The Number of Guillotine Partitions in d Dimensions*

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Abstract

Guillotine partitions play an important role in many research areas and application domains, e.g., computational geometry, computer graphics, integrated circuit layout, and solid modeling, to mention just a few. In this paper we present an exact summation formula for the number of structurally-different guillotine partitions in d dimensions by n hyperplanes, and then show that it is $\Theta\left(\left(2d-1+2\sqrt{d(d-1)}\right)^n/n^{3/2}\right)$.

Keywords: Combinatorial problems, guillotine partitions, binary space partitions.

1 Introduction

Given a d-dimensional box B in $\mathbb{R}^d$, a *guillotine partition* of B is a subdivision of B into smaller d-dimensional boxes obtained by first cutting B into two d-boxes by a hyperplane that is parallel to its axes, then recursively cutting these boxes in the same manner (changing the direction of the cut at will). Clearly, there are infinitely-many guillotine partitions with a given number of hyperplanes; however, if we look at the *structure* of the hierarchy that is formed by the hyperplanes, i.e., we care merely about the directions of the cuts rather than their exact positions, then we can count the finitely-many (structurally) different guillotine partitions by n hyperplanes in d dimensions. In this paper we are interested in finding this number, denoted $g_d(n)$; to the best of our knowledge, this problem for $d > 2$ has not been investigated to date (see [14] for a discussion of the case $d = 2$). For example, Figure 1 lists the $g_2(2) = 6$ possible guillotine partitions by $n = 2$ lines in the plane ($d = 2$). Table 1 provides the values of $g_d(n)$ for $d = 2, 3, 4$ and $n \leq 20$.

The hierarchical structure of guillotine partitions is useful in many areas, such as integrated circuit layout [8] (where $d = 2$) and approximation algorithms in computational geometry [3, 5, 6, 9]. Guillotine partitions are also the underlying structure of orthogonal *binary space partitions* (BSPs) which are widely used in computer graphics (e.g., for hidden-surface removal [4] and shadow generation [2]), solid modeling [13], motion planning [1], etc.

*Work on this paper by the first and second authors has been supported in part by AIM@SHAPE, a grant of the European Commission 6th Framework.

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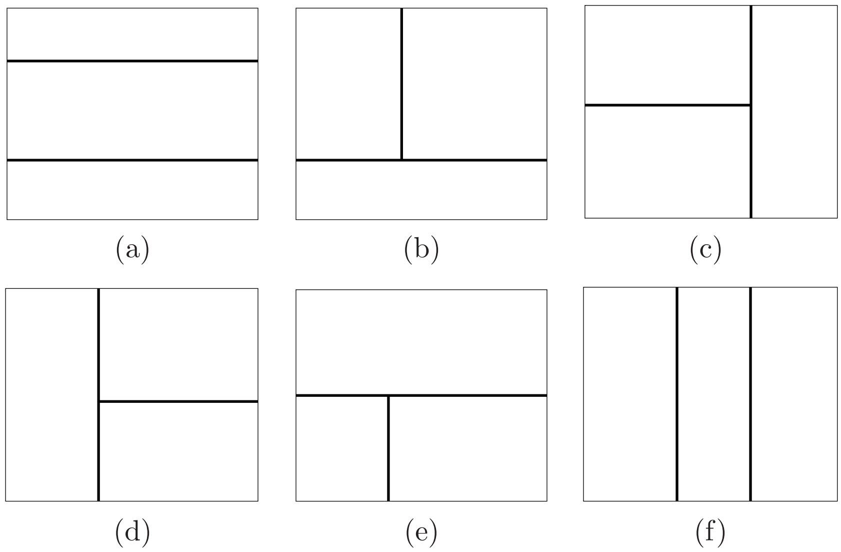

Figure 1: Guillotine partitions by two lines in the plane.

n	$d=2$	$d=3$	$d=4$
0	1	1	1
1	2	3	4
2	6	15	28
3	22	93	244
4	90	645	2,380
5	394	4,791	24,868
6	1,806	37,275	272,188
7	8,558	299,865	3,080,596
8	41,586	2,474,025	35,758,828
9	206,098	20,819,307	423,373,636
10	1,037,718	178,003,815	5,092,965,724
11	5,293,446	1,541,918,901	62,071,299,892
12	27,297,738	13,503,125,805	764,811,509,644
13	142,078,746	119,352,115,551	9,511,373,563,492
14	745,387,038	1,063,366,539,315	119,231,457,692,284
15	3,937,603,038	9,539,785,668,657	1,505,021,128,450,516
16	20,927,156,706	86,104,685,123,025	19,112,961,439,180,588
17	111,818,026,018	781,343,125,570,515	244,028,820,862,442,116
18	600,318,853,926	7,124,072,211,203,775	3,130,592,301,487,969,948
19	3,236,724,317,174	65,233,526,296,899,981	40,333,745,806,536,135,028
20	17,518,619,320,890	599,633,539,433,039,445	521,655,330,655,122,923,980

