Combinatorial Model and Bounds for Target Set Selection

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Abstract

The adoption of everyday decisions in public affairs, fashion, movie-going, and consumer behavior is now thoroughly believed to migrate in a population through an *influential network*. The same diffusion process when being imitated by intention is called *viral marketing*. This process can be modeled by a (directed) graph $G = (V, E)$ with a threshold $t(v)$ for every vertex $v \in V$, where v becomes *active* once at least $t(v)$ of its (in-)neighbors are already active. A *Perfect Target Set* is a set of vertices whose activation will eventually activate the entire graph, and the *Perfect Target Set Selection* Problem (PTSS) asks for the minimum such initial set. It is known [6] that PTSS is hard to approximate, even for some special cases such as bounded-degree graphs, or majority thresholds.

We propose a combinatorial model for this dynamic activation process, and use it to represent PTSS and its variants by linear integer programs. This allows one to use standard integer programming solvers for solving small-size PTSS instances. We also show combinatorial lower and upper bounds on the size of the minimum Perfect Target Set. Our upper bound implies that there are always Perfect Target Sets of size at most $|V|/2$ and $2|V|/3$ under majority and *strict* majority thresholds, respectively, both in directed and undirected graphs. This improves the bounds of $0.727|V|$ and $0.7732|V|$ found recently by Chang and Lyuu [5] for majority and strict majority thresholds in directed graphs, and matches their bound under majority thresholds in undirected graphs. Furthermore, our proof is much simpler, and we observe that some of these bounds are tight. One interesting and perhaps surprising implication of our lower bound for undirected graphs, is that it is easy to get a constant factor approximation for PTSS for "relatively balanced" graphs (e.g., bounded-degree graphs, nearly regular graphs) with a "more than majority" threshold (that is, $t(v) = \vartheta \cdot \deg(v)$, for every $v \in V$ and some constant $\vartheta > 1/2$), whereas no polylogarithmic-approximation exists for "more than majority" graphs.

1 Introduction

Social Networks, modeled by graphs with individuals or organizations as vertices, and relationships or interactions as edges, have long been a major scientific object in many science fields, including most social sciences [11, 19, 18], life sciences [7, 27] and medicine [7, 18, 22]. Social Network play a critical role in determining the way problems are solved, organizations are run, and the degree to which individuals succeed in achieving their goals. The adoption of everyday decisions in public affairs, fashion, movie-going, and consumer behavior is now thoroughly believed to migrate in a population through an *influential network*. The same diffusion process when being imitated

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by intention is called *viral marketing*. This have roots in [13] where serendipitous discovery that messages from the media may be further mediated by informal 'opinion leaders' who intercept, interpret, and diffuse what they see and hear to the personal networks in which they are embedded.

Viral Marketing has recently became a widespread technique for promoting novel ideas, marketing new products, or spreading innovation [8, 14]. In this method, one wishes to find a good set of individuals in a network, persuade them to adopt the idea, product or innovation, and wait for the 'word-of-mouth' process to take care of 'spreading the rumor'.

One model for Viral Marketing is the *threshold model* [12], where a graph represents the social network, and a threshold value for every vertex represents the influence its neighbors have on it. Initially, a subset of vertices, the *Target Set*, is selected to be *active*. Then, repeatedly, every non-active vertex whose number of active neighbors is at least its threshold becomes active itself, and, thus, can activate neighboring vertices in the following iterations of this process.

Formally, let $G = (V, E)$ be a directed graph, $S \subseteq V$, and let $t : V \to \mathbb{N}$ be a *threshold* function associated with the vertices of G. An *activation process in G starting at S* is a chain of vertex subsets $\mathsf{Active}[0] \subseteq \mathsf{Active}[1] \subseteq \ldots \subseteq V$, with $\mathsf{Active}[0] = S$, and for all $i > 0$, $\mathsf{Active}[i] = \{u \mid u \in \mathsf{Active}[i-1] \text{ or } t(u) \leq |\{v \in \mathsf{Active}[i-1] \mid (v,u) \in E\}|\}$. We say that v is *activated* at iteration i if $v \in \mathsf{Active}[i] \setminus \mathsf{Active}[i-1]$, $i > 0$. Since the graph is finite and $\mathsf{Active}[i-1] \subseteq \mathsf{Active}[i]$, there is an integer z for which $\mathsf{Active}[z] = \mathsf{Active}[j]$, for every $j > z$. Let z be the smallest such integer, then clearly $z < n$. We define $\mathsf{Active}[S] = \mathsf{Active}[z]$ and say that S *activates* $\mathsf{Active}[S]$ in G.

