The maximum number of edges in geometric graphs with pairwise virtually avoiding edges

Eyal Ackerman* Noa Nitzan† Rom Pinchasi†

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Abstract

Let G be a geometric graph on n vertices that are not necessarily in general position. Assume that no line passing through one edge of G meets the relative interior of another edge. We show that in this case the number of edges in G is at most $2n-3$.

1 Introduction

A *geometric graph* is a graph drawn in the plane such that its vertices are distinct points and its edges are straight-line segments connecting corresponding vertices. It is assumed that no edge of a geometric graph passes through a vertex of the graph (other than its endpoints). Two edges in a geometric graph are called *avoiding* (sometimes also *parallel*) if they form opposite sides of a convex quadrilateral. It was shown by Katchalski and Last [6] and Valtr [11] (see also [9]) that the maximum number of edges in a geometric graph on n vertices with no pair of avoiding edges is $2n-2$. Many such bounds on the size of geometric graphs avoiding certain geometric patterns were studies in the past years (see [5, §9.5 and §9.6] for some examples).

In most (perhaps even all) of these works the notion of a geometric graph usually refers to a graph on a set of vertices drawn as points in *general position*, that is, no three of the vertices are on a line. In this paper we will particularly be interested in the case where the vertex set of the graph may contain more than two points on a line, and investigate the following question: What is the maximum number of edges in a geometric graph G on n vertices in which every pair of edges are *virtually avoiding*?

Definition 1.1 (virtually avoiding edges)**.** *Let e_1 and e_2 be two edges in a geometric graph, and let ℓ_1 and ℓ_2 be the lines containing them, respectively. Then e_1 and e_2 are called* virtually avoiding *if ℓ_1 does not intersect the relative interior of e_2 and ℓ_2 does not intersect the relative interior of e_1. Otherwise, e_1 and e_2 are called* virtually crossing*.*

See Figure 1 for examples of these notions. Notice that when the points are in general position two edges are virtually avoiding if and only if they are avoiding or have a common vertex. However, for points that are not necessarily in general position, these notions are even further apart, as it may happen that a line containing one of two virtually avoiding edges meets a vertex of the other edge (see for example e_1 and e_6 in Figure 1).

*Department of Mathematics, Physics, and Computer Science, University of Haifa at Oranim, Tivon 36006, Israel. `ackerman@sci.haifa.ac.il`.

†Mathematics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel. `noanitzan@math.huji.ac.il, room@math.technion.ac.il`. Supported by ISF grant (grant No. 1357/ 12) and by BSF grant (grant No. 2008290).

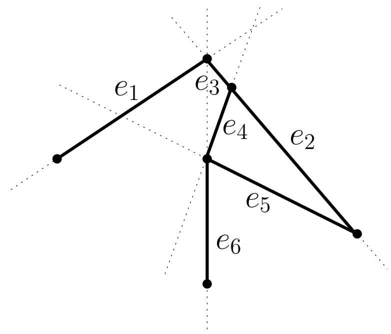


Figure 1: Both pairs of edges (e_1, e_4) and (e_2, e_6) are avoiding. Both (e_1, e_3) and (e_1, e_6) are virtually avoiding. (e_1, e_5), (e_2, e_3), and (e_1, e_7) are all virtually crossing.

Observe that virtually avoiding edges cannot be collinear and cannot cross each other. The latter property implies that a graph with pairwise virtually avoiding edges is planar and hence has at most $3n - 6$ edges, for $n \geq 3$. Note also that in a triangulation of a set of n vertices in (strictly) convex position every pair of edges are virtually avoiding and there are $2n - 3$ edges. It is easy to show by induction that no other graph G with pairwise virtually avoiding edges has more edges if its vertices are in general position.

To see the inductive argument observe that we may assume that the degree of each vertex is at least 3, or else we may conclude by the induction hypothesis. Let x be a vertex of G that is also extreme on the convex hull of the set of vertices of G. Because the degree of x is at least 3, there must be an edge e adjacent to x that is not an edge of the convex hull of the set of vertices of G. The line ℓ through e divides the rest of the vertices of G into two non-empty sets of cardinalities n_1 and n_2, respectively. Notice also that no edge of G crosses ℓ, as this will contradict our assumption that every pair of edges in G are virtually avoiding. We can therefore conclude the theorem by the induction hypothesis on the two parts of G defined by ℓ where the vertices of e are common to both parts: The number of edges in G is therefore at most

$$(2(n_1 + 2) - 3) + (2(n_2 + 2) - 3) - 1 = 2n - 3,$$

where the -1 in the left hand side is due to the fact that e is counted in both parts of G.