Table 1: First values of $g_d(n)$ for $d = 2, 3, 4$ and $n \leq 20$. The series $g_2(n)$ is the sequence of Schröder numbers [14].

Two-dimensional guillotine partitions are equivalent to what is known in integrated circuit layout as *slicing floor-plans* [8], whose number was shown to be the nth *Schröder* number [14]. These numbers arise in numerous other enumerative combinatorial problems [12, pp. 239–240]. One example is the number of paths on an orthogonal grid from $(0,0)$ to (n,n) that do not go above the line $y=x$ and use only the steps $(1,0)$, $(0,1)$, and $(1,1)$.

Recall that $g_d(n)$ denotes the number of guillotine partitions of a d-dimensional box in $\mathbb{R}^d$ by n hyperplanes. By analyzing the combinatorial properties of guillotine partitions, we show that $g_d(n) = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}(d-1)^k d^{n-k}$. Then, we analyze the asymptotic behavior of $g_d(n)$ and prove that it is $\Theta\Big(\big(2d-1+2\sqrt{d(d-1)}\big)^n/n^{3/2}\Big)$ for a fixed value of d. In fact, our analysis provides a rather accurate estimate of $g_d(n)$.

The paper is organized as follows. In Section 2 we compute a binomial-sum formula for $g_d(n)$ and provide its generating function, while in Section 3 we obtain an asymptotic value for $g_d(n)$ directly as well as by using the generating function. We end in Section 4 with some concluding remarks and a suggestion for a related open problem.

2 The Exact Number of Guillotine Partitions

In this section we derive a recursive formula, a binomial-sum formula, and a generating function for $g_d(n)$, the number of guillotine partitions by n hyperplanes in $\mathbb{R}^d$. We first define formally the notion of *structurally-different* guillotine partitions.

Let B be an axis-parallel d-dimensional box in $\mathbb{R}^d$. A *partition* (or *subdivision*) of B is a set S of $k>0$ interior-disjoint axis-parallel boxes $b_1, b_2, \ldots, b_k$ whose union equals B. A partition $S = \{b_1, b_2, \ldots, b_k\}$ of B is *guillotine* if $k=1$ or there are a hyperplane h and two disjoint non-empty subsets $S_1, S_2 \subset S$ such that:

1. h splits B into two interior-disjoint boxes B_1 and B_2;
2. S_1 is a guillotine partition of B_1; and
3. S_2 is a guillotine partition of B_2.

Clearly, the hyperplane h must be orthogonal to some axis x_i. We will assume without loss of generality that the interior of B_1 is below h with respect to x_i, and the interior of B_2 is above h. The definition of a guillotine partition implies a method by which one can obtain the partition of the box B into the small boxes $b_1, b_2, \ldots, b_k$: if $k=1$ then do nothing, otherwise cut B into two boxes B_1 and B_2 according to h, then continue by cutting recursively B_1 and B_2. One way to describe this cutting procedure is by a binary tree: The tree is a single node (and no edges) in case the d-dimensional box is not partitioned. Otherwise, the root of the tree contains the details of the hyperplane h; the left child of the root is the binary tree that corresponds to the guillotine partition of B_1; and the right child of the root is the binary tree that corresponds to the guillotine partition of B_2.

A guillotine partition can be represented by several trees: For example, an $n \times 1$ two-dimensional box (a rectangle) that is partitioned into n 1×1 boxes (squares) can be represented by C_{n-1} different trees (where C_n is the nth Catalan number). Another example is the two different trees representing a 2×2 two-dimensional box (square) partitioned into four 1×1 boxes (squares). For a *canonical* tree representation of a guillotine partition S of a box B, consider all the hyperplanes h that split B into two boxes B_1 and B_2 with the respective subpartitions $S_1, S_2 \subset S$ such that S_1 (respectively S_2) is a guillotine partition of B_1 (respectively B_2). Among these hyperplanes consider only those that are orthogonal to the axis x_i with the smallest index i, and among them choose the one which is above all the others with respect to x_i. The canonical tree will have this hyperplane as its root and the canonical trees representing guillotine partitions of the resulting two sub-boxes as children.

