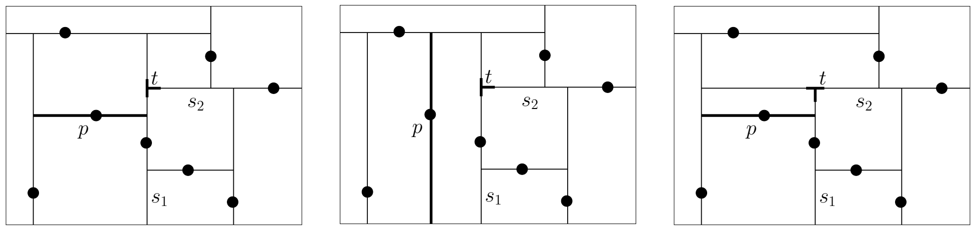
See Figure 2 for examples of the `Flip` and `Rotate` operators.

Given a set of (noncorectilinear) points P, we denote by $G(P) = (V, E)$ the graph of rectangulations of P, where $V = \{x : x \text{ is a rectangulation of } P\}$ and $E = \{(x_1, x_2) : x_2$ is reachable from x_1 by a single `Flip` or `Rotate` operation$\}$. $G(P)$ is undirected since both operators are clearly reversible.

Lemma 2.3 *Let P be a set of noncorectilinear points in the plane and let $G(P)$ be the graph of rectangulations of P. Then $G(P)$ is connected.*

Proof: Let x_1 and x_2 be two different rectangulations, and let x_v be the rectangulation in which all the segments are vertical. The rectangulation x_v can be reached from both x_1 and

[1] The bounding rectangle is obviously irrelevant to the number of rectangulation, so we sometimes omit it.



(a) A rectangulation x (b) $\texttt{Flip}(x, p)$ (c) $\texttt{Rotate}(x, t)$

Figure 2: Applying the `Flip` and `Rotate` operators

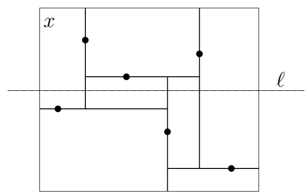


Figure 3: 10101101 represents the intersection of x and ℓ.

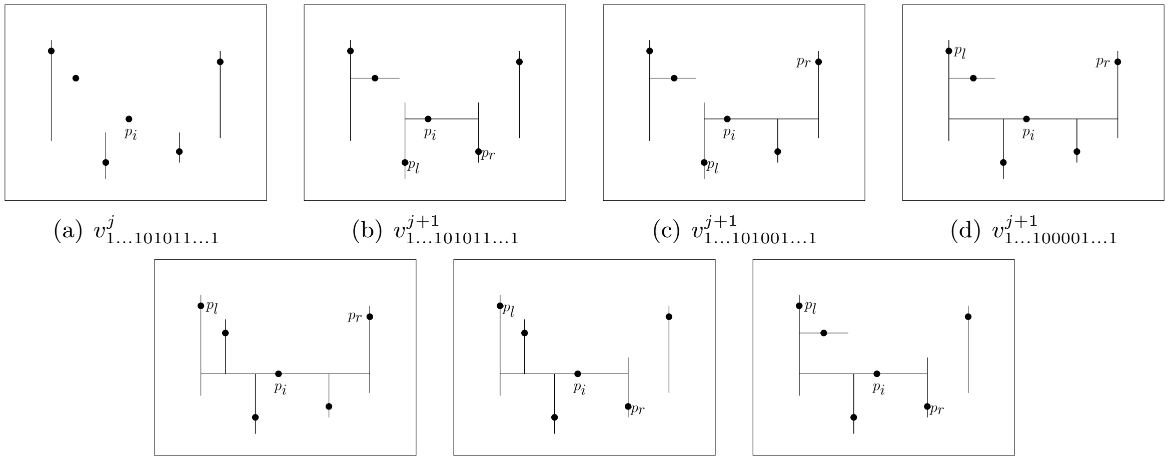
x_2 by a finite series of `Rotate` and `Flip` operations: Shorten every horizontal segment that contains endpoints of other segments by the `Rotate` operator, then turn it into a vertical segment by the `Flip` operator. Therefore there is a path between x_1 and x_2 (through x_v) in $G(P)$. □

It is thus possible to generate and iterate over all the rectangulations of P by traversing $G(P)$ by, say, a standard depth- (or breadth-) first search. Since the `Flip` and `Rotate` operations can be implemented in $O(1)$ time, exploring all the rectangulations in such a way takes $O(n\,\Xi(P))$ time and $O(n\,\Xi(P))$ space. Alternatively, it is possible to traverse a spanning tree of $G(P)$ using the *reverse search* method [2] in $O(\Xi(P)\log(n))$ time and $O(n^3)$ space. For details see Appendix A.1.

2.2 Fast Enumeration of Rectangulations

Let x be a rectangulation of P, a set of n points, and let ℓ be a horizontal line not containing any point from P. The intersection of x and ℓ can be represented by a binary word of length $n+2$, in which the $(i+1)$st bit (from left to right) is set if ℓ intersects a vertical segment that passes through the ith point (left-to-right) in P. (For convenience, the first and last bits of the word are always set, in order to represent the intersection of the sweeping line with the bounding rectangle.) See Figure 3 for an example. If we sweep ℓ from bottom to top (skipping over the points of P) we get a sequence of $n+1$ binary words of length $n+2$ that represents the rectangulation x. For example, the rectangulation in Figure 3 is represented by the sequence (10001001, 10001101, 10001101, 10101101, 10100101, 10100101, 10100101). This observation suggests a way of computing the number of rectangulations of P as follows. Define the following directed acyclic graph $G = (V, E)$:

1. Set two distinct vertices v_N and v_S, and $(n+1)2^n$ vertices of the form v_w^j, for every $w \in 1\{0,1\}^n 1$ and $1 \le j \le n+1$. For $1 \le j \le n$, a vertex of the form v_w^j corresponds to an intersection of the sweeping line just below the jth point (from bottom to top),



(a) $v^j_{1\ldots101011\ldots1}$ (b) $v^{j+1}_{1\ldots101011\ldots1}$ (c) $v^{j+1}_{1\ldots101001\ldots1}$ (d) $v^{j+1}_{1\ldots100001\ldots1}$

(e) $v^{j+1}_{1\ldots110001\ldots1}$ (f) $v^{j+1}_{1\ldots110011\ldots1}$ (g) $v^{j+1}_{1\ldots100011\ldots1}$

Figure 4: $v^j_{1\ldots101011\ldots1}$ and its neighbors according to rule 3(b)

resulting in the sequence w. A vertex of the form v^{n+1}_w corresponds to an intersection of the sweeping line just above the nth point (from bottom to top), resulting in the sequence w.

2. Set edges from v_S to v^1_w and from v^{n+1}_w to v_N, for every $w \in 1\{0,1\}^n1$.

3. Let $p_2, p_3, \ldots, p_{n+1}$ be the points of P, such that p_{k+1} is the kth point from left to right. Let p_i be the $(i-1)$st point from left to right, and the jth point from bottom to top. Denote by w_k the kth bit of w. Then, the neighbors of v^j_w are defined by the following rules:

   (a) If $w_i = 1$, then v^j_w has only one neighbor, v^{j+1}_w. This case corresponds to a vertical segment through p_i.

