On the Degenerate Crossing Number

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Abstract

The *degenerate crossing number* $\mathrm{cr}^*(G)$ of a graph G is the minimum number of crossing *points* of edges in any drawing of G as a simple topological graph in the plane. This notion was introduced by Pach and Tóth who showed that for a graph G with n vertices and $e \geq 4n$ edges $\mathrm{cr}^*(G) = \Omega(e^4/n^4)$. In this paper we completely resolve the main open question about degenerate crossing numbers and show that $\mathrm{cr}^*(G) = \Omega(e^3/n^2)$, provided that $e \geq 4n$. This bound is best possible (apart for the multiplicative constant) as it matches the tight lower bound for the standard crossing number of a graph.

Keywords: Crossing Lemma, crossing number, simple topological graphs.

1 Introduction

The graphs considered in this paper contain no loops or parallel edges. A *topological graph* is a drawing of a graph in the plane such that the vertices are drawn as distinct points and the edges are drawn as Jordan arcs connecting corresponding points without passing through other vertices of the graph. Two edges in a topological graph may intersect at a finite number of points, where in each intersection point they either share a common endpoint or properly cross each other. If every pair of edges intersect at most once, then the topological graph is called *simple*. One sometimes assumes that in a topological graph there are no three edges that cross each other at the same point. However, in this paper we are interested in topological graphs in which such crossings are allowed.

For a topological graph D we denote by $\mathrm{cr}(D)$ the number of crossings of pairs of edges in D. The *crossing number* of an abstract graph G, denoted by $\mathrm{cr}(G)$, is the minimum value of $\mathrm{cr}(D)$ taken over all drawings D of G as a topological graph. It is not hard to see that if $\mathrm{cr}(G) = \mathrm{cr}(D)$, then D is a simple topological graph.

Ajtai, Chvátal, Newborn, Szemerédi [1] and, independently, Leighton [4] proved that there is an absolute constant $c > 0$ such that for every abstract graph G with n vertices and e edges $\mathrm{cr}(G) \geq c\frac{e^3}{n^2}$, provided that $e \geq 4n$. This result is referred to as the *Crossing Lemma* and has numerous applications in combinatorial and computational geometry, number theory, and other fields of mathematics.

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Several works have considered different crossing numbers (see [2, 6, 8] and also [3, § 9.4], [5, § 5.3], and the references therein). One such example is the *degenerate* crossing number. This notion was first introduced in [7] following questions by G. Rote, M. Sharir, and others who asked what happens if we count crossing *points*, rather than crossings of pairs of edges? For example, if $k \geq 2$ edges in a topological graph D cross each other at the same point, then we count this point only once, instead of $\binom{k}{2}$ times as in $\mathrm{cr}(D)$.

Pach and Tóth [7] proved that in any drawing of a graph G with n vertices and e edges the number of crossing points is at least $\frac{1}{3}e - n + 2$, and that this bound cannot be improved by much. Namely, they showed that any graph G with e edges can be drawn as a topological graph containing at most $e - 1$ crossing points. However, in their construction a pair of edges may cross many times. Therefore, they have also considered the minimum number of crossing points in any possible drawing of a graph as a *simple* topological graph.

For a *simple* topological graph D, denote the number of crossing points of edges of D by $\mathrm{cr}^*(D)$. For an abstract graph G, we denote by $\mathrm{cr}^*(G)$ the *degenerate* crossing number of G. That is, the minimum value of $\mathrm{cr}^*(D)$ taken over all possible drawings of G as a simple topological graph D. In [7] it is shown that $\mathrm{cr}^*(G) = \Omega(\frac{e^4}{n^4})$, and the question remained whether the behavior of $\mathrm{cr}^*(G)$ is very much different from $\mathrm{cr}(G)$. Or, in the words of [5], "it is a challenging question to decide whether the $\frac{e^4}{n^4}$ term can be replaced by $\frac{e^3}{n^2}$, just like in the Crossing Lemma." In this paper we answer this question in the affirmative.

Theorem 1. *Let G be a simple topological graph with n vertices and $e \geq 4n$ edges. Then the number of points in which two or more edges of G cross each other is at least $\Omega(\frac{e^3}{n^2})$.*

2 The Proof

For a (simple) topological graph G and an integer $k \geq 2$, we denote the number of crossing points of precisely k edges in G by $t_k(G)$. Therefore, $\mathrm{cr}(G) = \sum_k \binom{k}{2} t_k(G)$ and $\mathrm{cr}^*(G) = \sum_k t_k(G)$. Roughly speaking, the idea of the proof is to show that $\sum_k t_k(G) \geq \Omega(\sum_k k t_k(G))$ and to also give a lower bound for $\sum_k k t_k(G)$ in terms of $\sum_k \binom{k}{2} t_k(G)$).