There are several interesting computational and combinatorial problems related to this activation process. The first is the Target Set Selection Problem, which is defined as follows.

Target Set Selection (TSS):

Input: Two integers k, l and a digraph $G = (V, E)$ with thresholds $t : V \to \mathbb{N}$.

Problem: Find a set $S \subseteq V$, such that $|S| \leq k$ and $|\mathsf{Active}[S]| \geq l$.

A simple reduction from Vertex Cover shows that it is NP-hard to decide whether such a target set exists [15]: set $l = n$ and $t(v) = \deg(v)$. Now G has a vertex cover of size k if and only if it has a target set of size k. Usually, one would like to have a small target set (since, for example, they will get a product for free) that will activate a large number of vertices. This motivates the following optimization versions of Target Set Selection.

Minimum Target Set:

Input: An integer l and a digraph $G = (V, E)$ with thresholds $t : V \to \mathbb{N}$.

Problem: Find the smallest set $S \subseteq V$, such that $|\mathsf{Active}[S]| \geq l$.

Maximum Active Set:

Input: An integer k and a digraph $G = (V, E)$ with thresholds $t : V \to \mathbb{N}$.

Problem: Find a set $S \subseteq V$ of size k, such that any other set $S' \subseteq V$ of size k satisfies $|\mathsf{Active}[S']| \leq |\mathsf{Active}[S]|$.

Given that Target Set Selection is NP-hard, one would like to obtain good approximations for Minimum Target Set and Maximum Active Set. However, both problems turned out to be hard to approximate. Kempe, Kleinberg and Tardos [15, 16] studied Maximum Active Set (under the name Target Set Selection) and showed that it is NP-hard to approximate within a factor of $n^{1-\epsilon}$, for any constant $\epsilon > 0$. For the special case where the thresholds are taken uniformly at random, they obtained a constant-factor approximation algorithm (see also [21]). Minimum Target Set was studied by Chen [6] (again, under the name Target Set Selection) when l is a constant fraction of all the vertices, and, again, inapproximability results, even for very special cases, were shown. In particular, no $O(2^{\log^{1-\epsilon}|V|})$-approximation algorithm exists, for any constant $\epsilon > 0$, under some reasonable complexity assumption, even when all thresholds are set to 2 and the graph is of constant degree. On the positive side, exact algorithms exist if the input graph is a tree [6] or has bounded tree-width [2].

Our first contribution is a combinatorial model for Target Set Selection, that is, a model in which no iterative process is involved. We then use this model to represent the optimization problems as binary integer linear programs (IP). Integer programs for NP-hard problems are useful because one can use standard and powerful IP solvers (e.g., CPLEX, MINTO, lp_solve) in order to solve small-size problems. Moreover, linear programming relaxations for IP are a common tool for obtaining approximation algorithms for NP-hard problems [26].

A target set is called *perfect* if it activates the entire graph. The term *irreversible dynamic monopoly (dynamo)* usually refers to a perfect target set under majority or strict majority thresholds.[1] Optimal or almost optimal bounds on the size of a minimum dynamo were obtained over the years for some special graph classes such as butterfly, cube-connected cycles, hypercube, and rings to name a few (see [9, 10, 17] and the references within). These classes are usually stem from networks topologies and were considered since the activation process described above also models the propagation of faults in a fault-tolerant majority-based distributed system. Chang and Lyuu have recently studied the size of a minimum dynamo in directed and undirected graphs. In [4] they gave an upper bound of $23|V|/27$ under strict majority thresholds in directed graphs. Later, in [5], they improved this bound to $0.7732|V|$ and $\lfloor |V|/2 \rfloor$ in directed and undirected graphs, respectively. For (simple) majority thresholds, they proved a $0.727|V|$ bound for directed graphs, and a $\lfloor |V|/2 \rfloor$ bound for undirected graphs.

