On Grids in Topological Graphs

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Abstract

A *topological graph* G is a graph drawn in the plane with vertices represented by points and edges represented by continuous arcs connecting the vertices. If every edge is drawn as a straight-line segment, then G is called a *geometric graph*. A k*-grid* in a topological graph is a pair of subsets of the edge set, each of size k, such that every edge in one subset crosses every edge in the other subset. It is known that every n-vertex topological graph with no k-grid has $O_k(n)$ edges.

We conjecture that the number of edges of every n-vertex topological graph with no k-grid such that all of its $2k$ edges have distinct endpoints is $O_k(n)$. This conjecture is shown to be true apart from an iterated logarithmic factor $\log^* n$. A k-grid is *natural* if its edges have distinct endpoints, and the arcs representing each of its edge subsets are pairwise disjoint. We also conjecture that every n-vertex geometric graph with no natural k-grid has $O_k(n)$ edges, but we can establish only an $O_k(n \log^2 n)$ upper bound. We verify the above conjectures in several special cases.

1 Introduction

The *intersection graph* of a set $\mathcal{C}$ of geometric objects has vertex set $\mathcal{C}$ and two objects are connected by an edge if and only if their intersection is nonempty. The problems of finding a maximum independent set and a maximum clique in the intersection graph of geometric objects have received considerable attention in the literature due to their applications in VLSI design [10], map labeling [1], frequency assignment in cellular networks [13], and elsewhere. Here we study the intersection graph of the edge set of graphs that are drawn in the plane. It is known that if such an intersection graph does not contain a large complete bipartite subgraph, then the number of edges in the original graph is small [8, 16]. We show that this statement remains true under much weaker assumptions.

A *topological graph* G is a graph drawn in the plane with points as vertices and edges as Jordan arcs between these vertices. We further assume that (1) no arc passes through any vertex different from its endpoints, (2) every pair of edges have only finitely many interior points in common, and (3) at each of these points the two edges properly cross. We only consider graphs without parallel edges or self-loops. A topological graph is *simple* if every pair of its edges intersect in at most one

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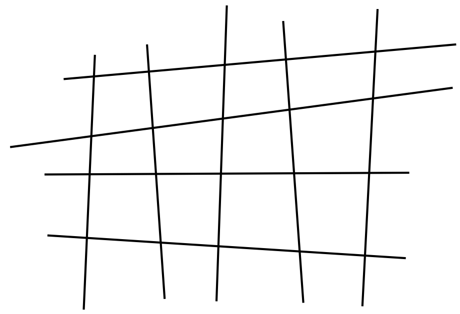


Figure 1: a "natural" grid

point, which is either a common endpoint or a proper crossing. If the edges are drawn as straight-line segments, then G is called a *geometric graph*.

Given a topological graph G, the intersection graph of its edge set is an abstract graph with vertex set $E(G)$, where two elements of $E(G)$ are connected by an edge if and only if they cross each other (a common endpoint of two edges does not count as a crossing). A complete bipartite subgraph in the intersection graph of $E(G)$ corresponds to a *grid* structure in G.

Definition 1.1 *A (k,l)-*grid *in a topological graph is a pair of edge subsets E_1, E_2 such that $|E_1| = k$, $|E_2| = l$, and every edge in E_1 crosses every edge in E_2. A k-*grid *is an abbreviation for a (k,k)-grid.*

Theorem 1.2 ([16]) *For given integers $k, l \geq 1$, there exists a constant $c_{k,l}$ such that any topological graph on n vertices with no (k,l)-grid has at most $c_{k,l}n$ edges.*

The proof of Theorem 1.2 in [16] actually guarantees that a graph with many edges must contain a grid in which all the edges of one of the subsets are adjacent to a common vertex. For two recent and different proofs of Theorem 1.2, see [9] and [8]. Tardos and Tóth [21] extended the result in [16] by showing that there is a constant c_k such that a topological graph on n vertices and at least $c_k n$ edges must contain three subsets of k edges each, such that every pair of edges from different subsets cross, and for two of the subsets all the edges within the subset are adjacent to a common vertex.

Note that, according to Definition 1.1, the edges within each subset of the grid are allowed to cross or share a common vertex, as is indeed required in the proofs of [16] and [21]. When we thinks of a "grid", usually a drawing similar to Figure 1 comes to our mind. Here the edges participating in the grid form a matching and each of the two edge sets consists of disjoint edges. More precisely, we define a *natural (k,l)-grid* in a topological graph G, as a pair of subsets $E_1, E_2 \subset E(G)$ with $|E_1| = k, |E_2| = l$ such that all $2(k+l)$ endpoints of $E_1 \cup E_2$ are distinct, the edges in E_1 are pairwise disjoint, and the edges in E_2 are pairwise disjoint.

Conjecture 1.3 *For given integers $k, l \geq 1$ there exists a constant $c_{k,l}$, such that any simple topological graph G on n vertices with no natural (k,l)-grid has at most $c_{k,l}n$ edges.*

Note that it is already not easy to show that an n-vertex geometric graph with no k pairwise disjoint edges has $O_k(n)$ edges (see [18] and [22]). Moreover, it is an open question whether a simple topological graph on n vertices and no k disjoint edges has $O_k(n)$ edges (the best upper bound, due to Pach and Tóth [17], is $O_k(n \log^{4k-8} n)$). Therefore, probably it is not an easy task to prove

Conjecture 1.3. Here we establish the following upper bounds for the number of edges of geometric and simple topological graphs that contain no natural k-grids.

Theorem 1.4

(i) Every n-vertex geometric graph with no natural k-grid has $O(k^2 n \log^2 n)$ edges.

(ii) Every n-vertex simple topological graph with no natural k-grid has $O_k(n \log^{4k-6} n)$ edges.

We phrased Conjecture 1.3 only for *simple* topological graphs, because it is false without this assumption. Indeed, one can draw the complete graph as a topological graph so that every pair of its edges intersect. (It can be done even in such a way that every pair of edges cross at most twice [17]). Therefore, for topological graphs we have to strengthen the condition by excluding all grids with distinct endpoints.

Conjecture 1.5 *For given integers $k, l \geq 1$, there exists a constant $c_{k,l}$ such that every n-vertex topological graph that contains no (k,l)-grid with distinct endpoints has at most $c_{k,l}n$ edges.*

However, we were able to prove only a slightly weaker upper bound (note also that the constant that we obtain is just barely superpolynomial in k).

Theorem 1.6 *Every n-vertex topological graph that contains no k-grid with distinct vertices has at most $c_k n \log^* n$ vertices, where $c_k = k^{O(\log\log k)}$ and $\log^*$ is the iterated logarithm function.*

We also settle Conjectures 1.3 and 1.5 in some special cases.

Theorem 1.7 *Every n-vertex topological graph that contains no $(k,1)$-grid with distinct vertices has $O_k(n)$ edges.*

Every n-vertex topological graph with no $(1,1)$-grid is planar, and hence has at most $3n-6$ edges, for $n > 2$. We prove Conjecture 1.3 for the first nontrivial case.

Theorem 1.8 *Every n-vertex simple topological graph with no natural $(2,1)$-grid has $O(n)$ edges.*

Many extremal problems on geometric graphs become easier for *convex* geometric graphs— geometric graphs whose vertices are in convex position. Indeed, it was already pointed out by Klazar and Marcus [11] that it is not hard to modify the proof of the Marcus-Tardos Theorem [14] and obtain a linear bound for the number of edges in an *ordered* graph that does not contain a certain ordered matching (see [11] for more details). Since the set of crossings in a convex geometric graph is completely determined by the order of the vertices, this linear bound also settles Conjecture 1.3 for convex geometric graphs.

Corollary 1.9 *For a given integer $k \geq 1$, there exists a constant c_k such that any convex geometric graph on n vertices with no natural k-grid has at most $c_k n$ edges.*

Note that the constants c_k appearing in different theorems are unrelated positive numbers. The constant c_k in Corollary 1.9 is huge. Using different techniques, we give improved upper bounds for the number of edges in convex geometric graphs avoiding natural $(2,1)$-, $(2,2)$-, or $(k,1)$-grids.

Organization. The rest of this paper is organized as follows. Theorems 1.4 and 1.6 are proved in Sections 2 and 3, respectively. In Section 4, we consider the special cases of Conjectures 1.3 and 1.5. We systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs. All logarithms in this paper are base 2.