   (b) Assume that $w_i = 0$. This case corresponds to a horizontal segment through p_i. The neighbors of v^j_w are all the vertices $v^{j+1}_{w'}$ that satisfy:

      i. $w'_i = 0$ (since the segment through p_i is horizontal); and

      ii. there are integers $1 \le l < i$ and $i < r \le n+2$ (representing the left and right endpoints of the horizontal segment through p_i) such that:

         A. $w_l = w'_l = w_r = w'_r = 1$;

         B. $w_s = w'_s$ for every $1 \le s < l$ and $r < s \le n+2$;

         C. $w'_s = 0$ for every $l < s < r$ such that p_s is below p_i; and

         D. $w_s = 0$ for every $l < s < r$ such that p_s is above p_i.

      See Figure 4 for an example of the neighbors of a certain vertex according to this rule.

Consequently, the number of rectangulations is the number of paths in G from v_S to v_N. Counting the number of rectangulations in this way can be implemented in $O(n^4 2^n)$ time and $O(n^3 2^n)$ space. For details see Appendix A.2.

3 An Upper Bound on the Number of Rectangulations

In this section we prove the following theorem:

Theorem 1 *The maximum number of rectangulations of n noncorectilinear points (by n segments) is at most $20^n/\binom{n+4}{4}$.*

Proof: Denote by $f(n)$ the maximum number of rectangulations of n points. Let P be a set of n noncorectilinear points within a rectangle R, such that $\Xi(P) = f(n)$, and let x be a rectangulation of (R, P). A *T-junction* is an endpoint of a segment on another segment, or on the boundary. The *degree* of a point $p \in P$ in x is the number of T-junctions on the segment that contains p. For example, the rightmost point in P in Figure 1(a) has degree 2 in the rectangulation of Figure 1(b) and degree 3 in the rectangulation of Figure 1(c). Let n_i^x be the number of points with degree i in x, then clearly $n = \sum_i n_i^x$.

Every segment is bounded by two T-junctions, thus every segment s contributes at most 4 to the total sum of degrees: 2 to the point it contains, and 1 to every point that is contained in a segment bounding s (if it is not a boundary segment). Thus, the total sum of degrees is $4n - b$, where b is the number of T-junctions on the boundary of R in x. It is easy to verify that if $n \geq 3$, then $b \geq 4$. Thus, for $n \geq 3$ we have

$$4n - 4 \geq \sum_i i \cdot n_i^x.$$

Easy manipulations show that

$$\begin{aligned} 4\sum_i n_i^x &\geq 4 + \sum_i i \cdot n_i^x, \\ \sum_i (4-i)n_i^x &\geq 4, \\ \sum_i (5-i)n_i^x &\geq 4 + \sum_i n_i^x = n + 4. \end{aligned}$$

Considering only the positive summands on the left-hand side of the last equation we have:

$$3n_2^x + 2n_3^x + n_4^x \geq n + 4. \tag{1}$$

Now, let $p \in P$ be a certain point and let r' be a rectangulation of $(R, P \setminus \{p\})$. We denote by h_i the number of rectangulations of (R, P) that we obtain by adding p to x' and "stretching" the segment through p such that the degree of p in the resulting rectangulation is i. Clearly, $h_2 = 2$, since the segment through p can be either vertical or horizontal and we must stop "stretching" it as soon as it hits another segment in each direction. In a similar way we have $h_3 \leq 4$ (see Figure 5) and $h_4 \leq 6$ (and in general $h_i \leq 2(i-1)$).

Let N_i be the number of points with degree i in all the rectangulations of (R, P). Then,

$$N_i \leq n \cdot h_i \cdot f(n-1),$$

since any fixed point can be inserted into any of at most $f(n-1)$ rectangulations of the remaining $n-1$ points in at most h_i different ways, such that its degree in the resulting

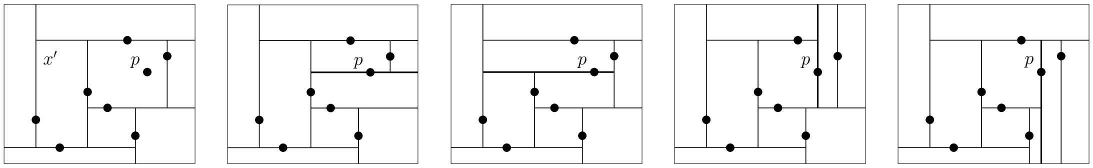


Figure 5: Four possible ways of adding p to r' such that the degree of p is 3

rectangulation is i. Specifically, we have $N_2 \leq 2n \cdot f(n-1)$, $N_3 \leq 4n \cdot f(n-1)$, and $N_4 \leq 6n \cdot f(n-1)$.

We now prove by induction on n that $f(n) \leq 20^n/\binom{n+4}{4}$. For $n = 0, 1, 2$ the claim holds trivially: $f(0) = 1 = 20^0/\binom{4}{4}$, $f(1) = 2 < 4 = 20^1/\binom{5}{4}$, and $f(2) = 6 < 26.666... = 20^2/\binom{6}{4}$. Now assume that the claim holds for all $n' \leq n$, for $n \geq 3$. By summing Equation 1 over all possible rectangulations, we have:

$$3N_2 + 2N_3 + N_4 \geq (n+4)f(n), \tag{2}$$

since we chose P such that $\Xi(P) = f(n)$. On the left-hand side of Equation 2 we have:

$$20n \cdot f(n-1) \leq 20n \frac{20^{n-1}}{\binom{n+3}{4}} = (n+4)\frac{20^n}{\binom{n+4}{4}}.$$

Hence, $f(n) \leq 20^n/\binom{n+4}{4}$, and the claim follows. □

4 Guillotine Rectangulations

In this section we consider a special class of rectangulations: guillotine rectangulations.

Definition 4.1 (Guillotine rectangulation) *In a* guillotine *rectangulation the segments can be ordered so that when the partition is executed according to that order, the current segment always partitions a rectangle into two rectangles.*

For example, the rectangulation in Figure 1(b) is guillotine, whereas the rectangulation in Figure 1(c) is not. In this section we consider the number of guillotine rectangulations. It is easy to see that this number depends only on the number of points in P. Let $\Gamma(n)$ be the number of guillotine rectangulations when $|P| = n$. Clearly, $\Gamma(n)/2$ guillotine rectangulations contain a vertical segment cutting the bounding rectangle into two rectangles, while the remaining $\Gamma(n)/2$ rectangulations contain a horizontal segment cutting the bounding rectangle into two. Considering only the first set and denoting by k the first point left to right, through which passes a vertical segment cutting the bounding rectangle into two, we derive the following recursive formula for $\Gamma(n)$:

$$\Gamma(n)/2 = \Gamma(n-1) + \sum_{k=2}^{n} \left(\frac{1}{2}\Gamma(k-1)\right)\Gamma(n-k),$$

where $\Gamma(0) = 1$ The formula holds since for $k = 1$ there are $\Gamma(n-1)$ guillotine rectangulations, while for $k > 1$ the segment through the kth point splits the bounding rectangle into two rectangles: the right one has $\Gamma(n-k)$ guillotine rectangulations, while the left one has $\Gamma(k-1)/2$ guillotine rectangulation as it must be cut into two by a horizontal segment.