The hierarchy structure of a guillotine partition is then the canonical tree representing this partition, in which each node that corresponds to a hyperplane h only records the index i such that h is orthogonal to the axis x_i. We refer to this tree as the *structure-tree* of the guillotine partition.

Definition 1 *Two guillotine partitions by n hyperplanes of a d-dimensional box in $\mathbb{R}^d$ are* structurally equivalent *if they have the same structure-tree representation.*

Clearly, the number of different guillotine partitions of a box $B \subset \mathbb{R}^d$ by n hyperplane does not depend on the dimensions of B. Hence, the notation $g_d(n)$ is used for this number.

Observation 2 *Let $\mathcal{T}_d(n)$ be the set of binary trees with n nodes, such that every node has a label $\ell \in \{1, 2, \ldots, d\}$ and the label of every right child is different from the label of its parent node. Then $|\mathcal{T}_d(n)| = g_d(n)$.*

Proof: Clearly, two (structurally) different guillotine partitions are represented by two different binary trees such that each node in a tree has a label from $\{1, 2, \ldots, d\}$. Additionally, it follows from the canonical tree representation of a guillotine partition that a structure-tree of a guillotine partition cannot have a node whose right child has the same label as its parent. Therefore, $g_d(n) \leq |\mathcal{T}_d(n)|$. On the other hand, given $t \in \mathcal{T}_d(n)$, one can easily construct a guillotine partition of a d-dimensional box B whose structure-tree is t: Let r be the root of t and let i be the label of r. Cut B into two equal halves by a hyperplane h orthogonal to x_i. Then, cut recursively the half of B below h according to the subtree whose root is the left child of r, and similarly cut the half of B above h according to the subtree whose root is the right child of r. Thus, $|\mathcal{T}_d(n)| \leq g_d(n)$. $\square$

From this observation we can deduce that

$$g_d(n) = \sum_{k=0}^{n-1} N_{n,k} (d-1)^k d^{n-k},$$

where $N_{n,k}$ is the number of binary trees with n nodes, k of which are right children. ($N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ are the well-known *Narayana* numbers; see, for example, [7, pp. 1033–1034].) This follows easily from the fact that every node that is a right child has one of $d-1$ possible labels, while every other node has one of d possible labels. Thus, we obtain the following binomial-sum formula for the number of guillotine partitions in d dimensions:

Theorem 3

$$g_d(n) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} (d-1)^k d^{n-k}.$$

Observation 2 also yields a recursive formula for $g_d(n)$:

$$g_d(n) = d \cdot \left(g_d(n-1) + \sum_{k=1}^{n-1} \frac{d-1}{d} g_d(k) g_d(n-1-k) \right), \qquad g_d(0) = 1, \tag{1}$$

where k represents the number of nodes in the right subtree of the root, and the number of these subtrees is $\frac{d-1}{d}\, g_d(k)$ since the root of such a subtree cannot have the same label as its parent node. By rearranging terms is Equation (1), we have:

$$g_d(n) = g_d(n-1) + (d-1) \sum_{k=0}^{n-1} g_d(k) g_d(n-1-k), \qquad g_d(0) = 1.$$

From this recursive formula we can also easily compute $f_d(z)$, the generating function of $g_d(n)$:

$$\begin{aligned} f_d(z) &= \sum_{n=0}^{\infty} g_d(n) z^n \\ &= 1 + \sum_{n=1}^{\infty} \left(g_d(n-1) + (d-1) \sum_{k=0}^{n-1} g_d(k) g_d(n-1-k) \right) z^n \\ &= 1 + \sum_{n=1}^{\infty} g_d(n-1) z^n + (d-1) \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} g_d(k) g_d(n-1-k) z^n. \end{aligned} \tag{2}$$

It is readily seen that

$$\sum_{n=1}^{\infty} g_d(n-1) z^n = z \sum_{n=1}^{\infty} g_d(n-1) z^{n-1} = z f_d(z) \tag{3}$$

and

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} g_d(k) g_d(n-1-k) z^n = z \sum_{n=0}^{\infty} \sum_{k=0}^{n} g_d(k) g_d(n-k) z^n = z(f_d(z))^2. \tag{4}$$