2 Graphs with no natural grids

In this section, we consider natural grids in geometric and simple topological graphs and prove Theorem 1.4.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum possible number of crossing edges in a drawing of G. The *bisection width*, denoted by $b(G)$, is defined for every simple graph G with at least two vertices. It is the smallest nonnegative integer such that there is a partition of the vertex set $V = V_1 \dot\cup V_2$ with $\frac{1}{3} \cdot |V| \le V_i \le \frac{2}{3} \cdot |V|$ for $i = 1, 2$, and $|E(V_1, V_2)| = b(G)$. Pach et al. [15] proved the following relationship between the crossing number and the bisection width of a graph.

Lemma 2.1 ([15]) *If G is a graph with n vertices of degrees $d_1, \ldots, d_n$, then*

$$b(G) \le 7\mathrm{cr}(G)^{1/2} + 2\sqrt{\sum_{i=1}^{n} d_i^2}.$$

Let m be the number of edges in G. Since $\sum_{i=1}^{n} d_i = 2m$, $d_i \le n$ for every i, and $2\sqrt{2} < 3$, we have

$$b(G) \le 7\mathrm{cr}(G)^{1/2} + 3\sqrt{mn}. \tag{1}$$

Theorem 1.4(i) gives an upper bound on the number of edges of an n-vertex geometric graph with no natural k-grid. The proof of this theorem is by induction on the number of vertices. In the case the bisection width of the considered graph is small, the induction hypothesis applied to each part gives the desired bound on the number of edges. Otherwise, the bisection width is large, and we conclude from (1) that the crossing number of the considered graph is large. In this case, there must be many pairs of crossing edges, and we will apply to the intersection graph of the edges the following lemma, which is tight apart from the constant factor.

Lemma 2.2 ([8]) *For every positive integer p, there is a constant c_p such that if t is a positive integer and H is a graph with n vertices, at least $c_p tn$ edges, and is an intersection graph of curves in the plane such that each pair of curves intersects in at most p points, then H contains the complete bipartite graph $K_{t,t}$ as a subgraph.*

We will only need to use the case $p = 1$. The last tool we use is an upper bound on the number of edges of a geometric graph with no k pairwise disjoint edges.

Lemma 2.3 ([22]) *Any geometric graph with n vertices and no k pairwise disjoint edges has at most $2^9(k-1)^2 n$ edges.*

We now prove Theorem 1.4(i). As the proofs of (i) and (ii) are very similar, we only give the details for (i) and discuss how to modify it to obtain (ii).

Proof of Theorem 1.4(i). Let $g_k(n)$ be the maximum number of edges of a geometric graph with n vertices and no natural k-grid. Let G be a geometric graph on n vertices with $m = g_k(n)$ edges and no natural k-grid. Let $c = \max(2^{20}c_1, 144)$, where c_1 is the constant with $p = 1$ from Lemma 2.2. We prove by induction on n that $g_k(n) \leq ck^2 n \log^2 n$. Suppose for contradiction that $g_k(n) > ck^2 n \log^2 n$. Let $\epsilon = 10^{-3} \log^{-2} n$. The proof splits into two cases, depending on whether or not the number of pairs of crossing edges of G exceeds ϵm^2.

Case 1: The number of pairs of crossing edges is smaller than ϵm^2. Then the crossing number of G is less than ϵm^2. By (1), there is a partition $V(G) = V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2n/3$ and the number of edges with one vertex in V_1 and one vertex in V_2 is

$$\begin{aligned} b(G) &\leq 7\mathrm{cr}(G)^{1/2} + 3\sqrt{mn} \\ &\leq 7\epsilon^{1/2} m + 3\sqrt{mn} = (7\epsilon^{1/2} + 3\sqrt{n/m})m. \end{aligned}$$

Let $n_1 = |V_1|$ and $n_2 = |V_2|$, so $n = n_1 + n_2$. Then we have

$$\begin{aligned} m = g_k(n) &\leq g_k(|V_1|) + g_k(|V_2|) + b(G) \\ &\leq g_k(n_1) + g_k(n_2) + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq ck^2 n_1 \log^2 n_1 + ck^2 n_2 \log^2 n_2 + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq ck^2 n \log^2(2n/3) + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq \left(ck^2 n \log^2 n\right)\left(1 - \frac{1}{2\log n}\right) + (7\epsilon^{1/2} + 3\sqrt{n/m})m. \end{aligned}$$

This implies

$$\begin{aligned} g_k(n) = m &\leq ck^2 n \log^2 n \left(\frac{1 - (\log^{-1} n)/2}{1 - 7\epsilon^{1/2} - 3\sqrt{n/m}}\right) \\ &< ck^2 n \log^2 n \left(\frac{1 - (\log^{-1} n)/2}{1 - (\log n)^{-1}/4 - 3c^{-1/2}k^{-1}\log^{-1} n}\right) \\ &\leq ck^2 n \log^2 n, \end{aligned}$$

where we use $3c^{-1/2}k^{-1} \leq 1/4$. This completes the proof in this case.

Case 2: The number of pairs of crossing edges is at least ϵm^2. Consider the intersection graph H of the edges of G where two edges are adjacent if they cross. Since H has m vertices and at least ϵm^2 edges, and each pair of edges of G intersect at most once, Lemma 2.2 implies it contains a complete bipartite graph with parts of size

$$t \geq \frac{\epsilon m}{c_1} \geq \frac{(\log n)^{-2} m}{1000 c_1} > \frac{c}{1000 c_1} k^2 n > 2^9 k^2 n,$$

where c_1 is the constant with $p = 1$ from Lemma 2.2. Therefore, G contains edge subsets E_1, E_2 with $|E_1| = |E_2| = t$ and every edge in E_1 crosses every edge of E_2, i.e., G contains a t-grid. Since $t > 2^9 k^2 n$, Lemma 2.3 implies that E_i contains k disjoint edges for $i = 1, 2$. Also, as the edges are segments, any pair of edges incident to the same vertex do not cross, and hence no edge in the first subset shares a vertex in common with an edge in the second subset. These two subsets of k disjoint edges cross each other and hence form a natural k-grid, completing the proof. □

To prove Theorem 1.4(ii), essentially the same proof works as above, except replacing the bound $O(k^2 n)$ of Tóth [22] on the number of edges of a geometric graph with no k disjoint edges by the bound $O(n \log^{4k-8} n)$ of Pach and Tóth [17] on the number of edges of a simple topological graph with no k disjoint edges.

3 Graphs with no grids on distinct vertices

In this section, we study topological graphs that contain no k-grid with distinct vertices, and prove Theorem 1.6. In the previous section, we only considered *simple* topological graphs, in which every pair of edges have at most one point in common. Here we allow two edges to cross an arbitrary number of times.

We use the following three results from different papers. A graph is a *string graph* if it is an intersection graph of a collection of curves in the plane. The first of these results shows that every dense string graph contains a large complete bipartite subgraph. Apart from a slightly weaker bound, it extends Lemma 2.2 from the previous section.

Lemma 3.1 ([6]) *Every string graph with m vertices and ϵm^2 edges contains the complete bipartite graph $K_{t,t}$ as a subgraph with $t \geq \epsilon^{c_1} \frac{m}{\log m}$, where c_1 is an absolute constant.*

The *pair-crossing number* pair-cr(G) of a graph G is the minimum possible number of unordered pairs of crossing edges in a drawing of G. We will use the following result of Kolman and Matoušek [12] which relates the pair-crossing number and the bisection width of a graph. Apart from the logarithmic factor, it extends Lemma 2.1 which relates the crossing number and the bisection width of a graph.

Lemma 3.2 ([12]) *There is an absolute positive constant c_2 such that if G is a graph with n vertices of degrees $d_1, \ldots, d_n$, then*

$$b(G) \leq c_2 \log n \left(\sqrt{\text{pair-cr}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2} \right).$$

It is shown in [7] that every topological graph G with n vertices and more than $n(\log n)^{O(\log h)}$ edges contains h pairwise crossing edges. Lemma 5.3 in [8] slightly strengthens this result. It states that under the same hypothesis, G has h pairwise crossing edges *with distinct vertices*. This stronger version was needed in the proof of an upper bound on the number of edges in a string graph with a forbidden bipartite subgraph. Here we need an even stronger version for the proof of Theorem 1.6.