This formula is equivalent to a recursive formula of the nth (large[2]) Schröder number:

$$r_n = r_{n-1} + \sum_{k=0}^{n-1} r_k r_{n-1-k}, \qquad r_0 = 1$$

Thus, we have:

Theorem 2 *Given a rectangle R which encloses a set P of n noncorectilinear points, the number of guillotine rectangulations of (R, P) is the nth Schröder number.*

The Schröder numbers arise in several enumerative combinatorial problems. One example is the number of paths on a grid from $(0,0)$ to (n,n), that stay strictly below the line $y = x+1$ and use only the steps $(1,0)$, $(0,1)$, and $(1,1)$. Other examples can be found in [24, pp. 239–240].

The nth Schröder number, r_n, also satisfies the following summation formula:

$$r_n = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k},$$

where C_n is the nth Catalan number. It can be shown (see, e.g., [22]) that $r_n = \Theta\left((3+\sqrt{8})n/n^{1.5}\right)$. The first Schröder numbers (starting from $n = 0$) are $\{1, 2, 6, 22, 90, 394, 1806, \ldots\}$.

5 The Exact Number of Rectangulations

In this section we investigate $\Xi(P)$—the exact number of rectangulations (guillotine and non-guillotine) of a set P of n noncorectilinear points within a rectangle R. We start by observing that $\Xi(P)$ depends only on the permutation of the points in P. Next, we show that for identity permutations the number of rectangulations equals the $(n+1)$st Baxter number. Finally, we generalize this result for the class of *separable* permutations.

A Baxter permutation on $[n] = \{1, 2, \ldots, n\}$ is a permutation $\pi = (\sigma_1\sigma_2\ldots\sigma_n)$ for which there are no four indices $1 \le i < j < k < l \le n$ such that

1. $\sigma_k < \sigma_i + 1 = \sigma_l < \sigma_j$; or

2. $\sigma_j < \sigma_l + 1 = \sigma_i < \sigma_k$.

For example, for $n = 4$, 3142 and 2413 are the only non-Baxter permutations. This class of permutations was introduced by Baxter [3] in the context of fixed points of the composite of commuting functions. The nth Baxter number, $\mathrm{B}(n)$, is the number of Baxter permutations on $[n]$. Chung et al. [7] proved that

$$\mathrm{B}(n) = \sum_{r=0}^{n-1} \frac{\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}}{\binom{n+1}{1}\binom{n+1}{2}}$$

[2]The nth small Schröder number counts the number of possible bracketing on a word of n letters. For $n > 1$ it is exactly half of the nth large Schröder number.

Dulucq and Guibert [10] showed bijections between Baxter permutations, *twin binary trees*, and some type of three non-intersecting paths on a grid. Shen et al. [22] analyzed the asymptotic behavior of the Baxter numbers and proved that $\mathrm{B}(n) = \Theta(8^n/n^4)$. The first Baxter numbers (starting from $n = 0$) are $\{0, 1, 2, 6, 22, 92, 422, 2074, \ldots\}$.

5.1 Rectangulations and Permutations

Definition 5.1 *Given a set P of noncorectilinear points, we refer to the relative order of the points in P as the permutation of P and denote it by $\pi(P)$.*

Representing the relative order of the points by a permutation $\pi = (\sigma_1\sigma_2 \ldots \sigma_n)$ is feasible since the points are noncorectilinear. By $\sigma_i = j$ we mean that the ith point along the x-axis is the jth point along the y-axis. It is easy to see that given two sets of points, P_1 and P_2, such that $\pi(P_1) = \pi(P_2)$, we always have $\Xi(P_1) = \Xi(P_2)$. In other words, the number of rectangulations does not depend on the actual point coordinates, it depends only on the permutation of points. Therefore, we will also use the notation $\Xi(\pi)$. However, computational enumerations we have performed showed that when $\pi(P_1) \neq \pi(P_2)$ it is possible to have $\Xi(\pi(P_1)) \neq \Xi(\pi(P_2))$. For example, $\Xi(1234) = 92$, whereas $\Xi(3142) = 93$.

5.2 The Number of Rectangulations of Identity Permutations

Lemma 5.2 *Let $\mathcal{I}_n$ be the identity permutation on $[n]$. Then $\Xi(\mathcal{I}) = B(n + 1)$.*

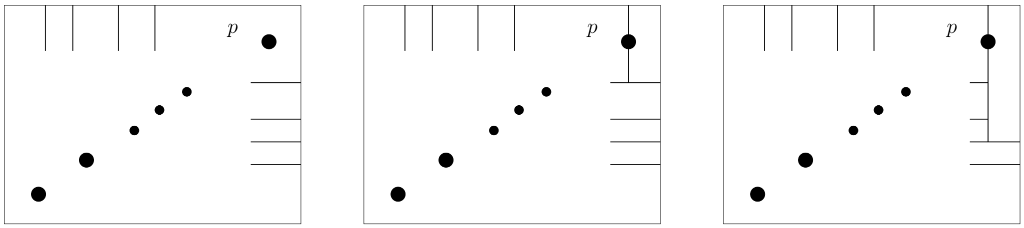
Proof: Given a rectangulation x we denote by *bottom*(x) (resp., *top*(x)) the set of vertical segments touching the bottom (resp., top) edge of the bounding rectangle R. Similarly, *left*(x) (resp., *right*(x)) denotes the set of horizontal segments touching the left (resp., right) edge of R. Let $T_n(i, j)$ be the number of different rectangulations x of n points with the identity permutation, such that $|\mathrm{top}(x)| = i$ and $|\mathrm{right}(x)| = j$. Then we can write the following recurrence relation for $n > 0$:

$$T\left(n+1, i+1, j+1\right) = \sum_{k=1}^{\infty} \left(T\left(n, i, j+k\right) + T\left(n, i+k, j\right)\right), \tag{3}$$

where $T_0(0, 0) = 1$ and $T_n(i, j) = 0$ for $n < 0$. To understand why this relation holds, note that we can create a rectangulation x of $n+1$ points such that $|\mathrm{top}(x)| = i+1$ and $|\mathrm{right}(x)| = j+1$ from a rectangulation x' of n points, such that $|\mathrm{top}(x')| = i$ and $|\mathrm{right}(x')| = j+k$ (for $k \geq 1$), by:

1. Adding an additional point p to the right and above all the points of x';
2. Setting a vertical segment s through p; and
3. Extending s downwards using `Rotate` operations until $k-1$ segments are removed from $\mathrm{right}(x)$.