By substituting Equations (3) and (4) in Equation (2), we obtain

$$f_d(z) = z f_d(z) + (d-1) z (f_d(z))^2 + 1. \tag{5}$$

One solution of Equation (5) is spurious, and thus we get

$$f_d(z) = \frac{1 - z - \sqrt{z^2 - 2(2d-1)z + 1}}{2(d-1)z}.$$

Aside from the intrinsic interest of this generating function, knowing the generating function of a combinatorial sequence is useful for obtaining information on the asymptotics of the sequence by the standard technique (which has been implemented in software) of analyzing the singularities of the function — see Remark 2 in Section 3. Generating functions are also a useful tool for manipulating combinatorial sequences, and for showing that they satisfy various identities, recurrence equations, etc.

3 The Asymptotic Number of Guillotine Partitions

In this section we compute the asymptotic number of guillotine partitions.

Theorem 4 *For every $d \in \mathbb{N}$, as $n \to \infty$*

$$g_d(n) = (1+o(1))\frac{\sqrt{2d(d-1)+(2d-1)\sqrt{d(d-1)}}}{2(d-1)\sqrt{\pi}} \cdot \frac{\left(2d-1+2\sqrt{d(d-1)}\right)^n}{n^{3/2}}. \qquad (6)$$

Proof: Let $g_d(n) = \sum_{k=0}^{n-1} b_d(n,k)$, where

$$b_d(n,k) = \frac{d^n (n!)^2 ((d-1)/d)^k (n-k)}{(k!)^2((n-k)!)^2 n(k+1)}.$$

By Stirling's formula, $m! = (1+O(m^{-1}))\sqrt{2\pi m}(m/e)^m$. Therefore,

$$b_d(n,k) = (1+O(k^{-1}+(n-k)^{-1}))\frac{n}{2\pi k^2}\exp(nQ_d(k/n)), \qquad (7)$$

where $Q_d : [0,1] \to [0,\infty)$ is the function defined by

$$Q_d(t) = \log d + t\log\frac{d-1}{d} - 2t\log t - 2(1-t)\log(1-t).$$

Let us analyze the properties of Q_d. Since

$$Q_d'(t) = \log\frac{d-1}{d} + 2\log\frac{1-t}{t} = \log\frac{(d-1)(1-t)^2}{dt^2},$$

it follows easily that Q_d has a unique maximum at $t = t_d := \sqrt{d(d-1)} - (d-1)$, with the value

$$Q_d(t_d) = \log(2d-1+2\sqrt{d(d-1)})$$

and second derivative

$$v_d := Q_d''(t_d) = \frac{-2}{1-t_d} - \frac{2}{t_d} = -2\cdot\frac{2d(d-1)+(2d-1)\sqrt{d(d-1)}}{d(d-1)}.$$

In other words, one has the Taylor expansion

$$Q_d(t) = \log(2d-1+2\sqrt{d(d-1)}) - \tfrac{2d(d-1)+(2d-1)\sqrt{d(d-1)}}{d(d-1)}(t-t_d)^2 + O((t-t_d)^3),$$

which is valid in some neighborhood of t_d.

Let $\varepsilon_n = n^{-5/12}$, and write

$$g_d(n) = \sum_{k=0}^{n-1} b_d(n,k) = \sigma_1 + \sigma_2,$$

where

$$\sigma_1 = \sum_{|k-t_d n|>n\varepsilon_n} b_d(n,k), \qquad \sigma_2 = \sum_{|k-t_d n|\le n\varepsilon_n} b_d(n,k).$$

First, we show that the contribution of σ_1 to $g_d(n)$ is negligible. Indeed, if $k \approx tn$, then

$$\frac{b_d(n,k-1)}{b_d(n,k)} = \frac{d(n-k-1)k^3}{(d-1)(n-k)(k+1)(n-k+1)^2} \approx \frac{dt^2}{(d-1)(1-t)^2}.$$