Theorem 3.3 *Let G be a topological graph with n vertices and with colored edges such that each color class forms a matching. If G has more than $n(\log n)^{c_3 \log h}$ edges, then it contains h pairwise crossing edges of different colors and with distinct vertices. Here c_3 is an absolute constant.*

The proof of Theorem 3.3 is so similar to the proof of the previous weaker versions that we only outline the proof idea, showing details only where they differ from the previous versions in [7] and [8]. The proof is by induction on n and h. Let m denote the number of edges of G. If the intersection graph of the m edges is sparse, i.e., there are at most $cm^2/(\log n)^4$ pairs of intersecting edges for some small absolute constant $c > 0$, then we apply Lemma 3.2 and find a partition of the vertices into two subsets with few edges between them. In this case, we are done by the induction hypothesis applied to each of these vertex subsets (this is similar to Case 1 in the proof of Theorem 1.4(i) in the previous section). If the intersection graph of the edges is dense, i.e., there are more than $cm^2/(\log n)^4$ pairs of edges that intersect, then using Lemma 3.1 we find two large edge subsets E_1, E_2 with $|E_1| = |E_2|$ such that every edge in E_1 crosses every edge in E_2. In [8], it is shown that one can pick $E_1' \subset E_1$ and $E_2' \subset E_2$ with $|E_1'| = |E_2'| \geq |E_2|/8$ such that no vertex of G is in an edge in E_1' and in an edge in E_2'. With the next lemma, with $A_i = E_i'$ for $i = 1, 2$, we find subsets $E_1'' \subset E_1'$ and $E_2'' \subset E_2'$ with $|E_1''| = |E_2''|$ such that each edge in E_1'' has different color from each edge in E_2''. We can apply this lemma here since a matching in an n-vertex graph has at most $n/2$ edges, and $|E_1'| = |E_2'| \geq |E_2|/8 \geq 2n$.

Lemma 3.4 *Let A_1, A_2 be two disjoint sets such that $|A_1| = |A_2| \geq 2n$. Suppose the elements of $A_1 \cup A_2$ are colored such that no color class has more than $n/2$ elements. Then there are subsets $A_1' \subset A_1$ and $A_2' \subset A_2$ with $|A_1'|, |A_2'| \geq |A_1|/4$ such that every element of A_1' has a different color from every element of A_2'.*

Proof. Let $c_1, \ldots, c_t$ be the colors. We construct the subsets A_1' and A_2' algorithmically in several steps as follows. Initially, both sets are empty. For $j = 1, 2, \ldots,$, at step j, as long as $|A_i'| < |A_i|/2$ for $i = 1, 2$, we do the following. If there are at least as many elements in A_1 of color c_j as there are in A_2 of color c_j, then we place the elements of color c_j which are in A_1 in A_1'. If the number of elements of color c_j which are in A_2 is more than the number of elements of color c_j in A_1, then we place all elements of color c_j which are in A_2 in A_2'. We stop this process when there is an i such that $|A_i'| \geq |A_i|/2$. This process stops after some step j_0, since at least half of the elements considered are placed in A_1' or A_2'. Suppose without loss of generality that $|A_1'| \geq |A_1|/2$. For $j_0 < j \leq t$, we also place all elements of A_2 of color c_j in A_2'. Since at most $|A_1'| \leq |A_1|/2 + n/2$ elements of A_2 are not in A_2', then $|A_2'| \geq |A_2| - |A_1|/2 - n/2 = |A_2|/2 - n/2 \geq |A_2|/4$. By construction, $|A_1'|, |A_2'| \geq |A_1|/4$, and no element of A_1' has the same color as an element in A_2'. □

We may assume that not both E_1'' and E_2'' contain $h/2$ pairwise crossing edges of distinct colors and distinct vertices, since otherwise together we would have h pairwise crossing edges of distinct colors and distinct vertices. The induction hypothesis therefore gives an upper bound on the size of E_1'', which, together with the lower bound estimate in the construction of E_1'', completes the proof of Theorem 3.3.

The next lemma is used twice in the proof of Lemma 3.6, which is the main lemma in the proof of Theorem 1.6. Suppose that every curve in a collection of pairwise intersecting curves is decomposed into at most two subcurves. We show that there are always two large subsets of subcurves such that every subcurve in one subset intersects every subcurve in the other subset. More formally, let $h(k)$ be the minimum positive integer h such that for every collection C of h pairwise intersecting curves, no matter how each of the curves is decomposed into one or two subcurves, there are k subcurves intersecting k other subcurves, and these $2k$ subcurves come from $2k$ distinct curves in C. Note that $h(1) = 2$.

Lemma 3.5 *For $k \geq 2$, we have $h(k) \leq c_4 k \log k$ for some absolute constant c_4.*

Proof. Let $h = c_4 k \log k$, where $c_4 = 16^{c_1+1} c_1$, where c_1 is the absolute constant in Lemma 3.1. Let C be a collection of h pairwise intersecting curves, each of which is decomposed into one or two subcurves. For each curve $\gamma \in C$, pick uniformly at randomly one of the (at most two) subcurves, and let C' be the collection of these picked subcurves. For each pair $\gamma, \gamma' \in C$, with probability at least $1/4$, the subcurve of γ in C' intersects the subcurve of γ' in C'. So the expected number of intersecting pairs of subcurves in C' is at least $\frac{1}{4}\binom{h}{2}$. Hence, there is a choice for the collection C' such that the number of intersecting pairs of subcurves in C' is at least $\frac{1}{4}\binom{h}{2}$. Since C' has cardinality h and at least $\frac{1}{4}\binom{h}{2} \geq \frac{1}{16}h^2$ pairs of intersecting subcurves, then by Lemma 3.1, the intersection graph of C', which is a string graph, contains a complete bipartite graph with parts of size

$$\left(\frac{1}{16}\right)^{c_1} \frac{h}{\log h} \geq k,$$

where the last inequality follows from our choice of c_4. □

Let $f_k(n)$ denote the maximum number of edges of a topological graph with n vertices and no k-grid with distinct vertices. The remainder of this subsection is devoted toward proving Theorem 1.6, which says that $f_k(n) \leq c_k n \log^* n$. It will be helpful to consider the following related functions. Let $f_k(n, \Delta)$ denote the maximum number of edges of a topological graph with n vertices, maximum degree at most Δ, and no k-grid with distinct vertices. We will also use $d_k(n) := f_k(n)/n$ and $d_k(n, \Delta) := f_k(n, \Delta)/n$, since it would be more convenient to bound the average degree rather than the maximum number of edges. Note that a planar graph on n vertices has at most $3n - 6$ edges for $n \geq 3$, and this bound is achieved by a triangulated planar graph. It follows that $d_1(n) = 3 - \frac{6}{n}$ for $n \geq 3$ and $d_k(n) \geq 1$ for $n \geq 3$. By Theorem 3.3, we have

$$d_k(n) \leq (\log n)^{c_3 \log 2k} \tag{2}$$

since a set of $2k$ pairwise crossing edges with distinct vertices in a topological graph contains a k-grid with distinct vertices. We will improve this bound significantly, namely show that $d_k(n) \leq c_k \log^* n$. This result will follow from the next two lemmas.

Lemma 3.6 *There is an absolute constant $c > 0$ such that for $\Delta = (\log n)^{c \log k}$ we have*

$$f_k(n) \leq f_k(n, \Delta) + k^{c \log \log k} n.$$

Lemma 3.7 *There is an absolute constant c' such that the following holds. For each k, n and Δ with $\Delta \geq k \geq 2$ and $n \geq \Delta^{c'}$, there is $n' \leq \Delta^{c'}$ such that $d_k(n, \Delta) \leq d_k(n') + 1$.*

To see that these two lemmas indeed imply Theorem 1.6, observe first that by Lemma 3.6 for $\Delta = (\log n)^{c \log k}$ we have that

$$d_k(n) \leq d_k(n, \Delta) + k^{c \log \log k}. \tag{3}$$

Combining inequality (3) and Lemma 3.7 we get that there is an absolute constant C such that

$$d_k(n) \leq d_k((\log n)^{C \log k}) + k^{C \log \log k}.$$

Iterating this inequality until $n \leq k^{2C \log \log k}$, and finally applying the trivial inequality $d_k(n) \leq n/2$, we get that $d_k(n) = O(k^{2C \log \log k} \log^* n)$, and hence

$$f_k(n) = O(k^{2C \log \log k} n \log^* n).$$

It remains to prove Lemmas 3.6 and 3.7. For a graph G and vertex sets A and B, let $e_G(A)$ denote the number of edges with both vertices in A and $e_G(A, B)$ denote the number of edges of G between A and B.