Figure 6 shows these steps. We can create in a similar way a rectangulation x of $n+1$ points, for which $|\mathrm{top}(x)| = i+1$ and $|\mathrm{right}(x)| = j+1$, from a rectangulation x' of n points, such that



(a) Adding an additional point (b) Setting a vertical segment (c) Extending the vertical segment

Figure 6: From $T_n(i, j+k)$ to $T_{n+1}(i+1, j+1)$

$|\mathrm{top}(x')| = i+k$ (for $k \geq 1$) and $|\mathrm{right}(x')| = j$, by passing a horizontal segment through a new point p. Clearly, every rectangulation x of $n+1$ points can be created from a rectangulation x' of n points as described above, and there are no two different rectangulations x'_1, x'_2 of n points that lead to the same rectangulation of $n+1$ points. Therefore,

$$\Xi(\mathcal{I}_n) = \sum_{i,j \geq 0} T_n(i,j), \tag{4}$$

which is exactly $\mathrm{B}(n+1)$ by [7]. $\square$

5.3 Separable Permutations and Their Number of Rectangulations

In this section we define the class of separable permutations and show that $\Xi(\pi) = \mathrm{B}(n+1)$ if π is a separable permutation.

5.3.1 Separable Permutations

Let $\pi' = (\alpha_1 \alpha_2 \dots \alpha_n)$ and $\pi'' = (\beta_1 \beta_2 \dots \beta_m)$ be two permutations on $[n]$ and $[m]$, respectively. We say that $\pi = (\sigma_1 \sigma_2 \dots \sigma_{n+m})$ is the result of concatenating π'' *above* π' if $\pi_i = \alpha_i$ for $1 \leq i \leq n$ and $\pi_{n+i} = n + \beta_i$ for $1 \leq i \leq m$. Likewise, we say that $\pi = (\sigma_1 \sigma_2 \dots \sigma_{n+m})$ is the result of concatenating π'' *below* π' if $\pi_i = m + \alpha_i$ for $1 \leq i \leq n$ and $\pi_{n+i} = \beta_i$ for $1 \leq i \leq m$.

Definition 5.3 (separable permutation) *A permutation π is a* separable *if either*

1. $\pi = (1)$*; or*
2. *There are two separable permutations π' and π'' such that π is the the concatenation of π'' above or below π'.*

Another characterization of separable permutations is in terms of *forbidden sub-sequences*. A permutation $\pi = (\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n) \in S_n$ *avoids* a certain sub-permutation $\tau \in S_k$ (for $k \leq n$) if it does not contain a sub-sequence $(\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k})$ with the same pairwise comparisons as τ. The set of permutations on $[n]$ avoiding τ is denoted by $S_n(\tau)$. Bose et al. [4] showed that the set of separable permutations is exactly $S_n(3142, 2413)$. Separable permutations are also the permutations that can be sorted by an unbounded sequence of pop-stacks [1] (in a pop-stack the pop operation unloads the entire stack). Shapiro and Stephens [21] showed that permutation matrices that eventually fill up under *bootstrap percolation*, are exactly those matrices representing separable permutations. The next observation follows from their results and the results of West [25]:

Observation 5.4 *The number of separable permutations on $[n]$ is the $(n-1)$st Schröder number.*

5.3.2 The Number of Rectangulations for Separable Permutations

In this section we prove that the number of rectangulations when the points are arranged in a separable permutation is $\mathrm{B}(n+1)$.

Let x be a rectangulation. The *interface* of x, denoted by $\mathcal{F}(x)$, is an ordered quadruple (l,t,r,b), such that $l = |\mathrm{left}(x)|$, $t = |\mathrm{top}(x)|$, $r = |\mathrm{right}(x)|$, and $b = |\mathrm{bottom}(x)|$. We denote by $\Xi(\pi,\mathcal{F})$ the number of rectangulations with permutation π and interface $\mathcal{F}$.

Lemma 5.5 *For every n,l,t,r,b, $\Xi\left(\mathcal{I}_n,(l,t,r,b)\right) = \Xi\left(\mathcal{I}_n,(l,b,r,t)\right)$.*

The proof of this property is not trivial and does not follow from simple symmetry arguments. Since it is rather long and technical, it appears in Appendix B.

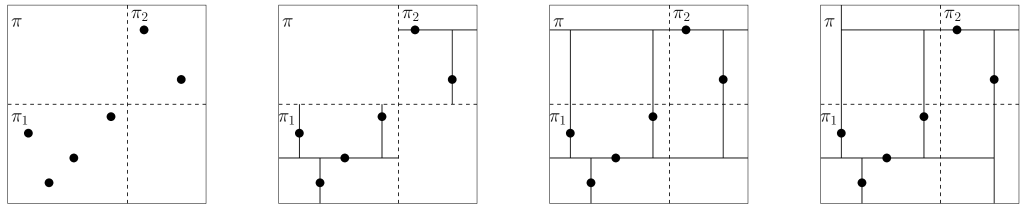
Corollary 5.6 *Let $\overline{\mathcal{I}}_n$ be the reverse identity permutation on $[n]$ $(n,n-1,\ldots,1)$, then for every n,l,t,r,b $\Xi\left(\mathcal{I}_n,(l,t,r,b)\right) = \Xi\left(\overline{\mathcal{I}}_n,(l,t,r,b)\right)$.*

Proof: Let x be a rectangulation of n points in the identity permutation, such that $\mathcal{F}(x) = (l,t,r,b)$. When x is reflected with respect to the x-axis we get a rectangulation x' of n points in the reverse identity permutation, such that $\mathcal{F}(x') = (l,b,r,t)$. The corollary follows directly from this fact and from Lemma 5.5. $\square$

Lemma 5.7 *Let π be a separable permutation of n points. Then for every interface $\mathcal{F}$, $\Xi(\pi,\mathcal{F}) = \Xi(\mathcal{I}_n,\mathcal{F})$.*

Proof: By induction on n. For $n = 1$ a permutation of one point is both the identity permutation and a separable permutation. Assume the claim is true for every separable permutation of $n' < n$ points, and let π be a separable permutation of n points. π may be a concatenation-above or a concatenation-below of two separable permutations. Suppose that π is the result of concatenating a separable permutation $\pi_2 \in S_{n-k}$ above another separable permutation $\pi_1 \in S_k$. Then all the rectangulations of π can be created by considering every pair of a rectangulation of π_1 and a rectangulation of π_2, and by combining every such pair in all the possible combinations (see Figure 7). Note that given x_1 and x_2, rectangulations of π_1 and π_2, respectively, the number of rectangulations of π that are created by combining x_1 and x_2 in all the possible combinations depends only on $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$. Moreover, the interface of every such combined rectangulation also depends only on $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ and the way they were combined.

According to the induction hypothesis, for every pair of interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ we have $\Xi(\pi_1,\mathcal{F}_1) = \Xi(\mathcal{I}_k,\mathcal{F}_1)$ and $\Xi(\pi_2,\mathcal{F}_2) = \Xi(\mathcal{I}_{n-k},\mathcal{F}_2)$. All the rectangulations of $\mathcal{I}_n$ can be



(a) π_2 concatenated above π_1

(b) Rectangulations of π_1 and π_2

(c) One combination of the rectangulations of π_1 and π_2

(d) Another combination of the rectangulations of π_1 and π_2

Figure 7: Rectangulations of a separable permutation

created by combining all the pairs of a rectangulation of $\mathcal{I}_k$ and a rectangulation of $\mathcal{I}_{n-k}$ in all possible combinations. Again, the number of combinations and the interface of every such combined rectangulation depends only on the interfaces of the rectangulations of $\mathcal{I}_k$ and $\mathcal{I}_{n-k}$, and on the way they were combined. Thus, for every concatenation-above separable permutation π and interface $\mathcal{F}$, $\Xi(\pi, \mathcal{F}) = \Xi(\mathcal{I}_n, \mathcal{F})$.