Using our new combinatorial formulation, and a straightforward randomized argument we derive some bounds on the size of the minimum perfect target set. We give a much simpler proof that the size of the minimum perfect target set is at most $2|V|/3$ under strict majority thresholds. This proof applies for both directed and undirected graphs, thus, it improves the bound of Chang and Lyuu in the case of directed graphs under strict majority thresholds. The same proof gives an upper bound of $|V|/2$ on the size of the minimum perfect target set under majority thresholds, both for directed and undirected graphs. This is an improvement over the $0.727|V|$ bound of Chang and Lyuu [5] for directed graphs, and basically matches their bound for undirected graphs.

We show some more bounds on the size of the minimum perfect target set for undirected graphs, using a potential function argument. Some of these bounds seem counter-intuitive in light of the hardness of approximation results of Chen [6]. For example, when $t(v) = \lceil 3/4 \cdot \deg(v) \rceil$, for every $v \in V$, it can be shown that Chen's inapproximability result holds. However, our combinatorial bounds imply that a trivial constant factor approximation exists if $\Delta(G)/\delta(G)$ is bounded ($\Delta(G)$

[1] In a majority threshold for every v we have $t(v) = \lceil \deg_{\text{in}}(v)/2 \rceil$, while in a strict majority threshold we have $t(v) = \lceil (\deg_{\text{in}}(v) + 1)/2 \rceil$.

and $\delta(G)$ are the maximum and minimum degrees in G, respectively).

Finally, we remark that the activation process we described is *monotone* in the sense that an active vertex remains active throughout the process. Non-monotone settings were also studied in the literature, see for example [3, 20, 23, 24].

Organization. In Section 2 we present a combinatorial and static model for Target Set Selection. Based on this model we suggest 0-1 integer linear programs for the two optimization problems that derive from Target Set Selection. Some combinatorial bounds for the minimum perfect target set are discussed in Section 3. All graphs in this paper are finite and simple. For a graph $G = (V, E)$ we use n to denote the number of vertices, that is $n = |V|$. We use $\deg_{\text{in}}(v)$ to denote the in-degree of a vertex v in G, and $\deg(v)$ to denote the degree of v in an undirected graph G.

2 A Combinatorial Model for TSS

Recall the Target Set Selection Problem.

Target Set Selection (TSS):

Input: Two integers k, l and a digraph $G = (V, E)$ with thresholds $t : V \to \mathbb{N}$.

Problem: Find a set $S \subseteq V$, such that $|S| \leq k$ and $|\mathsf{Active}[S]| \geq l$.

For a set $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by U. Following is an equivalent formulation of TSS.

Combinatorial Target Set Selection:

Input: Two integers k, l and a digraph $G = (V, E)$ with thresholds $t : V \to \mathbb{N}$.

Problem: Find a set $S \subseteq V$, such that $|S| \leq k$ and there is a set $A \subseteq V$ such that $S \subseteq A$, $|A| \geq l$, and one can remove edges such that $G[A]$ is acyclic and $\deg_{\text{in}}(v) \geq t(v)$ for every vertex $v \in A \setminus S$.

Lemma 2.1. *$S \subseteq V$ is a solution of* Target Set Selection *if and only if it is a solution of* Combinatorial Target Set Selection.

Proof. let S be a solution of Target Set Selection. Set $A = \mathsf{Active}[S]$ and remove every edge (u, v) for which there is no i such that $u \in \mathsf{Active}[i]$ and $v \notin \mathsf{Active}[i]$. Clearly, $G[A]$ contains no cycles. Consider a vertex $v \in A \setminus S$. When v became active at least $t(v)$ of its in-neighbors were already active. Thus, by construction v has at least $t(v)$ incoming edges in $G[A]$.

Let S be a solution of Combinatorial Target Set Selection, and consider the corresponding A and $G[A]$. Since $G[A]$ is acyclic, the vertices of A can be topologically sorted. Denote them by $a_0, a_1, \ldots, a_r$ according to this order. We prove by induction on i that $a_i \in \mathsf{Active}[i]$. For every vertex $v \in A$ we have $t(v) > 0$, therefore $\deg_{\text{in}}(v) = 0$ if and only if $v \in S = \mathsf{Active}[0]$. Thus, $a_0 \in \mathsf{Active}[0]$. Assume that the claim holds for every a_j, $0 \leq j < i$, and consider a_i, $i > 0$. By the induction hypothesis all of the at least $t(a_i)$ in-neighbors of a_i are in $\mathsf{Active}[i-1]$, therefore $a_i \in \mathsf{Active}[i]$. ☐

We will now use the new formulation of TSS to derive 0-1 integer linear programs for Minimum Target Set and Maximum Active Set.