Proof of Lemma 3.6. Let $G = (V, E)$ be a topological graph with n vertices, $f_k(n)$ edges, and no k-grid with distinct vertices. Partition $V = A \dot{\cup} B$, where A consists of those vertices with degree more than $\Delta = (\log n)^{c \log k}$, where c is a sufficiently large absolute constant. We construct a sequence of topological graphs G_i with vertex set A. Let G_0 simply be the induced subgraph of G with vertex set A. Suppose we have already defined the topological graph G_i. If there is a vertex $v \in B$ adjacent to two vertices $a_1, a_2 \in A$ which are not adjacent, then we replace the path of length two (a_1, v, a_2) by an edge (a_1, a_2), and let G_{i+1} be the resulting topological graph. We eventually stop at some step j when there are no vertices v, a_1, a_2 as above, and we have a topological graph G_j on A. Notice that at each step, we delete two edges from B to A and replace them by one edge between two vertices in A. For each vertex $v \in B$, the set A_v of vertices in A adjacent to v after constructing G_j form a clique in G_j. Indeed, otherwise v is adjacent to two vertices $a_1, a_2 \in A$ that are not adjacent in G_j, which contradicts that we stopped at step j. Note that G_j has j more edges than the subgraph of G induced by A.

We first provide an upper bound on the number of edges of G_j. Each edge in G_j corresponds to either an edge or a path of length two in G. We assign each edge of G_j a color, where each edge of G_j that is an edge of G gets its own color, and we color the edges (a_1, a_2) of G_j that correspond to a path (a_1, v, a_2) of length two in G by the middle vertex $v \in B$. Note that by construction this coloring of the edges of G_j has the property that each color class is a matching. So if there are $h(k)$ pairwise intersecting edges in G_j with distinct vertices and distinct colors, then G contains a k-grid with distinct vertices, a contradiction. By Theorem 3.3 and Lemma 3.5, we have

$$\begin{aligned} e_G(A) + j &= e_{G_j}(A) \leq |A|(\log |A|)^{c_3 \log h(k)} \\ &\leq |A|(\log n)^{c_3 \log(c_4 k \log k)} \leq |A|(\log n)^{c_6 \log k} \end{aligned}$$

for some absolute constant c_6.

As discussed above, for each vertex $v \in B$, the set A_v of vertices in A adjacent to v in G_j forms a clique in G_j. This clique can not have $h(k)$ pairwise intersecting edges with distinct vertices and distinct colors, otherwise it contains a k-grid with distinct vertices. By Theorem 3.3, we have

$$\binom{|A_v|}{2} \leq |A_v|(\log |A_v|)^{c_3 \log h(k)},$$

so dividing both sides by $|A_v|$ we get

$$|A_v| \leq 2(\log |A_v|)^{c_3 \log h(k)} + 1 = 2h(k)^{c_3 \log \log |A_v|} + 1.$$

Taking $\log\log$ of both sides we conclude that $\log \log |A_v| \leq c_7 \log \log h(k)$ for some absolute constant c_7. Therefore, substituting this estimate into the exponent, we get

$$|A_v| \leq 2h(k)^{c_3 c_7 \log \log h(k)} + 1.$$

Also using Lemma 3.5, we have

$$|A_v| \leq k^{c_8 \log \log k}$$

for some absolute constant c_8. The number $e_G(A, B)$ of edges of G with one vertex in A and the other vertex in B is

$$e_G(A, B) = 2j + \sum_{v \in B} |A_v| \leq 2j + |B| k^{c_8 \log \log k}.$$

Since each vertex in A has degree at least Δ in G, the number $e_G(A) + e_G(A,B)$ of edges in G containing at least one vertex in A is at least $|A|\Delta/2$. So

$$\begin{aligned} |A|\Delta/2 &\leq e_G(A) + e_G(A,B) \\ &\leq e_G(A) + 2j + |B|k^{c_8 \log\log k} \\ &\leq 2|A|(\log n)^{c_6 \log k} + |B|k^{c_8 \log\log k} \\ &\leq 2|A|(\log n)^{c_6 \log k} + nk^{c_8 \log\log k} \end{aligned}$$

If $nk^{c_8 \log\log k} \leq 2|A|(\log n)^{c_6 \log k}$, then we get

$$\Delta \leq 8(\log n)^{c_6 \log k},$$

which contradicts $\Delta = (\log n)^{c \log k}$ with c a sufficiently large constant. So $nk^{c_8 \log\log k} > 2|A|(\log n)^{c_6 \log k}$, and the number of edges in G containing a vertex in A is at most $2k^{c_8 \log\log k}n \leq k^{c \log\log k}n$. Note that every vertex in B has degree at most Δ in G, so $e_G(B) \leq f_k(|B|, \Delta) \leq f_k(n, \Delta)$, where the last inequality follows by adding isolated vertices to B to get a set of n vertices. Therefore, the number $f_k(n)$ of edges of G is at most $f_k(n, \Delta) + k^{c \log\log k}n$. □

In order to prove Lemma 3.7 we need the following lemma.

Lemma 3.8 *There are absolute constants c_9 and $c_{10} > 0$ such that for each k, n and Δ with $\Delta \geq k \geq 2$ and $n \geq \Delta^{c_9}$, there is $n_1 \leq 2n/3$ such that*

$$d_k(n_1, \Delta) \geq d_k(n, \Delta)\left(1 - n^{-c_{10}}\right).$$

Proof. Let G be a topological graph with n vertices, maximum degree at most Δ, and no k-grid with distinct vertices which has maximum possible average degree, which is $2d_k(n, \Delta)$, among all such topological graphs. Let $m = f_k(n, \Delta)$ be the number of edges of G.

Since each vertex has degree at most Δ, then G does not contain a $4k\Delta$-grid. Indeed, otherwise there would be disjoint edge subsets E_1, E_2 such that every edge in E_1 crosses every edge in E_2 and $|E_1| = |E_2| = 4k\Delta$. We greedily pick out a subset $E_1' \subset E_1$ with $|E_1'| = k$ such that for each new edge e added to E_1', we delete from $E_1 \cup E_2$ all other edges sharing a vertex with e. Less than 2Δ edges are deleted for each edge in E_1', so at least $4k\Delta - 2k\Delta = 2k\Delta$ edges remain in E_2. We similarly greedily pick out a subset $E_2' \subset E_2$ with $|E_2'| = k$ so that no two edges in E_2' share a common vertex. By construction, E_1' and E_2' form a k-grid with distinct vertices, a contradiction.

Let the number of crossing pairs of edges of G be ϵm^2, so the underlying graph of G has pair-crossing number at most ϵm^2. By Lemma 3.1, G has an ℓ-grid with $\ell \geq \epsilon^{c_1}\frac{m}{\log m}$. Therefore, we have the inequality $\epsilon^{c_1}\frac{m}{\log m} \leq 4k\Delta$, and we get $\epsilon \leq m^{-\frac{2}{3c_1}}$, where we use $4k\Delta \leq m^{1/6}$, which follows from $\Delta \geq k \geq 2$, $m \geq n \geq \Delta^{c_9}$, and c_9 is chosen sufficiently large, and $\log m \leq m^{1/6}$. By Lemma 3.2, there is an absolute constant c_2 such that if $d_1, \ldots, d_n$ is the degree sequence of G, then

$$\begin{aligned} b(G) &\leq c_2 \log n \left(\sqrt{\text{pair-cr}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2}\right) \\ &\leq c_2 \log n \left(\epsilon^{1/2}m + \Delta\sqrt{n}\right) \\ &\leq c_2 \log n \left(m^{1-\frac{1}{3c_1}} + \Delta\sqrt{n}\right) \leq m^{1-c_{10}} \end{aligned}$$

for some absolute constant $c_{10} > 0$.

Therefore, there is a partition $V(G) = V_1 \dot\cup V_2$ such that $|V_1|, |V_2| \le \frac{2}{3}n$ and $e_G(V_1, V_2) \le m^{1-c_{10}}$. Since G has m edges in total, there is an $i \in \{1, 2\}$ such that $e_G(V_i) \ge \frac{|V_i|}{n}\left(m - m^{1-c_{10}}\right)$. Therefore, the subgraph of G induced by V_i has average degree at least a fraction $1 - m^{-c_{10}} \ge 1 - n^{-c_{10}}$ of the average degree of G. Letting $n_1 = |V_i|$, we have $n_1 \le 2n/3$ and the subgraph of G induced by V_i also has maximum degree at most Δ and does not contain a k-grid with distinct vertices, completing the proof. □

By repeatedly applying Lemma 3.8, we prove Lemma 3.7 and thus complete the proof of Theorem 1.6.