Perhaps somewhat surprisingly the same question becomes significantly less trivial if we allow the set of vertices of G to be not in general position. The simple inductive argument will fail now if we wish to prove the same bound, because the line ℓ may contain more than just two vertices and these vertices will be taken into account when considering both parts of the graph.

The main result in this paper is that the same bound $2n - 3$ is valid also in the case where the set of vertices of G may not be in general position. Nicely enough, the proof is still elegant and at the same time non-trivial.

Theorem 1. *Let G be a geometric graph on n vertices that are not necessarily in general position. If every pair of edges of G are virtually avoiding, then G has at most $2n - 3$ edges for $n \geq 2$.*

In other words, we prove that a geometric graph on n vertices with no pair of virtually crossing edges has at most $2n-3$ edges. It would be interesting to consider geometric graphs in which there are no k pairwise virtually crossing edges, for some fixed number k (put differently, every set of k edges contains a pair of virtually avoiding edges). In particular, such a graph does not contain k pairwise crossing edges and therefore has at most $O(n \log n)$ edges by a result of Valtr [10]. It is a well-known conjecture that any graph with no k

pairwise crossing edges has at most $O(n)$ edges [7, Problem 3.3]. However, this conjecture was only verified for $k \le 4$ [1, 2, 3]. Therefore, it might be easier to prove:

Conjecture 2. *Let G be a geometric graph on n vertices (in general position). If G does not contain k pairwise virtually crossing edges, then G has at most $c_k n$ edges, where c_k is a constant that depends only on k.*

Indeed, it is not hard to show that every complete n-vertex geometric graph has $\Omega(n)$ pairwise virtually crossing edges, whereas it is conjectured that such a graph also contains $\Omega(n)$ pairwise crossing edges, however, the best known bound is only $\Omega(\sqrt{n})$[4].

Let us sketch the proof that a complete n-vertex geometric graph has $\lfloor \frac{n}{2} \rfloor$ pairwise virtually crossing edges, which is based on ideas from [8]. Assume that n is even and let ℓ be a directed line halving the set of vertices of the graph. Denote the two sets of vertices into which ℓ divides the graph by A_0 and B_0. Rotate ℓ clockwise until it touches the convex hulls of both A_0 and B_0 at two points a and b, respectively. Pick the edge (a, b) and define $A_1 = A_0 \setminus \{a\}$, $B_1 = B_0 \setminus \{b\}$. Now repeat the same procedure for A_1 and B_1. Continuing this way, after $\frac{n}{2}$ steps we pick $\frac{n}{2}$ edges of G that are pairwise virtually crossing. We leave it to the reader to verify this last assertion.

As an application of Theorem 1 we obtain the following:

Corollary 3. *Let $\mathcal{L}$ be a set of n lines in the plane, and suppose that for every line $L \in \mathcal{L}$ two consecutive intersection points on L are marked. Then at least $\frac{n+3}{2}$ intersection points are marked overall.*

Proof. Let V be the set of intersection points that are marked, and let E be the set of segments connecting two consecutive intersection points in V, that were marked on the same line. Observe that $G = (V, E)$ is a geometric graph in which every pair of edges are virtually avoiding. Thus, $n = |E| \le 2|V| - 3$. □

2 Proof of Theorem 1

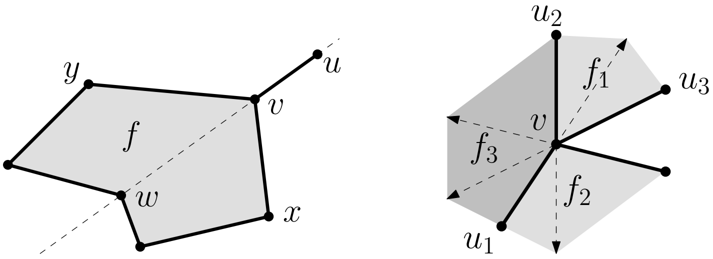
We prove the theorem by induction on n. The claim clearly holds for $n = 2$. Let G be a geometric graph on $n > 2$ vertices with pairwise virtually avoiding edges, and assume that the claim holds for any such graph with less than n vertices. If G has a vertex of degree at most two, then we can conclude the theorem by removing this vertex and applying the induction hypothesis. Therefore, we assume that the minimum degree of G is at least three.