Lemma 1. *Let G be a (connected) simple topological graph with $n > 2$ vertices and e edges. Then $t_2(G) + t_3(G) \geq \frac{1}{8} \sum_{k \geq 2} k t_k(G)$.*

Proof. Denote by $d_1, \ldots, d_n$ the degrees of the n vertices of G. Consider the planar map obtained by adding the crossing points of G as new vertices and subdividing the edges accordingly. Let V, E, and F, be the numbers of vertices, edges, and faces, respectively, of this planar map, and let f_k denote its number of faces with precisely k edges. Then: $V = n + \sum_k t_k(G)$, $\sum_i d_i + \sum_k 2k t_k(G) = 2E = \sum_k k f_k$, and $F = \sum_k f_k$.

From Euler's polyhedral formula in the plane ($V - E + F = 2$) we have:

$$6 = 3V - 3E + 3F = 3n + 3\sum_k t_k(G) - \frac{1}{2}\sum_i d_i - \sum_k k t_k(G) - \sum_k k f_k + 3\sum_k f_k.$$

Note that every face in the planar map has at least 3 edges because G is simple and $n > 2$. Therefore, after rearranging the above equality, we get

$$2t_2(G) + 2t_3(G) = t_2(G) + 2t_3(G) + \sum_{k \geq 4}(k-3)t_k(G) + \sum_{k \geq 3}(k-3)f_k + e - 3n + 6 \geq \frac{1}{4}\sum_{k \geq 2} k t_k(G),$$

and the lemma is proved. □

We have thus showed that $\sum_k t_k(G) \geq \Omega(\sum_k kt_k(G))$. Our next goal is to give lower bound for $\sum_k kt_k(G)$ in terms of $\sum_k \binom{k}{2} t_k(G)$. To this end, we first prove an upper bound on the number of incidences between edges of a topological graph and (crossing) points.

Lemma 2. *Let G be a simple topological graph with e edges and let X be a set of $|X|$ distinct points. The number of incidences $I(X, E(G))$ between the points of X and the edges of G satisfies $I(X, E(G)) = O(|X|^{2/3}(cr(G))^{1/3} + |X| + e)$.*

Proof. We define a simple topological graph H whose vertices are the points of X. We connect two points of X by an edge if they are consecutive points of X along an edge w of G. We draw this edge along the portion of w delimited by these two vertices.

Clearly, we have $I(X, E(G)) \leq |E(H)| + e$. Observe that H is a simple topological graph on $|X|$ vertices and has at most $\mathrm{cr}(G)$ pairs of crossing edges. It follows from the Crossing Lemma that $(|E(H)| - 4|X|)^3/|X|^2 = O(\mathrm{cr}(G))$. Therefore,

$$I(X, E(G)) \leq |E(H)| + e = O(|X|^{2/3}\mathrm{cr}(G)^{1/3} + |X| + e),$$

as claimed. □

The following lemma is a direct corollary of Lemma 2 and the fact that $\mathrm{cr}(G) = \sum_{k\geq 2} \binom{k}{2} t_k(G)$.

Lemma 3. *There is a constant c_1 such that for every simple topological graph G with e edges and for every integer $k' \geq 2$ we have:*

$$\sum_{k\geq k'} t_k(G) \leq c_1 \cdot \left(\frac{\sum_{k\geq 2} \binom{k}{2} t_k(G)}{k'^3} + \frac{e}{k'} \right).$$

Proof. Let X denote the set of all intersection points of edges of G through which at least k' edges of G pass. We have $|X| = \sum_{k\geq k'} t_k(G)$. Therefore,

$$k'|X| \leq I(X, E(G)) = O(|X|^{2/3}\mathrm{cr}(G)^{1/3} + |X| + e) = O\left(|X|^{2/3} \left(\sum_{k\geq 2} \binom{k}{2} t_k(G) \right)^{1/3} + |X| + e \right).$$

From here we deduce:

$$\sum_{k\geq k'} t_k(G) = |X| = O\left(\frac{\sum_{k\geq 2} \binom{k}{2} t_k(G)}{k'^3} + \frac{e}{k'} \right).$$

□

Using Lemma 3, we can now give a lower bound for $\sum_k kt_k(G)$ in terms of $\sum_k \binom{k}{2} t_k(G)$ and the maximum number of edges in G that cross at the same point.

