An Upper Bound on the Number of Rectangulations of a Point Set*

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Abstract. We consider the number of different ways to divide a rectangle containing n noncorectilinear points into smaller rectangles by n non-intersecting axis-parallel segments, such that every point is on a segment. Using a novel counting technique of Santos and Seidel [12], we show an upper bound of $O(20^n/n^4)$ on this number.

1 Introduction

Given a set P of n points within an axis-parallel rectangle R, a *rectangulation* of (R, P) is a set of non-intersecting segments that partitions R into smaller rectangles, such that every point in P is on a segment. See Figure 1 for examples of rectangulations.

The problem of finding a rectangulation with a minimum total length of the segments has attracted considerable attention in the literature. Lingas, Pinter, Rivest, and Shamir [10] introduced it as a special case of a problem with applications to VLSI design, and showed that it is NP-hard. Since then, several approximation algorithms have been suggested (e.g., [7–9]), including a polynomial-time approximation scheme [11]. De Meneses and de Souza [6] suggested integer-programming formulations and techniques to find exact solutions for medium sized instances of the problem.

When the points are *noncorectilinear*, i.e., no two points share the same x or y coordinate, the complexity class of the minimization problem is unknown. However, it can be shown [3] that the optimal solution in this case consists of exactly n segments. Hereafter, we consider only such rectangulations and investigate the following question:

> Given a set P of n noncorectilinear points within a rectangle R, how many different rectangulations (by n segments) of (R, P) are there?

A similar question, that of the number of *triangulations* of the convex hull of a set of n points in the plane, has attracted considerable attention in the

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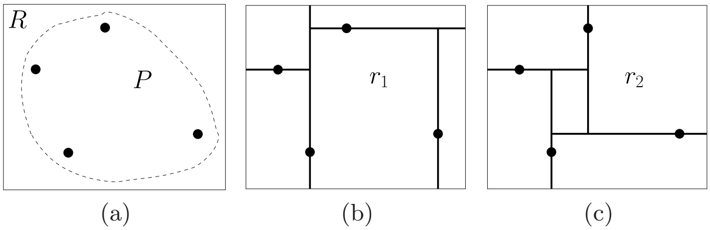

Fig. 1. Rectangulations of (R, P)

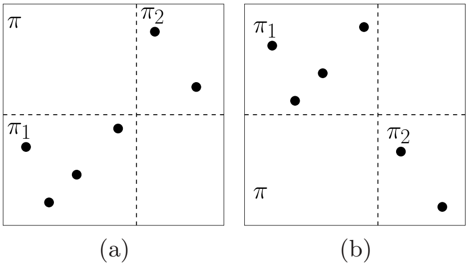

Fig. 2. Point sets in separable permutations

literature. The first singly-exponential upper bound on the number of triangulations, $O(173,000^n)$, was given by Smith [15]. The upper bound was improved by Seidel [13] to $O(2^{12.245113n-\Theta(\log n)}) \approx O(4,855^n)$ and by Denny and Sohler [5] to $O(2^{8.2n+O(\log n)}) \approx O(294^n)$. The best currently-known upper bound, $O(59^n/n^6)$, is due to Santos and Seidel [12].

In a previous paper [1] we observed that the number of rectangulations of a point set P depends only on the relative order of the points in P, which can be represented by a permutation on n. We proved that if the permutation of the points is *separable* [2][1], then the number of rectangulations is exactly the $(n+1)$st Baxter number, which is [4, 14]:

$$\mathrm{B}(n+1) = \sum_{r=0}^{n} \frac{\binom{n+2}{r}\binom{n+2}{r+1}\binom{n+2}{r+2}}{\binom{n+2}{1}\binom{n+2}{2}} = \Theta(8^n/n^4).$$

(In [1] we also observed that the number of separable permutations on n is the $(n-1)$st Schröder number $r_n = \sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n}{k-1}/n = \Theta((3+\sqrt{8})^n/n^{1.5})$. Thus,

[1] A separable permutation is either a permutation on one element or the concatenation of two separable permutations. Formally, let $\pi_1 = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ and $\pi_2 = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ be two permutations on n and m, respectively. We say that $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n+m})$ is the result of concatenating π_2 *above* π_1 if $\sigma_i = \alpha_i$ for $1 \leqslant i \leqslant n$ and $\sigma_{n+i} = n + \beta_i$ for $1 \leqslant i \leqslant m$ (see Figure 2(a)). Likewise, we say that $\pi = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n+m})$ is the result of concatenating π_2 *below* π_1 if $\sigma_i = m + \alpha_i$ for $1 \leqslant i \leqslant n$ and $\sigma_{n+i} = \beta_i$ for $1 \leqslant i \leqslant m$. (see Figure 2(b)). Then, a permutation π is a *separable* if 1. $\pi = (1)$; or 2. There are two separable permutations π_1 and π_2 such that π is the the concatenation of π_2 above or below π_1.

the portion of separble permutations out of the $n!$ permutations is asymptotically zero.)