Proof of Lemma 3.7. Recall that we would like to show that there is a constant c' such that the following holds. For each k, n and Δ with $\Delta \ge k \ge 2$ and $n \ge \Delta^{c'}$, there is $n' \le \Delta^{c'}$ such that $d_k(n') \ge d_k(n, \Delta) - 1$.

Let $n_0 = n$. After one application of Lemma 3.8, we get $d_k(n_1, \Delta) \ge d_k(n, \Delta)\left(1 - n^{-c_{10}}\right)$ for some $n_1 \le 2n/3$. After j applications of Lemma 3.8, we get $d_k(n_j, \Delta) \ge d_k(n, \Delta)\prod_{i=1}^{j}\left(1 - n_{i-1}^{-c_{10}}\right)$ for some $n_j \le (2/3)^j n$. Let i_0 be the first value such that $n_{i_0} \le \Delta^{c'}$, where c' is a sufficiently large constant. Let $n' = n_{i_0}$.

We have

$$\begin{aligned}
d_k(n') &\ge d_k(n', \Delta) \\
&\ge d_k(n, \Delta)\prod_{i=1}^{i_0}\left(1 - n_{i-1}^{-c_{10}}\right) \\
&\ge d_k(n, \Delta)\left(1 - \sum_{i=1}^{i_0} n_{i-1}^{-c_{10}}\right) \\
&\ge d_k(n, \Delta)\left(1 - n_{i_0-1}^{-c_{10}}\sum_{i=0}^{\infty}(2/3)^{c_{10}i}\right) \\
&\ge d_k(n, \Delta)\left(1 - n_{i_0-1}^{-c_{10}}\frac{1}{1-(2/3)^{c_{10}}}\right) \\
&\ge d_k(n, \Delta)\left(1 - (\Delta^{c'})^{-c_{10}}\frac{1}{1-(2/3)^{c_{10}}}\right) \\
&\ge d_k(n, \Delta)(1 - \frac{1}{\Delta}).
\end{aligned}$$

Since the average degree of a graph of maximum degree at most Δ is at most Δ, we have $d_k(n, \Delta) \le \Delta$. Summarizing,

$$d_k(n') \ge (1 - \frac{1}{\Delta})d_k(n, \Delta) \ge d_k(n, \Delta) - 1.$$

□

4 Special cases

In this section, we consider some special cases of Conjectures 1.3 and 1.5, for which we can prove better upper bounds than the ones in Theorems 1.4 and 1.6, respectively.

4.1 Topological graphs with no $(k,1)$-grid with distinct vertices

Here we prove Theorem 1.7. Let $G=(V,E)$ be a topological graph. For every edge $e \in E$ define $X(e)$ to be set of edges in E that cross e and share no common vertex with it. Given a set of edges $E' \subset E$, the *vertex cover number* of E' is the minimum size of a set of vertices $V' \subset V$ such that every edge in E' has at least one of its endpoints in V'. Theorem 1.7 will follow from the next lemma, whose proof is due to Rom Pinchasi [19].

Lemma 4.1 *For every fixed integer k there is a constant c_k such that the following holds. If $G=(V,E)$ is a topological graph on n vertices such that for every $e \in E$ the vertex cover number of $X(e)$ is at most k, then G has at most $c_k n$ edges.*

Proof. We apply the probabilistic method in a setting similar to one described in [20]. Let m be the number of edges in G, and let $0 < q < 1$ be a constant. Let G' be the induced subgraph obtained from G by taking every vertex of G independently with probability q. Call an edge e' in G' *good* if there is no edge f' in G' that crosses e' and shares no vertex with it. Denote by n^* and m^* the expected number of vertices and good edges in G', respectively. Clearly, $n^* = qn$. The probability that an edge e is good is at least $q^2(1-q)^k$, thus $m^* \geq q^2(1-q)^k m$. Observe that two good edges may cross only if they share a vertex. Thus, the good edges form a planar graph by the Hanani-Tutte Theorem (see, e.g., [23]). Therefore, $q^2(1-q)^k m \leq m^* \leq 3n^* = 3qn$, and thus, $m \leq \frac{3}{q(1-q)^k} n$. □

Now let G be an n-vertex topological graph with no $(k,1)$-grid with distinct vertices. We claim that for every $e \in E(G)$ the vertex cover number of $X(e)$ is at most $2k-2$, and therefore it has at most $c_{2k-2}n$ edges, where c_{2k-2} is the constant from Lemma 4.1. Assume for the sake of contradiction that there is an edge $e \in E(G)$ such that the vertex cover number of $X(e)$ is at least $2k-1$. Pick an edge $(u,v) \in X(e)$ and remove all the other edges in $X(e)$ that are covered by v or u. This can be done k times, for otherwise $X(e)$ can be covered by at most $2k-2$ vertices. The edges we picked along with the edge e form a $(k,1)$-grid with distinct vertices. This proves Theorem 1.7.

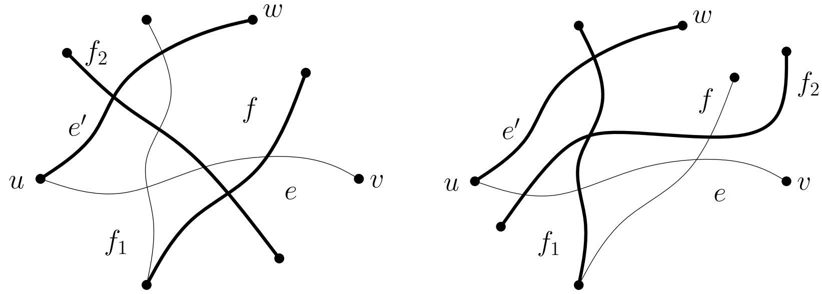
4.2 Graphs with no natural $(2,1)$-grids: proof of Theorem 1.8

Let $G=(V,E)$ be a simple topological graph on n vertices without a natural $(2,1)$-grid. For every $e \in E$ assign e the color *red* if $X(e)$ has vertex cover number at most 3, otherwise assign e the color *blue*. It follows from Lemma 4.1 that G has at most $29n$ red edges (by picking $q=1/4$). Therefore, it remains to bound the number of blue edges. To this end, we partition them into disjoint subsets such that edges from different subsets do not cross and edges within the same subset either cross or share a vertex. Then, it follows that the number of blue edges is proportional to the sum of the number of vertices in each of the subgraphs induced by the edge subsets. We bound this sum by drawing a certain planar multigraph.

The next claim is crucial for obtaining a partition of the edges with the above properties. For $F \subseteq E$, denote by $V(F)$ the set of vertices induced by F.

Proposition 4.2 *Let $e=(u,v)$ be a blue edge, and let $f_1 \in X(e)$. If there is an edge $e'=(u,w)$ such that $w \notin V(X(e))$ and e' crosses f_1, then e' crosses every edge $f \in X(e)$.*

Proof. Assume by contradiction that there is an edge $f \in X(e)$ such that e' and f do not cross. Note that e' and f must be disjoint since $w \notin V(X(e))$. If f and f_1 are disjoint, then e,f, and f_1 form



(a) f_2 crosses e'

(b) f_2 and e' are disjoint

Figure 2: Illustrations for the proof of Proposition 4.2

a natural $(2,1)$-grid. If f and f_1 cross, then e',f, and f_1 form a natural $(2,1)$-grid. Thus, f and f_1 must share a vertex, and $|V(\{f\} \cup \{f_1\})| = 3$. Since e is blue, there must be an edge $f_2 \in X(e)$ that does not share an endpoint with f or f_1 (and does not share an endpoint with e', since $w \notin V(X(e))$). Therefore, f_2 must cross both f and f_1. If f_2 crosses e' then f_2, e', and f form a natural $(2,1)$-grid (see Figure 2(a)). Otherwise, if f_2 and e' are disjoint, then f_2, e', and f_1 form a natural $(2,1)$-grid (see Figure 2(b)). □

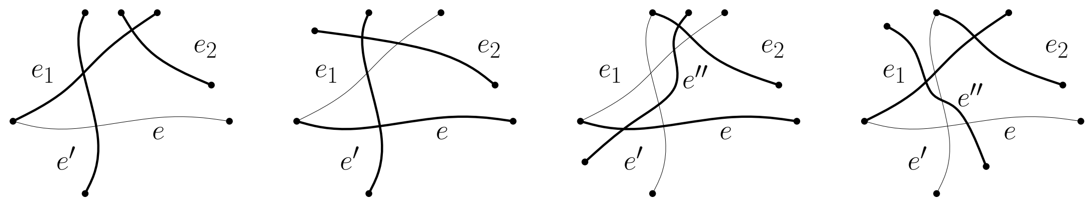
Next, we remove all red edges and process the blue edges in an arbitrary order. Initially, all blue edges are unmarked. Eventually, each blue edge will be either marked or deleted. Let B be the set of the currently unmarked and undeleted blue edges, and let $e = (u,v)$ be an edge in B. Delete (from the graph and from B) all edges that have one endpoint in $V(X(e) \cap B)$ and the other endpoint in $\{u,v\}$. Let E_u be the set of edges $(u,x) \in B$ such that $x \notin V(X(e) \cap B)$ and there is an edge $e' \in X(e) \cap B$ that crosses (u,x). Analogously, let E_v be the set of edges $(v,x) \in B$ such that $x \notin V(X(e) \cap B)$ and there is an edge $e' \in X(e) \cap B$ that crosses (v,x). Assume without loss of generality that $|E_u| \geq |E_v|$ and remove the edges E_v (from B as well). Recall that, according to Proposition 4.2, if there is an edge (u,x) such that $x \notin V(X(e))$, and (u,x) crosses some edge in $X(e)$, then (u,x) crosses every edge in $X(e)$.