Let $G = (V, E)$ be a digraph, and let E' be the set of non-edges, i.e., the set $\{(u, v) \mid (u, v) \notin E\}$. For every vertex $v \in V$ the variable s_v encodes whether v is selected to the target set. The threshold of a vertex v is $t_v = t(v)$. We would like to have a subset of $E \cup E'$ that yields a *tournament* (an acyclic digraph whose underlying undirected graph is complete). For every (non-)edge $(u, v) \in E \cup E'$ the variable e_{uv} encodes whether (u, v) belongs to this subset. The integer linear program for Minimum Target Set is then:

$$\begin{array}{lll} \min & \sum_{v\in V} s_v & \\ \text{s.t.} & \sum_{(u,v)\in E} e_{uv} \geq t_v \cdot (1 - s_v) & \forall v \in V \\ & e_{uv} + e_{vu} = 1 & \text{for every distinct } u, v \in V \\ & e_{uv} \in \{0, 1\} & \forall (u, v) \in E \cup E' \\ & s_v \in \{0, 1\} & \forall v \in V \\ & e_{uv} + e_{vw} + e_{wu} \leq 2 & \text{for every distinct } u, v, w \in V \end{array} \tag{Min Target Set}$$

The last constraint ensures that the graph induced by the edges and non-edges we pick is acyclic. Indeed, any maximal acylic subgraph of G can be extended to a tournament using the non-edges (this is basically a linear extension of a partial order of the vertices). Otherwise, if the edges we picked from E already induce a directed cycle, then there must be a directed cycle on three vertices no matter which of the non-edges were picked. This follows from the fact that a chord in a directed cycle creates a shorter cycle, no matter what is its orientation.

For Maximum Active Set we introduce another variable for every vertex $v \in V$, a_v, that encodes whether v is in the set A.

$$\begin{array}{lll} \max & \sum_{v\in V} a_v & \\ \text{s.t.} & \sum_{v\in V} s_v \leq k & \\ & \sum_{(u,v)\in E} e_{uv} \geq t_v \cdot (a_v - s_v) & \forall v \in V \\ & e_{uv} + e_{vu} = 1 & \text{for every distinct } u, v \in V \\ & e_{uv} \in \{0, 1\} & \forall (u, v) \in E \cup E' \\ & a_v, s_v \in \{0, 1\} & \forall v \in V \\ & e_{uv} + e_{vw} + e_{wu} \leq 2 & \text{for every distinct } u, v, w \in V \\ & a_v \geq s_v & \forall v \in V \end{array} \tag{Max Active Set}$$

Note that the second constraint guarantees that every vertex in A is in S or has enough incoming edges, while the last constraint ensures that the vertices of S are also counted as vertices in A. In both programs the number of variables is $\Theta(n^2)$ and the number of constraints is $\Theta(n^3)$.

3 Combinatorial Bounds for Perfect TSS

In this section we derive some combinatorial bounds on the size of the minimum perfect target set in terms of the vertices' degrees and thresholds.

Consider the definition of Combinatorial Target Set Selection and assume that a set A is known, but S is not known. We can find a set S that activates A as follows: start by taking a random permutation π of the vertices in A, then remove the edges in $G[A]$ that violate this order of the vertices, that is, edges (u, v) such that $\pi(u) > \pi(v)$. Now for a vertex $v \in A$, it should be in S or have at least $t(v)$ incoming edges in $G[A]$. Let S denote the set of vertices in A that do not satisfy the latter, then clearly $A \subseteq \mathsf{Active}[S]$.

The expected number of vertices in S is

$$\mathbb{E}[|S|] = \sum_{v \in A} \frac{t(v)}{\deg_{\text{in}}(v) + 1} \tag{1}$$

since there are $t(v)$ 'bad' spots for v out of the $\deg_{\text{in}}(v) + 1$ possible spots it has in the permutation of v and its in-neighbors. Therefore (1) gives an upper bound on the size of S in terms of $t(\cdot)$ and $\deg_{\text{in}}(\cdot)$. However, in general we do not know the set A, and thus, cannot compute such a set S, that activates it.