We may also assume that G is 2-connected. Indeed, if G is not connected then we conclude the theorem by applying the induction hypothesis on each of its connected components. Otherwise, suppose that G has a cut vertex v. Let $V_1, \ldots, V_k$, $k \ge 2$, be the vertex sets of the connected components of $G \setminus \{v\}$, and let G_i be the subgraph of G induced by $V_i \cup \{v\}$, for $i = 1, \ldots, k$. Then by the induction hypothesis G_i has at most $2(|V_i| + 1) - 3$ edges. Thus, the number of edges of G is at most

$$\sum_{i=1}^{k}(2(|V_i| + 1) - 3) = \sum_{i=1}^{k}(2|V_i| - 1) = 2(n - 1) - k < 2n - 3.$$

Recall that G, viewed as a geometric graph drawn in the plane, is a planar map. Let m be the number of its edges, let F be the number of its faces, and for every $k \ge 3$ let f_k be the number of faces of G of size k, that is, faces with k edges (notice that no face can have only two edges, as $n \ge 3$). Then:

$$m = 3n - 6 - \sum_{k}(k - 3)f_k. \qquad (1)$$



(a) v is a needy vertex of f, u is a witness for that.

(b) A non-reflex vertex which is not on the unbounded face is a needy vertex of at least three faces.

Figure 2: Needy vertices.

Indeed, Equation (1) follows easily from Euler's Formula, since:

$$3m = 3n + 3F - 6 = 3n - 6 + \sum_k k \cdot f_k - \sum_k (k-3) f_k = 3n - 6 + 2m - \sum_k (k-3) f_k.$$

Therefore, it is enough to prove that $\sum_k (k-3) f_k \geq n - 3$. This is done using the *discharging method*: First, we assign a *charge* of $k - 3$ to every face of size k. Then, we redistribute the charges (*discharge*) such that every vertex has charge at least 1, the unbounded face has charge -3, and every bounded face has a non-negative charge. This implies that the initial charge $\sum_k (k-3) f_k$ is at least $n - 3$.

Before describing the (dis)charging scheme, we first make some observations (we postpone for later the proofs of some of which). Observe that since G is 2-connected and $n \geq 3$, the boundary of each of the faces of G is a simple polygon. Recall that a vertex v of a face f is called a *reflex* vertex of f if the internal angle at v is greater than π. Otherwise v is called a *convex* vertex of f. (Note that an internal angle that equals π implies two virtually crossing edges.)

Definition 2.1 (needy vertex). *Let v be a convex vertex of a bounded face f and let x and y be its two neighbors on the boundary of f. If v has a neighbor $u \notin \{x, y\}$ such that x and y are separated by the line through v and u, then we say that v is a* needy *vertex of f and that u is a* witness *for that (see Figure 2(a)).*

Proposition 2.2. *Let f be a bounded face of G such that the number of convex vertices in f is three. Then f has neither needy nor reflex vertices (in particular f must be a triangle).*

Proposition 2.3. *Let f be a bounded face of G such that the number of convex vertices in f is four.*

1. *If f has a reflex vertex, then it has no needy vertices and no other reflex vertices (in particular f is a pentagon in this case).*
2. *If f has a needy vertex, then f is a convex quadrilateral and has at most one other needy vertex.*

Charging and discharging. As mentioned above, for every $k \geq 3$ we assign a *charge* of $k - 3$ to every face of size k in G. In the discharging phase, the unbounded face sends one unit of charge to each vertex on its boundary. Every bounded face sends one unit of charge to every reflex vertex on its boundary, and $\frac{1}{3}$ units of charge to each of its needy vertices.

Proposition 2.4. *After applying the discharging rules the following holds:*

1. *The charge at every vertex is at least* 1*;*
2. *The charge of the unbounded face of G is -3; and*
3. *The charge of every bounded face is non-negative.*

Proof. Let v be a vertex in G. If v is on the unbounded face or v is a reflex vertex of some bounded face, then its charge is clearly at least 1. Suppose that v is not on the unbounded face and is not a reflex vertex of any bounded face. Then v lies in the convex hull of its neighbors, and therefore (by Carathéodory's Theorem) in the convex hull of three of its neighbors which we denote by u_1, u_2, and u_3. For $i = 1, 2, 3$ let f_i denote the face to which the ray $\overrightarrow{u_i v}$ enters at v. Notice that f_1, f_2, and f_3 are distinct faces (because G is 2-connected) and v is a needy vertex with respect to each f_i (having u_i as a witness to this, see Figure 2(b)). Therefore, the final charge at v is at least one unit, as each f_i donates $\frac{1}{3}$ of a unit of charge to v. This proves (1).