Suppose now that π is the result of concatenating a separable permutation $\pi_2 \in S_{n-k}$ below another separable permutation $\pi_1 \in S_k$. It follows from Corollary 5.6 that for every pair of interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$, $\Xi(\mathcal{I}_k, \mathcal{F}_1) = \Xi(\overline{\mathcal{I}}_k, \mathcal{F}_1)$ and $\Xi(\mathcal{I}_{n-k}, \mathcal{F}_2) = \Xi(\overline{\mathcal{I}}_{n-k}, \mathcal{F}_2)$. Using the induction hypothesis we conclude that for every pair of two interfaces $\mathcal{F}_1$ and $\mathcal{F}_2$, $\Xi(\pi_1, \mathcal{F}_1) = \Xi(\overline{\mathcal{I}}_k, \mathcal{F}_1)$ and $\Xi(\pi_2, \mathcal{F}_2) = \Xi(\overline{\mathcal{I}}_{n-k}, \mathcal{F}_2)$. Then, according to the combination arguments given above and by using Corollary 5.6, for every concatenation-below separable permutation π and interface $\mathcal{F}$, $\Xi(\pi, \mathcal{F}) = \Xi(\overline{\mathcal{I}}_n, \mathcal{F}) = \Xi(\mathcal{I}_n, \mathcal{F})$.

In conclusion, the claim holds for all separable permutations. □

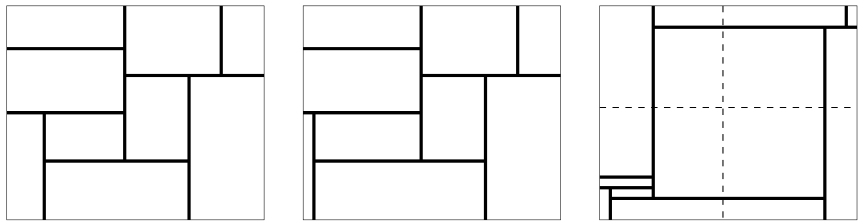
Theorem 3 *Given a rectangle R which encloses a set P of n noncorectilinear points, such that $\pi(P)$ is a separable permutation on $[n]$, $\Xi(R, P) = B(n+1)$.*

Proof: The claim follows from Lemmata 5.2 and 5.7. □

6 Rectangulations and Floorplans

Recall that a "point-free" rectangulation, that is, a subdivision of a rectangle into smaller rectangles by n non-intersecting axis-parallel segments is equivalent to what is known in integrated circuits design as mosaic floorplans [14]. Yao et al. [26] proved that the number of mosaic floorplans by n segments is $\mathrm{B}(n+1)$. In this section we prove that given a set of points P in a separable permutation and a mosaic floorplan f by n segments, there is a unique way of "combining" P and f into a rectangulation.

A mosaic floorplan is characterized by the relations between segments and rectangles it defines: We say that a segment s and a rectangle r in a mosaic floorplan f hold a *top*-, *left*-, *right*-, or *bottom-seg-rect* relation if s supports r from the respective direction. Two floorplans are considered equivalent if there is a labeling of their rectangles and segments such that they hold the same seg-rect relations.



(a) (b) (c)

Figure 8: Illustrations for the proof of Theorem 4.

Theorem 4 *Given a mosaic floorplan f with n segments and a set P of n points arranged in a separable permutation π, there is a unique rectangulation of P, x, such that $x \setminus P$ is equivalent to f.*

Proof: We will show that it is possible to create a rectangulation of a set of points whose permutation is π and its underlying mosaic floorplan is f. It then follows that an equivalent rectangulation can be created for P. Since by Theorem 3 the number of rectangulations of a set of n points in a separable permutation is $\mathrm{B}(n+1)$ and this is also the number of mosaic floorplans with n segments [26], it follows that the combination is unique.

We now prove by induction on n that for every mosaic floorplan f by n segments and a separable permutation $\pi \in S_n$ it is possible to create a rectangulation x of a set of n points whose permutation is π such that the underlying floorplan of x is f. Examining the bottom-left rectangle in f note that its top-right corner is either of the form $\dashv$ or T. In the first case by 'sliding' the horizontal segment creating the '$\dashv$'-junction downwards (resp., upwards) while 'stretching' (resp., 'shrinking') the vertical segments attached to it (if such exist) we create a floorplan which is equivalent to f. In the second case one can slide the vertical segment of the 'T'-junction leftwards or rightwards. Note that if a segment is shifted until it hits the boundary then we obtain a mosaic floorplan of $n-1$ segments.

Now suppose π can be formed by concatenating the permutation $\pi_2 \in S_{n-k}$ above the permutation $\pi_1 \in S_k$. We create two sets of the segments of f in the following manner. We start by shrinking the bottom-left rectangle in f by sliding one of its edges as described above. We stop sliding this edge when it is in a small distance $\epsilon > 0$ from the boundary (see Figures 8(a) and 8(b)), but we notice that if this edge vanishes in the boundary then it is possible to slide another segment in a similar manner.

We continue by sliding this segment until it is within a distance of 2ϵ from the boundary. Likewise, we slide each of the next $k-2$ segments: the ith segment is shifted either leftwards or downwards until it is within $i\epsilon$ distance from the boundary. This ensures we maintain a valid floorplan equivalent to f.

In a similar way we can 'group' the other $n-k$ segments near the top-right corner of f. See Figure 6 for illustrations of this process. Now divide f into four parts by drawing a vertical and a horizontal line through its center. Every segment in the top-left and the bottom-right parts is partly contained in either the bottom-left or the top-right parts as well. Additionally, the bottom-left part is actually a floorplan with k segments whereas the top-right part is a floorplan with $n-k$ segments. By induction it is possible to embed a set of k points whose permutation is π_1 into the first floorplan, and a set of $n-k$ points whose permutation is π_2 into

n	B$(n+1)$	Minimum number of rectangulations	Maximum number of rectangulations
4	92	93	93
5	422	424	428
6	2,074	2,080	2,122
7	10,754	10,776	11,092
8	58,202	58,290	60,524
9	326,240	326,608	342,938

Table 1: Empirical results of the number of rectangulations for non-separable permutations

the second floorplan. Therefore it is possible to embed a set of n points whose permutation is π into f.

The case in which π is concatenation-below permutation is handled in a similar manner (this time the segments are grouped at the top-left and bottom-right corners). □

7 Conclusions

We showed that the number of rectangulations (by n segments) of a set P of n noncorectilinear points depends only on the permutation in which the points are arranged. For any arrangement the number of guillotine rectangulations is always the nth Schröder number and the total number of rectangulation is $O(20^n/n^4)$.