A *thrackle* is a simple topological graph in which every pair of edges meet exactly once, either at a vertex or at a crossing point. It is known that a thrackle on n vertices has at most $3(n-1)/2$ edges [4] and it is a famous open problem (known as Conway's Thrackle Conjecture) to show that the tight bound is n. Set $\text{thrackle}(e) = B \cap (\{e\} \cup X(e) \cup \{(u,x) \mid \exists e' \in X(e) \text{ that crosses } (u,x)\})$. Mark all blue edges in thrackle(e), and continue to create thrackles as long as there is an unmarked blue edge.

Proposition 4.3 *thrackle(e) is a thrackle.*

Proof. By definition, e meets every other edge in thrackle(e). A pair of edges in $X(e)$ cannot be disjoint, for otherwise they will form a natural $(2,1)$-grid with e. Finally, by Proposition 4.2 every edge in thrackle(e) of the form (u,x) such that $x \notin V(X(e))$ must cross all the edges in $X(e)$. □

Proposition 4.4 *If $e_1 \in$ thrackle(e) and $e_2 \notin$ thrackle(e) then e_1 and e_2 do not cross.*



(a) Case 1: e_2 and e' are disjoint

(b) Case 2: e_2 and e' cross

(c) Case 3a: e_2 and e' share a vertex and e_2 crosses e''

(d) Case 3b: e_2 and e' share a vertex and e_2 and e'' are disjoint

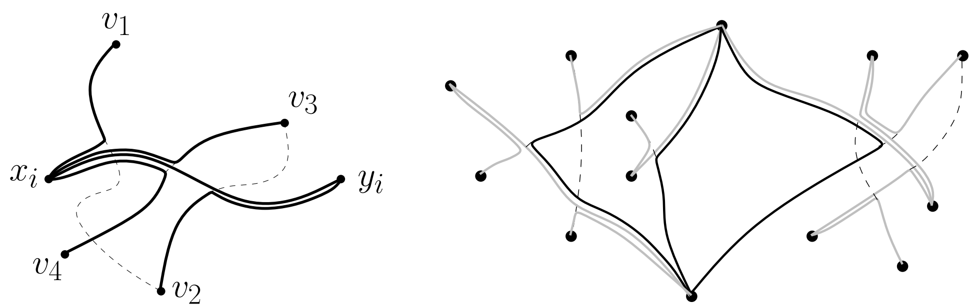
Figure 3: Illustrations for the proof of Proposition 4.4

Proof. Assume that the claim is false and let e_1 and e_2 be the first such pair along the process of creating the thrackles. Then, without loss of generality e_2 is unmarked when thrackle(e) is created. Clearly $e_1 \neq e$ for otherwise $e_2 \in X(e)$. If $e_1 \in X(e)$ then e_2 does not share a vertex with e, for otherwise it would have been added to thrackle(e) or removed. Thus, e, e_1, and e_2 form a natural $(2,1)$-grid. Otherwise, e_1 shares a vertex with e and there is an edge $e' \in X(e)$ that crosses e_1. Note that e_2 cannot share a vertex with e, since if it shares the same vertex as e_1 then they cannot cross, and otherwise it would have been removed. There are three possible cases to consider (see Figure 3):
Case 1: e_2 and e' are disjoint. Then e_2, e', and e_1 form a natural $(2,1)$-grid.
Case 2: e_2 and e' cross. Then e_2, e', and e form a natural $(2,1)$-grid.
Case 3: e_2 and e' share a vertex. Since e is blue there must an edge $e'' \in X(e)$ that do not share a vertex with e' or e_2. By Proposition 4.2 e'' must cross e_1. If (a) e_2 crosses e'', then e_2, e'', and e form a natural $(2,1)$-grid. Otherwise, if (b) e_2 and e'' are disjoint then e_2, e'', and e_1 form a natural $(2,1)$-grid. □

Since any newly created thrackle contains no edges of a previous thrackle, we obtain a partition of the edges that were not deleted into thrackles $t_1, t_2, \ldots, t_j$. Let $t_i = \text{thrackle}\left((u_i, v_i)\right)$ and denote by V_i the vertex set of t_i. When t_i was created, at most $2|V_i|$ edges (x_i, y_i) such that $x_i \in \{u_i, v_i\}$ and $y_i \in V\left(X\left((u_i, v_i)\right)\right)$ were deleted. We also removed edges (x_i, y_i) such that $x_i \in \{u_i, v_i\}$ and $y_i \notin V\left(X\left((u_i, v_i)\right)\right)$. Since we removed the smaller set among E_{u_i} and E_{v_i}, and each of these edge subsets is a star, it follows that at most $|V_i|$ such edges were deleted. The number of edges in t_i is at most $3(|V_i| - 1)/2$, thus, it remains to show that $\sum_{i=1}^{j} |V_i| = O(n)$.

To this end we draw a new planar multigraph G' with the same vertex set V. For every thrackle $t_i = \text{thrackle}\left((x_i, y_i)\right)$ we draw a crossing-free tree T_i with $|V_i| - 1$ edges as follows. First, draw the edge from x_i to y_i. Next, for every vertex $v \in V_i \setminus T_i$ pick one edge $e \in t_i$ that has v as one of its endpoints. Follow e from v until it either hits a vertex (necessarily x_i or y_i) or crosses an already drawn edge $e' \in T_i$. In the first case draw an edge identical to e. In the second case draw the segment of e from v almost until its crossing point with e', then continue the edge very close to e' (in one of the directions) until a vertex is reached. See Figure 4(a) for an example.

It follows from Proposition 4.4 and the construction of G' that G' is planar. Note that it is possible for G' to contain parallel edges (see Figure 4(b) for an example). However, it can be shown that they can be eliminated by removing at most half of the edges in G'. It follows from the standard proof for



(a) Constructing G' (b) G' might contain parallel edges

Figure 4: The graph G'

the maximum number of edges in a planar graph (see, e.g., [3]), that a planar multigraph on $n \geq 3$ vertices with no faces of size 2 (2-*faces*) has at most $3n-6$ edges. The next Proposition will be useful in showing that G' has not too many 2-faces.

Proposition 4.5 *Let $t_i = thrackle(e)$ be a thrackle and let p and q be two points on edges of t_i. Then there is a path on edges of t_i between p and q that does not go through any vertex.*

Proof. It is enough to show that there is a path from p to e. Let e_p be the edge that contains p. If $e_p = e$ then we are done. If e_p crosses e then the segment of e_p between p and the crossing point is the required path. Finally, if e_p does not cross e, then there is an edge $e' \in t_i$ that crosses both e and e_p. The segment of e_p from p to the crossing point of e_p and e' along with the segment of e' from that crossing point to the crossing point of e and e'' create the required path. □

Let t_1, t_2, t_3 be three different thrackles that yield three parallel edges c_1, c_2, c_3 in G' between two vertices u, v. The closed curve $c_1 \cup c_2$ splits the plane into two regions, one containing the interior of c_3. Then this region must contain every vertex in $V_3 \setminus \{u, v\}$. For otherwise, let $w \in V_3 \setminus \{u, v\}$ be a vertex outside that region and let p be some point on c_3. It follows from Proposition 4.5 that there is a path on edges of t_3 between p and w. This path must cross c_1 or c_2 at a point different from u and v, hence there are edges from different thrackles that cross, contradicting Proposition 4.4.