Observe that (2) holds by definition, therefore it remains to show (3) that the final charge of every bounded face is non-negative. Let f be a bounded face of size k with r reflex vertices. Thus, f sends at most $r + (k - r)/3$ units of charge to its vertices. If $r \leq k - 5$, then f sends at most $k - 5 + 5/3 < k - 3$ units of charges and thus remains with a positive charge. Since every face has at least three convex vertices, it remains to consider the cases $r = k - 3$ and $r = k - 4$. If f has exactly three convex vertices, then by Proposition 2.2 it is a triangle and it does not send any charge to its vertices and therefore ends up with its initial charge (zero). Finally, suppose that f has exactly four convex vertices and it sends some charge to its vertices. Then by Proposition 2.3 either f is a convex quadrilateral with at most two needy vertices, or f is a pentagon with exactly one reflex vertex and no needy vertices. In both cases f sends at most one unit of charge to its vertices and ends up with a non-negative charge. □

It remains to prove Propositions 2.2 and 2.3. To this end we first prove the next proposition. Given two vertices u, v on the boundary of a polygon (or a face) P we denote by $P_{u,v}$ the clockwise path (polygonal chain) from u to v on the boundary of P.

Proposition 2.5. *Let p and q be two adjacent vertices on a boundary of a simple polygon P. If P contains exactly one convex vertex v different from p and q, then either P is a triangle, or there are two edges of P different than pq that are virtually crossing.*

Proof. Let ℓ be the line passing through the two neighbors x and y of v on the boundary of P. Assume without loss of generality that p, q, v is the clockwise order of p, q and v along the boundary of P. Assume ℓ is horizontal and v lies below ℓ. If P is not a triangle, then one of p and q must lie above ℓ, or else P has another convex vertex (the highest vertex of P). Without loss of generality assume that p lies above ℓ and that x follows v in $P_{v,p}$ (refer to Figure 3(a)). Let z denote the other neighbor of x on the boundary of P. The vertex z must also lie above ℓ or else P has another convex vertex (the lowest vertex of P on the clockwise path from x to p along the boundary of P). Note the z must be a reflex vertex since $P_{v,p}$ does not contain any convex vertex apart from its endpoints. Now it is easy to see that the edges xz and vy are virtually crossing (see Figure 3(a)). □

Proof of Proposition 2.2: Let p, q, and w denote the three convex vertices of f and assume that this is their clockwise order on the boundary of f. Replace $f_{p,q}$ with an edge and notice that we are left with a simple polygon. Indeed, otherwise there are at least 4 convex vertices, namely p, q, and the two vertices of f furthest from the line pq in each of the two half-planes bounded by it. Keeping in mind that no two edges of G are virtually crossing, we conclude

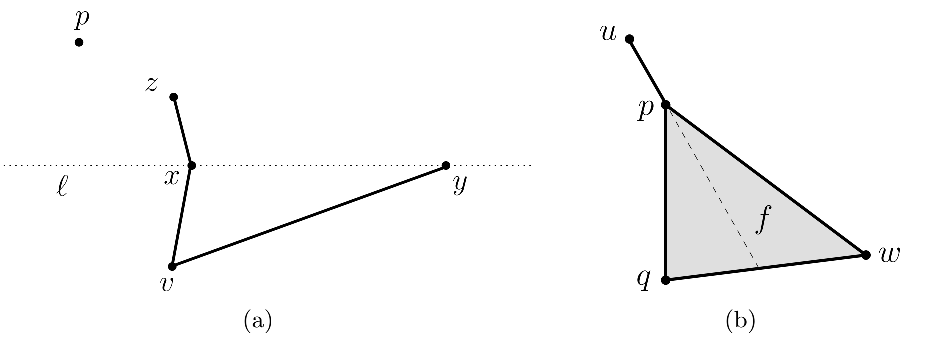


Figure 3: Illustrations for Propositions 2.5 and 2.2.