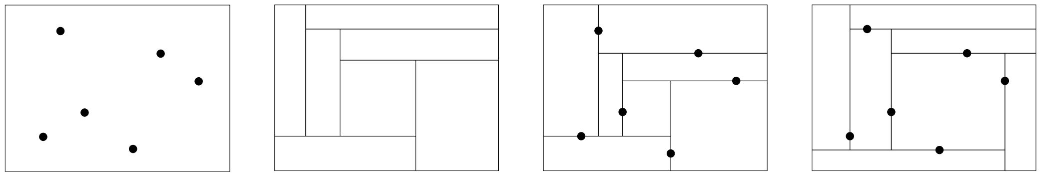
For point sets in a separable permutation we proved that the number of rectangulations is the $(n+1)$st Baxter number. Moreover, for every mosaic floorplan f with n segments there is a unique way to embed a set of n points, arranged in a separable permutation, in f. This strengthens a result of de Fraysseix et al. [8]: they showed that every bipartite planar graph can be represented as the contact graph[3] of a set of non-intersecting vertical and horizontal segments in the plane. It follows from our results that given a set P of n noncorectilinear points in the plane, arranged in a separable permutation, and a planar bipartite graph $G = (V, E)$ such that $|V| = n$, then it is possible to represent G as a contact graph of a set S of n vertical and horizontal segments such that every segment in S contains a single point from P.

Counting the number of rectangulations for non-separable permutations is still an open question. Our computations have led us to the following conjecture:

Conjecture 7.1 *Given a set P of n noncorectilinear points, such that $\pi(P)$ is a non-separable permutation on $[n]$, $\Xi(P) > B(n+1)$. Moreover, there is at least one way of embedding P in any mosaic floorplan containing n segments.*

For example, when $n = 4$ there are two non-separable permutations (3142 and 2413), and for both of them (not surprisingly, since one is the reverse of the other) the number of rectangulations is 93 (as opposed to B(5) = 92 for separable permutations). For $n = 5$ the number of rectangulations varies from 424 to 428 (as opposed to B(6) = 422 for separable permutations), but some values appear and some do not. Our empirical results are listed in Table 1.

[3]In a contact graph there is an edge between two touching elements.



(a) A set of points P in a non-separable permutation $\pi = 263154$

(b) A mosaic floorplan f

(c) A possible embedding of P into f

(d) A different embedding of P into f

Figure 9: Two possible embeddings for a non-separable permutation.

Perhaps the extra number of rectangulations for non-separable permutations can be computed by counting the number of different ways in which they can be embedded in some mosaic floorplans. Figure 9 shows, for example, two possible ways of embedding a set of points in a non-separable permutation into a certain mosaic floorplan.

Other questions of interest are:

1. Improve the upper bound of $O(20^n/n^4)$, perhaps to $O(16^n/n^4)$ by showing that for every mosaic floorplan and a set of points once the orientations of the segments through every point are set then there is at most one way of embedding the points into the floorplan.

2. What is the number of rectangulations when the problem is generalized to higher dimensions?

3. The original minimum edge-length partitioning problem (RGNLP). Furthermore, what is its computational complexity when restricted to monotone (or separable) permutations?

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A Enumeration-related Implementation Issues

A.1 Enumerating Rectangulations by Reverse Search

In Section 2.1 we described the `Flip` and `Rotate` operators and proved that the graph of rectangulations (of a set of points P) defined by these operations is connected. Thus, the number of rectangulation can be computed by a standard DFS (or BFS) on this graph. In this section we describe a more efficient way of traversing the graph of rectangulations. It is based on a method of Avis and Fukuda [2] known as *reverse search*. The key observation is that in order to visit all the vertices (rectangulations) of a graph, it is enough to traverse a *spanning tree* of the graph. This saves time (we do not explore all the edges) and space (there is no need to keep a record of the already-visited vertices).

Given a set of points P, let r_h be the rectangulation of P in which all the segments are horizontal. For every rectangulation except r_h we designate one of its neighbors to be its "parent" in such a way that every rectangulation is a descendent of r_h. The parent of a rectangulation $r \neq r_h$ is defined as follows: Let s be the leftmost vertical segment in r. If the operator `Flip` can be applied on s, then the result of applying it is the parent of s. Otherwise, we can apply the `Rotate` operator and shorten s. If we can shorten s from below, then the rectangulation we get as a result is the parent of r. Otherwise, the rectangulation we get by shortening s from above is the parent of r. Clearly, every rectangulation (except r_h) has a parent, and every rectangulation is a descendent of r_h.

In order to find the children of a certain rectangulation r we can keep a pointer to the leftmost vertical segment in r, and a sorted list of flippable horizontal segments that are to the left of it (that is, segments that pass through points which are left of the vertical segment). The children of r are obtained by either:

1. Flipping one of the horizontal segments. In this case the flipped segment becomes the leftmost vertical segment and the list of flippable horizontal segments is the list of flippable horizontal segments to the left of it that do not contain endpoints of the flipped segment.
2. Extending the leftmost vertical segment using a `Rotate` operation (downwards if it is possible, or upwards if it is possible and it is impossible to shorten it from below by a `Rotate` operation). In this case the leftmost vertical segment remains the same. Additionally, at most one segment is added to the list of flippable horizontal segments and at most one segment is removed from this list.

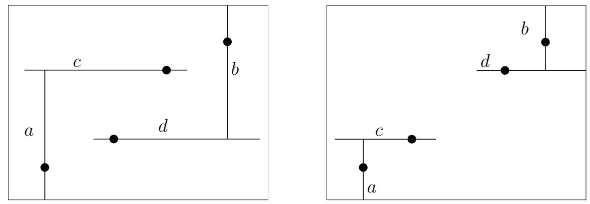
Updating the sorted list of segments can be performed in $O(\log n)$ using a (slightly modified) deterministic skip list [18]. Therefore, the time complexity of enumerating (by generating) all the rectangulations is $O(\Xi(P) \log n)$. The depth of the spanning tree is bounded by $O(n^2)$, since, when traversing from parent to child, the leftmost vertical segment is either extended or replaced by a vertical segment to the left of it. Thus, the space complexity is $O(n^3)$.

A.2 Faster Enumeration of Rectangulations

This section refers to the faster enumeration method described in Section 2.2.