We observed empirically that the number of rectangulations of all other sets of n points in non-separable permutations is strictly larger than the $(n+1)$st Baxter number. This was done by counting systematically all the rectangulations of sets of up to 10 points in all possible permutations. Nevertheless, we were unable to prove that this is true, that is, that the $(n+1)$st Baxter number is a lower bound on the number of rectangulations of all point sets of size n.

It is easy to show super-exponential upper bounds. For example, assume (without loss of generality) that there are fewer vertical segments than horizontal segments in any rectangulation. Then choose the endpoints of the at most $n/2$ vertical segments; for each such segment there are no more than $\binom{n+1}{2}$ options. After determining the vertical segments, all the horizontal segments are unique: they extend on both sides of the yet unused points until hitting the *interior* of the first vertical segment (or the bounding rectangle). This yields the upper bound $O(\binom{n+1}{2}^{n/2})$, which is $O(n^n)$.

Another method uses the fact that the number of "point-free" rectangulations (also known as *floor-plans* – subdivisions of a rectangle into smaller isothetic rectangles) is also related to Baxter numbers [16] and is thus $\Theta(8^n/n^4)$. Each such floor-plan can be trivially associated with at most $n!$ permutations, hence we obtain the slightly upper bound $O(n!\,8^n/n^4)$.

An even better – but still super-exponential – upper bound can be obtained from the fact that in any rectangulation there always exists a segment s that touches at most three other segments. By removing s and the point p on it (and extending the segment supported by s, if such segment exists), we obtain a rectangulation of $n-1$ points. Now, there are exactly six possible ways of inserting s into this rectangulation. Suppose s was horizontal. Then, if s touched exactly two vertical segments, we stretch a horizontal segment from p, in both directions, until hitting a vertical segment. If s touched exactly three vertical segments, then there are two possibilities: s must "chop" the first vertical segment either to its right or to its left. Since s might be vertical, we have a total of six possibilities. To be able to construct the rectangulation all we need to store is the way every point and segment are added and the order of the points. Thus the number of rectangulations is $O(n!\,6^n)$.

In the following section we show that the number of rectangulations of a set of n noncorectilinear points (arranged in any arbitrary permutation) is $O(20^n/n^4)$. This is the first proven singly-exponential upper bound on the number of rectangulations of any point set.

2 The Upper Bound

Our main result is:

Theorem 1. *The maximum number of rectangulations of n noncorectilinear points (by n segments) is $O(20^n/n^4)$.*

Proof. The proof follows the structure of the proof of the upper bound on the number of triangulations of a planar point set, given in [12]. We denote by $f(n)$ the maximum number of rectangulations of n points. Let P be a set of n noncorectilinear points within a rectangle R, and let r be a rectangulation of (R, P). A *T-junction* is an endpoint of a segment on another segment, or on the boundary. The *degree* of a point $p \in P$ in r is the number of T-junctions on the segment that contains p. For example, the rightmost point in P in Figure 1 has degree 2 in r_1 and degree 3 in r_2. Let n_i^r be the number of points with degree i in r, then clearly $n = \sum_i n_i^r$.

Every segment is bounded by two T-junctions, thus every segment s contributes at most four to the total sum of degrees: two to the point it contains, and one to every point that is contained in a segment bounding s (if it is not a boundary segment). Note that the point on s might have a degree greater than four, however we charge other segments for their contribution to this degree. Therefore, the total sum of degrees is $4n - b$, where b is the number of T-junctions on the boundary of R in r. It is easy to verify that if $n \geq 3$, then $b \geq 4$. Thus, for $n \geq 3$ we have

$$4n - 4 \geq \sum_i i \cdot n_i^r.$$

Easy manipulations show that

$$\begin{aligned} 4\sum_i n_i^r &\geq 4 + \sum_i i \cdot n_i^r, \\ \sum_i (4 - i) n_i^r &\geq 4, \text{ and} \\ \sum_i (5 - i) n_i^r &\geq 4 + \sum_i n_i^r = n + 4. \end{aligned}$$