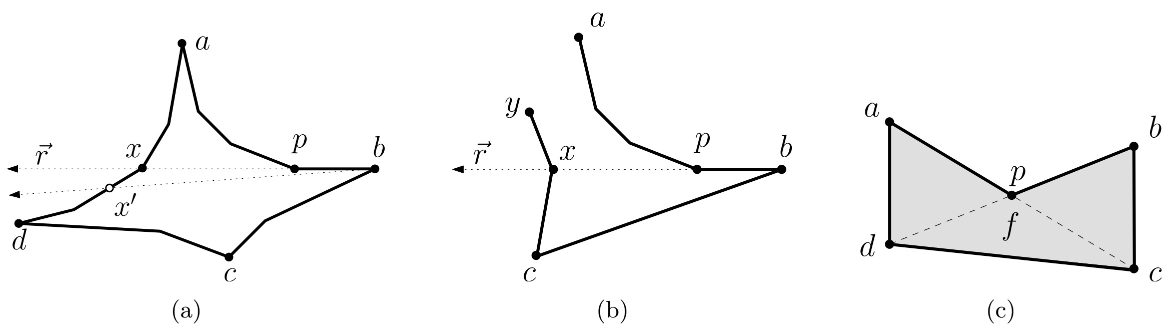


Figure 4: Illustrations for Proposition 2.3.

from Proposition 2.5 that both p and q are neighbors of w. Similarly one shows that p is a neighbor of q. Hence, f is a triangle.

Suppose that f has a needy vertex, say, p, and that u is a witness for that (see Figure 3(b)). Then pu and qw are virtually crossing, a contradiction. □

Proof of Proposition 2.3: Let a, b, c, and d denote the four convex vertices of f and assume that they appear in this cyclic clockwise order along the boundary of f. If f does not have any reflex vertex, then f is a convex quadrilateral. In this case only one of every pair of opposite vertices of f may be a needy vertex. Indeed, let x and y be two opposite vertices of f. If both are needy, let u be the witness for x and v be the witness for y. Then ux and vy must be collinear edges, which is a contradiction.

Suppose that f has a reflex vertex p, and assume without loss of generality that that p lies on $f_{a,b}$. We may assume that p is a neighbor of b. Let $\vec{r}$ be the ray through p with apex at b, and let x be the first point on $\vec{r}$ in which it crosses $f_{b,a}$.

We claim that $x = d$. Suppose for contradiction that $x \neq d$. Clearly $x \notin f_{b,c}$, for otherwise there must be a convex vertex on $f_{b,c}$ different from b and c. Suppose that $x \in f_{d,a}$. If we slightly rotate $\vec{r}$ counterclockwise, then it will cross $f_{d,a}$ at a point x' which slightly precedes x on $f_{d,a}$ (see Figure 4(a)). By applying Proposition 2.5 on the polygon consisting of the chain $f_{x',b}$ and the segment $\overline{bx'}$, we conclude that b and a are neighbors in this polygon, which is a contradiction (note that in this case x is a reflex vertex and remains so in the new polygon).

Therefore, $x \in f_{c,d}$. If $f_{b,x}$ touches (but does not cross) $\vec{r}$ at a point y that is before x on $\vec{r}$, then the open chains $f_{b,y}$ and $f_{y,x}$ must both contain a convex vertex. However, c is the only convex vertex in the open chain $f_{b,x}$. Thus, the polygon consisting of $f_{b,x}$ and $\overline{xb}$ is simple, and it follows from Proposition 2.5 that its vertices are b, c, and x. In a similar way we can deduce that a and d are neighbors in f.

Denote by y the other neighbor of x in f. The vertices y and c lie on different sides of $\vec{r}$, since $\vec{r}$ crosses $f_{b,a}$ at x. If $x \neq d$ then x must be a reflex vertex, and then yx and bc are virtually crossing (see Figure 4(b)), which is a contradiction. Therefore, $x = d$.

By applying Proposition 2.5 to the polygon consisting of $f_{b,d}$ and $\overline{bd}$, it follows that $f_{b,d}$ consists of the segments $\overline{bc}$ and $\overline{cd}$. Similarly, by applying Proposition 2.5 to the polygon consisting of $f_{d,p}$ and $\overline{dp}$, it follows that $f_{d,p}$ consists of the segments $\overline{ad}$ and $\overline{ap}$. Therefore, f must be a pentagon such that the edge ap is collinear with c and the edge bp is collinear with d (see Figure 4(c)). Notice that in this case no vertex of f may be needy. □

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