It follows that there are no two adjacent 2-faces (that is, sharing an edge) in G'. Consider the parallel edges between two vertices in G' according to their order around one of the vertices, and remove every other edge. The remaining graph has at least half of the edges of G' and no 2-faces, thus it has at most $3n$ edges. Therefore, G' has at most $6n$ edges, and thus the number of edges in all the thrackles is at most $9n$ and the total number of blue edges is at most $36n$. We conclude that the original graph G has at most $65n$ edges.

4.3 Convex geometric graphs with no natural grids

For some values of k or l, we are able to provide tighter bounds (in terms of the constant $c_{k,l}$) for the number of edges in convex geometric graphs avoiding natural (k, l)-grids, than the ones guaranteed by Theorem 1.8 and Corollary 1.9.

Theorem 4.6 *An n-vertex convex geometric graph with no natural $(2,1)$-grid has less than $5n$ edges.*

Theorem 4.7 *An n-vertex convex geometric graph with no natural $(2,2)$-grid has less than $8n$ edges.*

Theorem 4.8 *A convex geometric graph with $n \geq 3k$ vertices and no natural $(k,1)$-grid has at most $6kn - 12k^2$ edges.*

We mention first some basic notions and facts before moving to the proofs. Let G be a convex geometric graph. We denote by $d_G(v)$ the degree of the vertex $v \in V(G)$, and by $\delta(G)$ the minimum degree of the vertices of G. For $u, v \in V(G)$, we say that v and u are *consecutive vertices* if they appear next to each other on the convex hull of the vertices of G. For $u, v \in V(G)$ we denote by $R(u,v) \subset V(G)$ the set of vertices from u to v in clockwise order, not including u and v. A convex geometric graph G' is a *geometric minor* of G if G' can be obtained from G by performing a finite number of the following two operations:

1. Vertex deletion.
2. Consecutive vertex contraction, i.e., only consecutive vertices can contract. Recall that the contraction of two vertices x and y replaces x and y in G with a vertex v such that v is adjacent to all the neighbors of x and y.

Notice that if G' contains a natural (k,l)-grid, then so does G. Assume that G is a convex geometric graph with n vertices and at least cn edges. Let G' be a minimal geometric-minor of G such that $|E(G')|/|V(G')| \geq c$. Then we can conclude that:

1. $\delta(G') \geq c$ (otherwise vertex deletion can be applied); and
2. every consecutive pair of vertices v and u must have at least $c-1$ common neighbors (otherwise consecutive vertex contraction can by applied).

Proof of Theorem 4.6. Let G be such a graph such that $|E(G)|$ is maximum, and suppose that $|E(G)| \geq 5n$. Let G' be a minimal geometric-minor of G such that $|E(G')|/|V(G')| \geq 5$. Note that $|V(G')| \geq 11$. For a vertex $u \in V(G')$ denote by $u_1, u_2, \ldots$ the neighbors of u in clockwise order. Note that u_1 immediately follows u in clockwise order, since a straight-line segment connecting two consecutive vertices in G cannot be crossed by any edge of G, and hence we can assume without loss of generality that it is an edge of G. Let $v \in V(G')$ be the vertex such that:

$$|R(v_3, v)| = \min_{u \in V(G')} |R(u_3, u)|$$

Since $\delta(G') \geq 5$, u_3 exists for every u. Since v_1 and v are consecutive vertices they share at least 4 common neighbors. Hence, v_1 and v are both adjacent to a vertex $a \in V(G')$ such that $a \notin \{v_2, v_{k-1}, v_k\}$, where $k = d_{G'}(v)$. By minimality of $|R(v_3, v)|$, v_k has at least three neighbors in $R(v_k, v_3)$. Thus v_k has a neighbor $b \in R(v_k, v_3)$ other than v and v_1. Hence we have a natural $(2,1)$-grid with edges $(v, v_{k-1}), (v_1, a)$, and (v_k, b) in G' (see Figure 5), and hence in G. □

Proof of Theorem 4.7. Assume that $|E(G)| \geq 8n$. Let G' be a minimal geometric-minor of G with $|E(G')|/|V(G')| \geq 8$. Note that $|V(G')| \geq 17$, $\delta(G') \geq 8$, and every pair of consecutive vertices in G' share at least 7 common neighbors. Let (x, x') and (y, y') be a pair of disjoint edges such that:

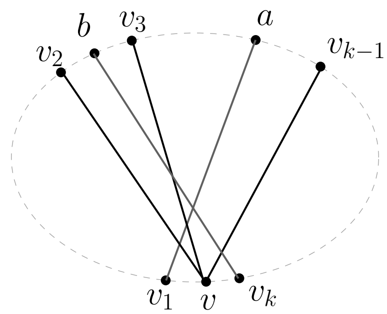


Figure 5: An illustration for the proof of Theorem 4.6

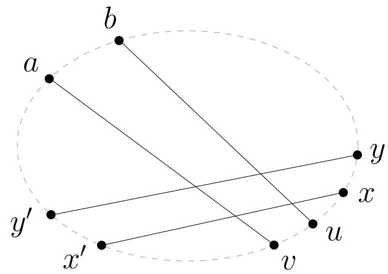


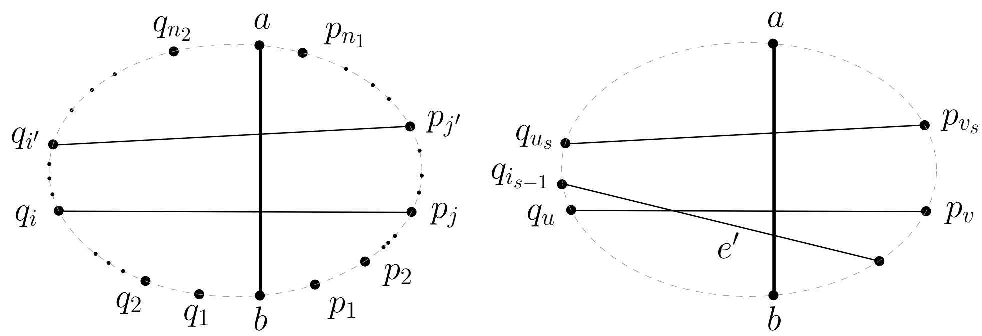
Figure 6: An illustration for the proof of Theorem 4.7

1. x and y are consecutive vertices with x following y in clockwise order;
2. $|R(x, x')|, |R(y', y)| \geq 2$; and
3. $|R(y', y)|$ is maximized subject to (1) and (2) above.

This is possible since consecutive vertices share at least 7 common neighbors. Now let u, v be the next two vertices after x in clockwise order. Since u and v are consecutive, we know that they share at least 7 common neighbors. Now u and v can have at most 3 common neighbors in $R(v, y') \cup \{y'\}$, since otherwise we would contradict the maximality of $|R(y', y)|$. Hence u and v must have two common neighbors $a, b \in R(y', y)$. See Figure 6. Hence $(x, x'), (y, y'), (u, a), (v, b)$ forms a natural $(2, 2)$-grid in G', and hence in G. □

Proof of Theorem 4.8. Our proof uses the technique from [5]. Let $k \geq 1$ be fixed. We will prove the theorem by induction on the number of vertices n. For $n = 3k$ we need to show that $|E(G)| \leq 6k^2$, however, there are at most $\binom{3k}{2} \leq \frac{9k^2}{2}$ edges. Assume now that the claim is true when the number of vertices is smaller than n and let G be an n-vertex convex geometric graph with no natural $(k, 1)$-grid.

If there is no edge whose endpoints are separated by at least $2k$ vertices along (both arcs of) the boundary of the n-gon, then $|E(G)| \leq 2kn \leq 6kn - 12k^2$ since $n \geq 3k$. So we may assume that there exists such an edge $e = ab$. Assume without loss of generality that e is vertical. Let $p_{n_1}, \ldots, p_1$ denote the vertices on the right-hand side of (a, b) and let $q_1, \ldots, q_{n_2}$ denote the vertices on its left-hand side, both in clockwise order. Define a partial order $\prec$ on the set of edges that cross (a, b) as follows: $q_i p_j \prec q_{i'} p_{j'} \Leftrightarrow i < i'$ and $j < j'$ (see Figure 7(a)). We denote by $rank(q_i p_j)$ the largest integer r such that there is a sequence of edges $q_{i_1} p_{j_1} \prec q_{i_2} p_{j_2} \prec \cdots \prec q_{i_r} p_{j_r} = q_i p_j$.