Lemma A.1 *Given a set of n points P, let $G = (V, E)$ be the corresponding DAG of rectangulations. Then, $|E| = \Theta(n^3 2^n)$.*

Proof: Let p_{j+1} be the jth point left-to-right and the kth point bottom-to-top, and let e_k be the number of edges of the form $(v_w^k \rightarrow v_{w'}^{k+1})$. Then $e_k = 2^{n-1} + (j-1)(n-j)2^{n-3} + (j-1)2^{n-2} + (n-j)2^{n-2} + 2^{n-1}$. The first summand stands for all the words w in which the jth bit is set (i.e., there is a vertical segment through p_{j+1}), from which there is only one out-edge. The second summand represents all the cases in which the endpoints of the horizontal segment through p_{j+1} are set by coordinates of other points



(a) c is not lower than d (b) c is lower than d

Figure 10: Illustrations for the proof of Proposition B.1

from P: one to the left of p_{j+1} and the other to the right of p_{j+1}. There are $(j-1)(n-j)$ options to choose such a pair and 2^{n-3} options to set the other bits in w and w' (if a bit in w can be either 0 or 1, then the corresponding bit in w' has only one option, and vice versa). The rest of the summands are for the cases in which one or two of the endpoints of the horizontal segment through p_{j+1} are on the bounding rectangle. Therefore,

$$|E| = 2 \cdot 2^n + \sum_{k=1}^{n} e_k = \Theta(n^3 2^n).$$

$\square$

Constructing G takes $O(n^2 2^n + n|E|)$ time since computing the neighbors of every vertex v_w^j can be performed in $O(n + d_{\text{out}}(v_w^j))$ time. Computing the number of paths from a source vertex to a sink vertex in a DAG takes $O(|E|)$ time. Since every vertex is represented by an n-bit word,[4] the time complexity of this enumeration algorithm is $O(n^4 2^n)$. Considering the space complexity, note that the DAG is composed of $n+3$ "levels" and the edges are only between consecutive levels. Thus, it is enough to hold in memory only two consecutive levels and therefore the space complexity is $O(n^3 2^n)$.

B Proof of Lemma 5.5

In this section we consider only point sets that are arranged in the identity permutation. Let x be a rectangulation of a point set P that lies within a rectangle R. We denote by $\mathcal{H}(x)$ (resp.,$\mathcal{V}(x)$) the set of segments in x touching both vertical (resp., horizontal) edges of R. Clearly, $\mathcal{H}(x) \neq \emptyset$ implies $\mathcal{V}(x) = \emptyset$ and $\mathcal{V}(x) \neq \emptyset$ implies $\mathcal{H}(x) = \emptyset$.

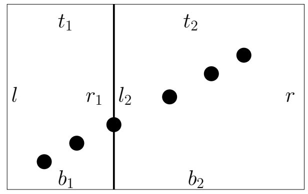
Given a rectangulation x, we call a pair of segments $s_1 \in \text{top}(x)$ and $s_2 \in \text{bottom}(x)$, such that s_1 is to the left of s_2, *$\natural$-segments*. If, in addition, there is no other segment $s \in \text{top}(x) \cup \text{bottom}(x)$ between s_1 and s_2, we say that s_1 and s_2 are *adjacent* $\natural$-segments. The next observation will be useful in the sequel.

Proposition B.1 *Given a rectangulation x of (R, P), such that $\pi(P) = \mathcal{I}_n$ and $\mathcal{H}(x) = \mathcal{V}(x) = \emptyset$:*

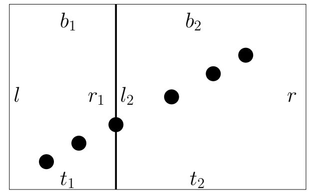
1. *There is a pair of $\natural$-segments in x; or*
2. *There are segments $s_1 \in$ left(x) and $s_2 \in$ right(x) such that s_1 is above s_2.*

Proof: $\mathcal{H}(x) = \emptyset$ implies that $\text{top}(x) \neq \emptyset$ and $\text{bottom}(x) \neq \emptyset$. Suppose that there is no pair of $\natural$-segments in x. That is, all the segments in top(x) are to the right of all the segments in bottom(x). Let a be the rightmost segment in bottom(x), and let b be the leftmost segment in top(x). Let c and d be the horizontal segments terminating a and b, respectively (there must be such segments since $\mathcal{V}(x) = \emptyset$). Suppose further that the height of c is at least the height of d (see Figure 10(a)). Then,

[4] Likewise, a factor of $\log n$ should be added in the analysis of the previous algorithm; however, we follow the common practice and omit this factor.



(a) x



(b) $\psi(x)$

Figure 11: $\psi(x)$ when $\mathcal{V}(x) \neq \emptyset$

as we now show, there must be a horizontal segment in left(x) whose height is at least the height of c. We traverse c to the left. If we reach the left edge of R, then c is the sought segment. Otherwise, we reach a vertical segment e that terminates c. It must be that $e \notin$ top(x), since all the segments in top(x) are to the right of a. Therefore, there is a horizontal segment f that terminates e from above. We proceed this way leftward and upward until we reach the left edge of R. Thus, there is a segment $s_1 \in$ left(x) which is not lower than c. Using the same arguments one can show that there exists a segment $s_2 \in$ right(x) which is not higher than d. Thus, s_1 and s_2 are the segments we seek.

The other case in which c is lower than d, (see Figure 10(b)) is handled in an similar manner. The claim follows. □

Proposition B.2 *Let X be the set of all the rectangulations of (R, P) when $\pi(P) = \mathcal{I}_n$. Then there is a mapping $\psi : X \to X$ such that for every rectangulation $x \in X$:*

1. *$|\mathcal{H}(x)| = |\mathcal{H}(\psi(x))|$ and $|\mathcal{V}(x)| = |\mathcal{V}(\psi(x))|$; and*
2. *if $\mathcal{F}(x) = (l, t, r, b)$ then $\mathcal{F}(\psi(x)) = (l, b, r, t)$.*

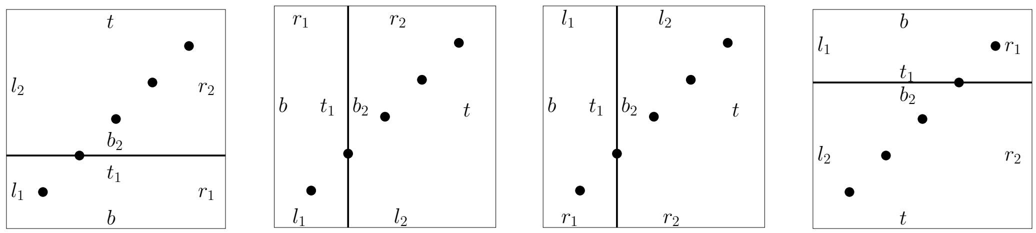
According to the these properties, $\psi(x)$ has the same number of segments crossing from left to right and from bottom to top as x, and the same interface as x except the numbers of top-touching and bottom-touching segments which are interchanged. Note that these properties are not trivial and do not follow from simple symmetry arguments.

Proof: We will build such a mapping by induction on n. When $n = 1$ there are only two possible rectangulations, each one corresponding to itself. Assume that such a mapping ψ exists for all the rectangulations of $n' < n$ points. Let x be a rectangulation of n points arranged in the identity permutation, such that $\mathcal{F}(x) = (l, t, r, b)$. There are three cases:

1. $\mathcal{V} \neq \emptyset$;
2. $\mathcal{H} \neq \emptyset$; or
3. $\mathcal{V} = \mathcal{H} = \emptyset$.