Lemma 4. *If G is a simple topological graph with e edges and $B \geq 2$ is an integer such that no more than B edges of G cross at the same point, then*

$$\sum_k k t_k(G) \geq \frac{1}{8c_1} \sum_k \binom{k}{2} t_k(G) - eB,$$

where c_1 is the constant from Lemma 3.

Proof. Recall that by Lemma 3 for every $k' \geq 2$ we have

$$\sum_{k \geq k'} t_k(G) \leq c_1 \left(\frac{\sum_{k\geq 2} \binom{k}{2} t_k(G)}{k'^3} + \frac{e}{k'} \right).$$

Let $s = 8c_1$. We have

$$\begin{aligned}
\sum_{k\geq 2} \binom{k}{2} t_k(G) &= \sum_{k=2}^{2^{\lfloor \log_2 s \rfloor}} \binom{k}{2} t_k(G) + \sum_{i=\lfloor \log_2 s \rfloor}^{\lceil \log_2 B \rceil} \sum_{k=2^i+1}^{2^{i+1}} \binom{k}{2} t_k(G) \\
&\leq \frac{s}{2} \sum_{k\geq 2} k t_k(G) + \sum_{i=\lfloor \log_2 s \rfloor}^{\lceil \log_2 B \rceil} \binom{2^{i+1}}{2} \sum_{k \geq 2^i} t_k(G) \\
&\leq \frac{s}{2} \sum_{k\geq 2} k t_k(G) + \sum_{i=\lfloor \log_2 s \rfloor}^{\lceil \log_2 B \rceil} \frac{2^{2i+2}}{2} c_1 \left(\frac{\sum_{k\geq 2} \binom{k}{2} t_k(G)}{2^{3i}} + \frac{e}{2^i} \right) \\
&\leq \frac{s}{2} \sum_{k\geq 2} k t_k(G) + 4c_1 \frac{\sum_{k\geq 2} \binom{k}{2} t_k(G)}{s} + 4c_1 eB \\
&\leq 4c_1 \sum_{k\geq 2} k t_k(G) + \frac{1}{2} \sum_{k\geq 2} \binom{k}{2} t_k(G) + 4c_1 eB. \qquad (1)
\end{aligned}$$

From (1) we conclude:

$$\sum_{k\geq 2} \binom{k}{2} t_k(G) \leq 8c_1 \sum_{k\geq 2} k t_k(G) + 8c_1 eB,$$

and the lemma follows. □

For technical reasons we will have to assume that the number of edges in the graphs that we consider is at least eight times the number of their vertices. For sparser graphs we will use the following weak bound on the degenerate crossing number.

Lemma 5. *Let G be a simple topological graph with $n > 2$ vertices and e edges. Then $cr^*(G) \geq e - 3n + 6$.*

Proof. Let G' be the plane graph we obtain by turning every crossing point of G into a new vertex and subdividing the edges accordingly. Denote by n' and e' the number of vertices and edges of G', respectively, and let $x = n' - n$ be the number of crossing points in G. Then $e' \geq e + 2x$, since every new vertex we add, subdivides at least two edges. The graph G' has no parallel edges, since G is a simple topological graph. Therefore, $e + 2x \leq e' \leq 3n' - 6 = 3(n + x) - 6$. Hence, $x \geq e - 3n + 6$. □

We note that Lemma 5 is also a slight improvement, for the case of simple topological graphs, of the result in [7], where it is shown that the number of crossing points of edges in G is at least $\frac{1}{3}e - n + 2$.

We are now ready to prove Theorem 1.

Proof of Theorem 1: Let G be a simple topological graph with n vertices and $e \geq 4n$ edges. If $4n \leq e < 8n$, then the claimed bound in Theorem 1 follows from Lemma 5.

Therefore, from now on, assume that $e \geq 8n$. Notice that we may also assume that G has at least n distinct crossing points of edges. Indeed, otherwise turn all crossing points of edges in G into vertices and G becomes a planar graph G' with at most $2n - 1$ vertices. The number of edges in G is at most the number of edges in G'. This in turn is at most $3(2n-1) - 6 < 8n$ contradicting our assumption that the number of edges in G is at least $8n$.