(a) $q_ip_j \prec q_{i'}p_{j'}$

(b) The second case in the proof of Proposition 4.9

Figure 7: Illustrations for the proof of Theorem 4.8

Since G has no natural $(k, 1)$-grid, every edge that crosses ab has rank at most $k-1$. Next, we define a convex geometric graph G_1 with n_2+k+1 vertices $\{a, p^*_{k-1}, \ldots, p^*_1, b, q_1, \ldots, q_{n_2}\}$ (in clockwise order). Let G_1 be the same as G when restricted to the vertices $\{a, b, q_1, \ldots, q_{n_2}\}$. Then let $q_ip^*_r$ be in $E(G_1)$ if and only if there is an edge $q_ip_j \in E(G)$ whose rank is r. First, we will show that if there are t pairwise disjoint edges in G_1 with their *left endpoints* inside an interval (q_i, q_j), then there are t pairwise disjoint edges in G with their *left endpoints* inside the interval (q_i, q_j).

Proposition 4.9 *Let $q_{i_1}p^*_{r_1}, \ldots, q_{i_t}p^*_{r_t}$ be t pairwise disjoint edges in G_1 that cross ab. Then there are t pairwise disjoint edges $q_{u_1}p_{v_1}, \ldots, q_{u_t}p_{v_t}$ such that*

1. *$u_t = i_t$.*
2. *$u_x \geq i_x$ for $x = 1, \ldots, t-1$.*
3. *$rank(q_{u_x}p_{v_x}) = r_x$, for $x = 1, \ldots, t$.*

Proof. The proof is by reverse induction on x. In G, we can pick an edge $q_{i_t}p_{v_t}$ that has rank r_t. We know that such an edge exists, since $q_{i_t}p^*_{r_t}$ exists in G_1. Assume that we have already found edges $q_{u_x}p_{v_x}$ for $x = t, t-1, \ldots, s > 1$, satisfying the above requirements. Let q_up_v be an edge of rank r_{s-1} such that $q_up_v \prec q_{u_s}p_{v_s}$. If $u \geq i_{s-1}$, then we can pick q_up_v as the next edge. Otherwise, let e' be an edge of rank r_{s-1} with $q_{i_{s-1}}$ as an endpoint. Since e' and q_up_v have the same rank, they must cross, which implies that $e' \prec q_{u_s}p_{v_s}$, and so we can pick e' to be the next edge. See Figure 7(b). □

Proposition 4.10 G_1 *does not contain a natural $(k, 1)$-grid.*

Proof. Assume that G_1 contains a natural $(k, 1)$-grid. Then, by considering the possible edges involved in such a grid and using Proposition 4.9 above, one can conclude that there is a natural $(k, 1)$-grid in G, which is a contradiction. □

We also define a convex geometric graph G_2 with n_1+k+1 vertices $\{a, p_{n_1}, \ldots, p_1, b, q^*_1, \ldots, q^*_{k-1}\}$ (in clockwise order). Let G_2 be the same as G when restricted to the vertices $\{a, p_{n_1}, \ldots, p_1, b\}$. Then let (q_r^*, p_j) be in $E(G_2)$ if and only if there is an edge $(q_i, p_j) \in E(G)$ whose rank is r. By similar arguments, G_2 does not contain a natural $(k,1)$-grid. Let E_r denote the set of edges in G with rank r, $1 \le r \le k-1$.

Proposition 4.11 $|E_r| \le d_{G_1}(p_r^*) + d_{G_2}(q_r^*) - 1$.

Proof. The edges in E_r cannot form a cycle. Indeed, consider a path $q_{i_1}p_{j_1}, q_{i_2}p_{j_1}, q_{i_2}p_{j_2}, \ldots$ and assume without loss of generality that $i_1 < i_2$. Then we have $j_2 < j_1$, for otherwise $q_{i_1}p_{j_1}$ and $q_{i_2}p_{j_2}$ are disjoint. Analogously, we have $i_l > i_{l-1}$ and $j_l < j_{l-1}$, for any $l > 1$, therefore the path cannot form a cycle. Since there are $d_{G_1}(p_r^*) + d_{G_2}(q_r^*)$ vertices that are endpoints of edges in E_r, the claim follows. □

Denote by E'_1 the set of edges in G_1 that do not cross ab and by E'_2 the edges in G_2 that do not cross ab (note that $ab \in E'_i$, $i = 1, 2$). Recall that ab has at least $2k$ vertices on each of its sides, therefore, $|V(G_1)|, |V(G_2)| \ge 3k$. Then we have

$$
\begin{aligned}
|E(G)| &= |E'_1| + |E'_2| - 1 + \sum_{r=1}^{k-1} |E_r| \\
&= |E'_1| + |E'_2| - 1 + \sum_{r=1}^{k-1} \Big(d_{G_1}(p_r^*) + d_{G_2}(q_r^*) - 1 \Big) \\
&= |E(G_1)| + |E(G_2)| - k \\
&\overset{\text{ind hyp}}{\le} (6k(n_1 + k + 1) - 12k^2) \\
&\quad + (6k(n_2 + k + 1) - 12k^2) - k \\
&= 6kn - 12k^2 - k \le 6kn - 12k^2
\end{aligned}
$$

This completes the proof of Theorem 4.8. □

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References

[1] P. K. Agarwal, M. van Kreveld, and S. Suri, Label placement by maximum independent set in rectangles, *Comput. Geom. Theory Appl.* **11** (1998), 209–218.

[2] P. Brass, G. Károlyi, and P. Valtr, A Turán-type extremal theory for convex geometric graphs, *Discrete and Computational Geometry—the Goodman-Pollack Festschrift* (B. Aronov et al., eds.), Springer 2003, 275–300.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, 1976.

[4] G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete and Computational Geometry* **23** (2000), 191–206.

[5] V. Capoyleas and J. Pach, A Turán-type problem on chords of a convex polygon, *J. Comb. Theory Ser. B.* **56** (1992), 9–15.

[6] J. Fox and J. Pach, String graphs and incomparability graphs, *Adv. Math.,* **230** (2012), 1381–1401.

[7] J. Fox and J. Pach, Coloring K_k-free intersection graphs of geometric objects in the plane, *J. of Combinatorial Theory Ser. A* **117** (2010), 776–782.

[8] J. Fox and J. Pach, A separator theorem for string graphs and its applications, *Combin. Probab. Comput.* **19** (2010), 371–390.

[9] J. Fox, J. Pach, and Cs. D. Tóth, A bipartite strengthening of the Crossing Lemma, *J. Combinatorial Theory Ser. B* **100** (2010), 23–35.

[10] D. S. Hochbaum and W. Maass, Approximation schemes for covering and packing problems in image processing and VLSI, *J. ACM* **32** (1985), 130–136.

[11] M. Klazar and A. Marcus, Extensions of the linear bound in the Füredi-Hajnal conjecture, *Adv. in Appl. Math.* **38** (2006), 258–266.

[12] P. Kolman and J. Matoušek, Crossing number, pair-crossing number, and expansion, *J. Combin. Theory Ser. B.* **92** (2004), 99–113.

[13] E. Malesinska, Graph-theoretical models for frequency assignment problems, Ph.D. Thesis, Technische Universität Berlin, 1997.

[14] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture, *J. Comb. Theory Ser. A.* **107** (2004), 153–160.

[15] J. Pach, F. Shahrokhi, and M. Szegedy, Applications of the crossing number, *Algorithmica* **16** 1996, 111–117.

[16] J. Pach, R. Pinchasi, M. Sharir, and G. Tóth, Topological graphs with no large grids, *Graphs and Combinatorics* **21** (2005), 355–364.

[17] J. Pach and G. Tóth, Disjoint edges in topological graphs, *Journal of Combinatorics* **1** (2010), 335–344.

[18] J. Pach and J. Törőcsik, Some geometric applications of Dilworth's theorem, *Discrete and Computational Geometry* **12** (1994), 1–7.

[19] R. Pinchasi, personal communication, October 2006.

[20] M. Sharir, The Clarkson-Shor technique revisited and extended, *Combinatorics, Probability and Computing* **12** (2003), 191–201.

[21] G. Tardos and G. Tóth, Crossing stars in topological graphs, *SIAM Journal on Discrete Mathematics* **21** (2007), 737–749.

[22] G. Tóth, Note on geometric graphs, *J. Comb. Theory Ser. A.* **89** (2000), 126–132.

[23] W. T. Tutte, Toward a theory of crossing numbers, *J. Comb. Theory* **8** (1970), 45–53.