Since t_d has precisely the property that $dt^2/((d-1)(1-t)^2) < 1$ for $t < t_d$ and $dt^2/((d-1)(1-t)^2) > 1$ for $t > t_d$, this implies in particular that $b_d(n,k-1) < b_d(n,k)$ for all $k < (t_d - \varepsilon_n)n$, and $b_d(n,k-1) > b_d(n,k)$ for all $k > (t_d + \varepsilon_n)n$, and, therefore,

$$\begin{aligned}
\sigma_1 &\le n \cdot \max\Big(b_d(n, \lfloor (t_d-\varepsilon_n)n \rfloor), b_d(n, \lceil (t_d+\varepsilon_n)n \rceil)\Big) \\
&\le O\Big[\exp(n \max(Q_d(t_d-\varepsilon_n), Q_d(t_d+\varepsilon_n)))\Big] \\
&= (2d-1+2\sqrt{d(d-1)})^n O\Big[\exp\Big(-v_d n\varepsilon_n^2 + O(n\varepsilon_n^3)\Big)\Big] \\
&= o\Big((2d-1+\sqrt{d(d-1)})^n / n^{3/2}\Big).
\end{aligned}$$

Second, turn to σ_2, which contributes the bulk of the value of $g_d(n)$. Use Equation (7), noting that in this range of values of k, $O(k^{-1} + (n-k)^{-1}) = O(n^{-1})$, uniformly, and denoting $k = t_d n + u\sqrt{n}$:

$$\begin{aligned}
\sigma_2 &= (1+O(n^{-1})) \sum_{|k-t_d n|<n\varepsilon_n} \frac{1}{2\pi(t_d+u/\sqrt{n})^2 n} \exp(nQ_d(t_d+u/\sqrt{n})) \\
&= (1+O(n^{-1}+\varepsilon_n)) \sum_{|k-t_d n|<n\varepsilon_n} \frac{\big(2d-1+2\sqrt{d(d-1)}\big)^n}{2\pi t_d^2 n} \cdot \exp(-v_d u^2 + O(n\varepsilon_n^3)) \\
&= (1+O(n^{-1}+n^{-5/12}+n^{-1/4})) \frac{\big(2d-1+2\sqrt{d(d-1)}\big)^n}{2\pi t_d^2 n^{3/2}} \int_{-\infty}^{\infty} e^{-v_d u^2} du \\
&= (1+o(1)) \frac{\big(2d-1+2\sqrt{d(d-1)}\big)^n}{2\pi t_d^2 n^{3/2}} \cdot \frac{\sqrt{\pi}}{\sqrt{v_d}}.
\end{aligned}$$

This, upon some simple manipulations, gives Equation (6). □

Remarks:

1. More careful estimates can be used to improve the $(1+o(1))$ term to $(1+O(n^{-1}))$. A complete asymptotic expansion in powers of n^{-1} can also be obtained.
2. Relation (6) can be obtained from the generating function $f_d(z)$ using standard saddle-point techniques, as described, e.g., in [10]. The Maple package `gdev` [11] produces such asymptotic estimates automatically. Upon loading the package and typing the command

```
> equivalent(1-z-sqrt(z^2-2*(2*d-1)*z+1))/(2*(d-1)*z),z,n);
```

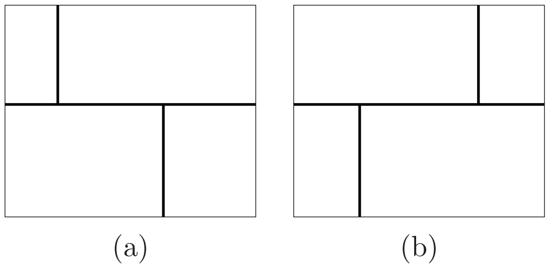

Figure 2: What is the number of guillotine partitions when these are considered different partitions?

one obtains an output, which after some reformatting and simplification, is seen to be equivalent to Equation (6).

When d is considered constant, Equation (6) readily yields:

Corollary 5 $g_d(n) = \Theta\left(\frac{\left(2d-1+2\sqrt{d(d-1)}\right)^n}{n^{3/2}}\right)$.

For example, there are $\Theta((3+2\sqrt{2})^n/n^{3/2}) \approx \Theta(5.828^n/n^{3/2})$ (resp., $\Theta((5+2\sqrt{6})^n/n^{3/2})$ $\approx \Theta(9.900^n/n^{3/2})$ structurally-different guillotine partitions with n lines (resp., planes) in the plane (resp., in 3-space).

4 Conclusion

In this paper we give a tight asymptotic estimation of the number of guillotine partitions in any dimension using any number of hyperplanes. This provides context to an enumeration of all guillotine partitions, for example, for finding the one that optimizes a given measure.

An interesting related open problem arises when we also care about the *relative order* between guillotine cuts on opposite sides of their parent cut; for example, we distinguish between the partitions shown in Figures 2(a) and 2(b). This gives rise to new modeling and enumeration problems which have some bearing on floor-planning of integrated circuits.

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