Considering only the positive summands on the left-hand side of the last equation we have:

$$3n_2^r + 2n_3^r + n_4^r \geq n + 4. \tag{1}$$

Denote by h_i the maximum number of rectangulations of (R, P) that one can obtain by adding some point $p \in P$ to a rectangulation r' of $(R, P \setminus \{p\})$ and "stretching" the segment through p such that the degree of p in the resulting rectangulation is i. Clearly, $h_2 = 2$, since the segment through p can be either vertical or horizontal and we must stop "stretching" it as soon as it hits another segment in each direction. Similarly, $h_3 = 4$, since when the orientation of the segment through p is horizontal (resp., vertical), then we must "chop" the first segment either to the left (resp., below) or to the right (resp., above) of p. Note that segments that were supported by the chopped part of the segment are extended until they hit another segment or the boundary (see Figures 3(d,e) for examples). Likewise, $h_4 \leq 6$ and in general $h_i \leq 2(i - 1)$.

Let N_i be the number of points with degree i in all the rectangulations of (R, P). Then,

$$N_i \leq n \cdot h_i \cdot f(n - 1),$$

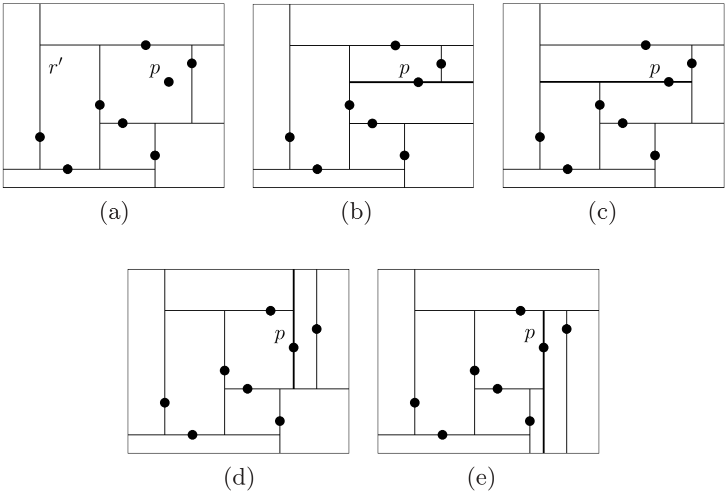

Fig. 3. Four possible ways of adding p to r' such that the degree of p is 3

Table 1. Empirical results of the maximum number of rectangulations

n	B$(n+1)$	Maximum number of rectangulations
4	92	93
5	422	428
6	2,074	2,122
7	10,754	11,092
8	58,202	60,524
9	326,240	342,938
10	1,882,960	2,000,856

and specifically $N_2 \leq 2n \cdot f(n-1)$, $N_3 \leq 4n \cdot f(n-1)$, and $N_4 \leq 6n \cdot f(n-1)$.

We now prove by induction on n that $f(n) \leq 20^n / \binom{n+4}{4}$. For $n = 0, 1, 2$ the claim holds trivially ($f(0) = 1 = 20^0/\binom{4}{4}$, $f(1) = 2 < 4 = 20^1/\binom{5}{4}$, and $f(2) = 6 < 26.666... = 20^2/\binom{6}{4}$). Now assume that the claim holds for all $n' < n$, for $n \geq 3$. By summing Equation 1 over all possible rectangulations, we have:

$$3N_2 + 2N_3 + N_4 \geq (n+4)f(n) \tag{2}$$

On the left-hand side of Equation 2 we have:

$$20n \cdot f(n-1) \leq 20n \frac{20^{n-1}}{\binom{n+3}{4}} = (n+4)\frac{20^n}{\binom{n+4}{4}}.$$

Hence $f(n) = O(20^n/n^4)$, and the claim follows.

3 Conclusions

We have showed that the number of rectangulations of a set of n noncorectilinear points is $O(20^n/n^4)$. However, according to our experiments for small values of n (see Table 1), it seems that the maximum number of rectangulations is much closer to the $\mathrm{B}(n+1) = \Theta(8^n/n^4)$ lower bound from [1]. As mentioned in the introduction, we also believe that for *every* set of n (noncorectilinear) points, the number of rectangulations is at least the $(n+1)$st Baxter number.

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