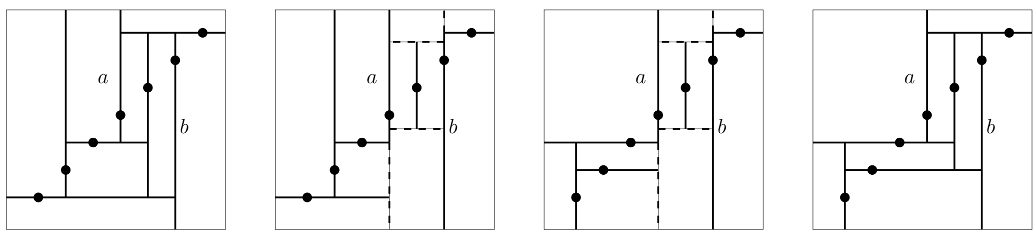
We now describe $\psi(x)$ in each of these cases.

1. $\mathcal{V}(x) \neq \emptyset$. Let s be the leftmost segment in $\mathcal{V}(x)$. We find the corresponding rectangulations for the points to the left and to the right of s, and concatenate them to create $y = \psi(x)$ (see Figure 11). Clearly, $\mathcal{F}(y) = (l, b, r, t)$, $|\mathcal{H}(y)| = |\mathcal{H}(x)|$, and $|\mathcal{V}(y)| = |\mathcal{V}(x)|$.
2. $\mathcal{H}(x) \neq \emptyset$. Let s be the lowest segment in $\mathcal{H}(x)$. Let x' be the rectangulation we get when we reflect x with respect to the primary diagonal (along the points). The points of x' are arranged in the identity permutation, $\mathcal{V}(x') \neq \emptyset$ and $\mathcal{F}(x') = (b, r, t, l)$, thus x' qualifies for the previous case. Let y be the rectangulation we get when reflecting $\psi(x')$ with respect to the secondary diagonal. Clearly, $\mathcal{F}(y) = (l, b, r, t)$, $|\mathcal{H}(y)| = |\mathcal{H}(x)|$, and $|\mathcal{V}(y)| = |\mathcal{V}(x)|$. See Figure 12 for an illustration of these steps.



(a) x (b) $x' := x$ reflected w.r.t. the primary diagonal (c) $\psi(x')$ (d) $\psi(x) := \psi(x')$ reflected w.r.t. the secondary diagonal

Figure 12: $\psi(x)$ when $\mathcal{H}(x) \neq \emptyset$



(a) x (b) Divide x into two sub-problems (c) Solve the sub-problems (d) Combine the partial rectangulations

Figure 13: $\psi(x)$ when x contains $\natural$-segments

3. $\mathcal{H}(x) = \mathcal{V}(x) = \emptyset$. In this case there are two subcases:
   (a) There is a pair of $\natural$-segments in x; or
   (b) There is no such pair of segments.

   Proposition B.1 guarantees that in the second subcase there are segments $s_1 \in \text{left}(x)$ and $s_2 \in \text{right}(x)$, such that s_1 is higher than s_2. By following the same series of steps described above (see Figure 12), we can reduce this subcase to the first subcase.

   Let us, then, consider the first subcase of the current case. Let (a, b) be the leftmost pair of adjacent $\natural$-segments. Let $x(-a)$ be the rectangulation induced by the points to the left of a, and let $x(a-b)$ and $x(b-)$ be the rectangulations induced by the points between a and b, and the points to the right of b, respectively. We construct $\psi(x)$ by concatenating $\psi(x(-a))$, $x(a-b)$, and $\psi(x(b-))$. However, since a and b do not cut R we need to "combine" $\text{right}(\psi(x(-a)))$ with $\text{bottom}(x(a-b)) \cup \{a, b\}$, and $\text{top}(x(a-b)) \cup \{a, b\}$ with $\text{left}(\psi(x(b-)))$ in order to create a valid rectangulation. Here are the details of this combination: Suppose the ith (bottom to top) segment in $\text{right}(x(-a))$ is terminated by the jth (left to right) segment in $\text{bottom}(x(a-b)) \cup \{a, b\}$. We stretch the ith segment in $\text{right}(\psi(x(-a)))$ until the jth segment in $\text{bottom}(x(a-b)) \cup \{a, b\}$, and vice versa. We do the same in order to combine $\text{top}(x(a-b)) \cup \{a, b\}$ with $\text{left}(\psi(x(b-)))$. The result is a rectangulation y such that $\mathcal{F}(y) = (l, b, r, t)$ and $\mathcal{V}(y) = \mathcal{H}(y) = \emptyset$. Figure 13 shows an example of the steps in this case.

□

It is not hard to prove the next property of ψ (e.g., by induction on the number of points).

Observation B.3 *ψ preserves pairs of adjacent $\natural$-segments (although their dimensions might change) and does not introduce such new pairs.*

Proposition B.4 *ψ is one-to-one.*

Proof: We show the claim by induction on n. For $n = 1$, ψ is one-to-one. Let us assume that ψ is one-to-one for every $n' < n$. Let x_1 and x_2 be two different rectangulations of n points, and let $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. We consider the different cases as in the definition of ψ.

1. $\mathcal{V}(x_1) \neq \emptyset$. If $\mathcal{V}(x_2) = \emptyset$ then clearly $y_1 \neq y_2$ since by Proposition B.2 we have $\mathcal{V}(y_1) \neq \emptyset$ and $\mathcal{V}(y_2) = \emptyset$. Otherwise, if $\mathcal{V}(x_2) \neq \emptyset$, then if the leftmost vertical segment in $\mathcal{V}(x_2)$ is different from the leftmost vertical segment in $\mathcal{V}(x_1)$, then $y_1 \neq y_2$ since applying ψ on a rectangulation x does not change the leftmost vertical segment in $\mathcal{V}(x)$. If the same segment is the leftmost segment both in $\mathcal{V}(x_1)$ and $\mathcal{V}(x_2)$, then we can conclude by the induction hypothesis.
2. $\mathcal{H}(x_1) \neq \emptyset$. This case is similar to the previous case, and is thus omitted.
3. $\mathcal{H}(x_1) = \mathcal{V}(x_1) = \emptyset$. As in the definition of ψ, in this case we consider two subcases:
   (a) There is a pair of $\natural$-segments in x; or
   (b) There is no such pair of segments.

   According to Observation B.3, if x_1 contains $\natural$-segments and x_2 does not, then $y_1 \neq y_2$. Assume that (a_1, b_1) and (a_2, b_2) are the leftmost pairs of adjacent $\natural$-segments in x_1 and x_2, respectively, and let p_1, q_1, p_2, q_2 be the points through which a_1, b_1, a_2, b_2 pass, respectively. If $p_1 \neq p_2$ or $q_1 \neq q_2$, then by Observation B.3 $y_1 \neq y_2$. Otherwise, one of the induced rectangulations in x_1 must be different from its corresponding induced rectangulation in x_2, or x_1 and x_2 are different in the way the induced rectangulations are "combined." In the first case, it follows from the induction hypothesis and the definition of ψ that $y_1 \neq y_2$. In the second case, after applying ψ on the induced rectangulations of x_1 and x_2, they are "combined" in a similar way as in x_1 and x_2, therefore again we have $y_1 \neq y_2$.

   The second subcase is handled in a similar manner, and is thus omitted.

□

Corollary B.5 (Lemma 5.5) *For every n, l, t, r, b, $\Xi\left(\mathcal{I}_n, (l,t,r,b)\right) = \Xi\left(\mathcal{I}_n, (l,b,r,t)\right)$.*