Let $P_1, P_2, P_3, \ldots$ be the crossing points of edges in G, and let g_i be the number of edges that cross each other at P_i. Assume without loss of generality that $g_1 \geq g_2 \geq g_3 \geq \ldots$. We start by preforming the following change in the graph G: Each of the n crossing points $P_1, P_2, \ldots, P_n$ becomes a vertex and subdivides the edges containing it accordingly. Denote the resulting graph by G' and note that the number n' of vertices in G' satisfies $n' = 2n$. Notice also that with each P_i we added at least $B = g_n$ new edges to the graph and therefore the number e' of edges in G' satisfies $e' \geq e + nB$. It is very important to notice that no more than B edges in G' may cross at the same point.

Observe that $\sum_{k \geq 2} \binom{k}{2} t_k(G') = \sum_{k=2}^{B} \binom{k}{2} t_k(G')$ counts the number of pairs of crossing edges of G' and therefore,

$$\sum_{k \geq 2} \binom{k}{2} t_k(G') \geq c\frac{e'^3}{n'^2}, \tag{2}$$

where c is the constant from the Crossing Lemma (observe that $e' \geq 4n'$ as $e' \geq e \geq 8n = 4n'$).

Let $c_0 > 0$ be an absolute constant to be determined later. If $B \leq c_0$, then (2) implies that

$$\binom{c_0}{2} \sum_{k \geq 2} t_k(G') = \binom{c_0}{2} \sum_{k=2}^{B} t_k(G') \geq \sum_{k=2}^{B} \binom{k}{2} t_k(G') \geq c\frac{e'^3}{n'^2}.$$

Hence,

$$\mathrm{cr}^*(G) \geq \mathrm{cr}^*(G') = \sum_{k \geq 2} t_k(G') \geq \frac{c}{\binom{c_0}{2}} \frac{e'^3}{n'^2} \geq \frac{c}{\binom{c_0}{2}} \frac{e^3}{4n^2}$$

and thus Theorem 1 is proved in this case.

We therefore assume that $B > c_0$. Lemmas 1 and 4 together with (2) imply:

$$t_2(G') + t_3(G') \geq \frac{1}{8} \sum_{k \geq 2} k t_k(G') \geq \frac{1}{64c_1} \sum_{k \geq 2} \binom{k}{2} t_k(G') - \frac{1}{8} e' B \geq \frac{c}{64c_1} \frac{e'^3}{n'^2} - \frac{1}{8} e' B. \tag{3}$$

We claim that

$$\frac{1}{8} e' B \leq \frac{1}{2} \cdot \frac{c}{64c_1} \frac{e'^3}{n'^2}. \tag{4}$$

Indeed, if (4) is false, then $e' < 4n'\sqrt{c_1 B/c}$. However, recall that $e' \geq Bn = Bn'/2$, and therefore we get that $B < 64c_1/c$. This leads to a contradiction once we choose $c_0 = 64c_1/c$, as we assume that $B > c_0$.

Once we established (4), then together with (3), we have

$$\mathrm{cr}^*(G) \geq t_2(G) + t_3(G) \geq t_2(G') + t_3(G') \geq \frac{c}{128c_1}\frac{e'^3}{n'^2} \geq \frac{c}{2^9 c_1}\frac{e^3}{n^2},$$

where the last inequality is because $e' \geq e$ and $n' = 2n$. □

Remarks. Note that if we had not added the new vertices $P_1, P_2, \ldots, P_n$ to G, then we would have had to assume that $e \geq \Omega(n^{3/2})$ for the proof to hold.

In our proof of Theorem 1 we did not make any use of the fact that G is a *simple* topological graph except for when we argued that no face in the planar map induced by G is a digon, that is, a face with two edges. This implies that our Theorem 1 is valid also for non-simple topological graphs G, and $\mathrm{cr}^*(G) = \Omega(\frac{e^3}{n^2})$ also for those graphs, provided that there are no digons in the planar map induced by G. It follows that the construction in [7] of a topological graph with at most $e - 1$ crossing points is possible only if many digons are introduced, and indeed, not very surprisingly, this is the case there.

As an easy corollary of Theorem 1 we get:

Corollary 1. *There is a constant c' such that for every integer $k > 0$ the following holds. If G is a simple topological graph with n vertices and e edges in which every edge contains at most k crossing points, then $e \leq c'n\sqrt{k}$.*

Proof. By choosing $c' > 4$ the upper bound holds trivially if $e < 4n$. Otherwise, it follows from Theorem 1 that there is a constant c^* such that $\mathrm{cr}^*(G) \geq c^* \frac{e^3}{n^2}$. On the other hand, every edge of G contains at most k crossing points, therefore $\mathrm{cr}^*(G) \leq ke$. Combining these two estimates, we conclude that $e \leq n\sqrt{k/c^*}$. □

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