The Minimum Perfect Target Set Problem asks for a minimum target set that activates the entire graph, i.e., it is a special case of Minimum Target Set with $l = n$ or, equivalently, $A = V$. Applying (1) we get an upper bound on S for this case as well. Moreover, since A is known in this case, we can compute a target set S whose size is at most the guaranteed bound. Since the conditional expectations can easily be computed, we can also do that deterministically, by the method of conditional expectation (see [25] for an introduction of the method).

Recall that under strict majority thresholds $t(v) = \lceil \frac{\deg_{\text{in}}(v)+1}{2} \rceil$ for every $v \in V$, and observe that in this case the ratio $(\lceil \frac{\deg_{\text{in}}(v)+1}{2} \rceil)/(\deg_{\text{in}}(v)+1)$ in (1) gets its worst value, $2/3$, when $\deg_{\text{in}}(v) = 2$. Similarly, with majority thresholds we have $(\lceil \frac{\deg_{\text{in}}(v)}{2} \rceil)/(\deg_{\text{in}}(v) + 1) \geq 1/2$.

Corollary 3.1. *Let G be a (directed) graph with strict majority thresholds, such that every vertex has a positive (in-)degree. Then there is an algorithm which finds in polynomial time a target set of size at most $2n/3$.*

Corollary 3.2. *Let G be a (directed) graph with majority thresholds, such that every vertex has a positive (in-)degree. Then there is an algorithm which finds in polynomial time a target set of size at most $n/2$.*

Remark: The upper bound described in (1) is tight, as can be seen by the following construction: Take an undirected graph with n/k non-adjacent k-cliques and thresholds $k - 1$. A perfect target set S contains at least $k - 1$ vertices from every clique. Thus, $|S| \geq \frac{n(k-1)}{k}$, which is the upper bound from (1) in this case. In particular, for strict (resp., simple) majority thresholds, the bound in Corollary 3.1 (resp., 3.2) is tight as is demonstrated by a set of disjoint triangles (resp., edges).

3.1 More-than-majority thresholds

Chen [6] studied Minimum Target Set for various threshold functions. If the threshold of a vertex is equal to its degree, for all the vertices, then, as mentioned in the Introduction, the problem is equivalent to the Vertex Cover problem, and hence has a good approximation factor [1]. On the other side of the scale, in the case where all thresholds are equal to 1, it is trivial to see that one vertex will activate its connected component, thus, the problem can be solved in linear time. When all the threshold are 2, the problem becomes hard to approximate within a polylogarithmic factor. The same lower bound applies for majority thresholds, i.e., when $t(v) = \lceil \deg(v)/2 \rceil$, for all $v \in V$. However, here we show that for undirected graphs, when the thresholds are only slightly bigger, namely when $t(v) \geq \deg(v)/2 + 1$ for every $v \in V$, then the size of the minimum perfect target set is at least n/T, where $T = \max_{v \in V} t(v)$. This implies that the algorithm from the previous section that in this case finds a perfect target set of size at most $2n/3$ is a $2T/3$-approximation. Moreover, it follows that for every graph with polylogarithmic average degree, Chen's hardness result does not apply.

Theorem 3.3. *Let $G = (V, E)$ be a graph and $t : V \to \mathbb{N}$ a threshold function on its vertices, such that $t(v) \geq \deg(v)/2 + 1$ for every $v \in V$. If $S \subseteq V$ is a set such that $\mathsf{Active}[S] = V$, then $|S| \geq n/T$, where $T = \max_{v \in V} t(v)$.*

Proof. Let z be the smallest integer such that $\mathsf{Active}[z] = V$. For every i, $0 \leq i \leq z$, define a potential function

$$\Phi(i) = \sum_{v \notin \mathsf{Active}[i]} t(v) + |E\left[\mathsf{Active}\left[i\right]\right]|,$$

where $E[U]$ denotes the set of edges induced by a vertex set $U \subseteq V$.

Note that $\Phi(i+1) \geq \Phi(i)$ and that $\Phi(z) = |E|$, therefore, $\Phi(0) \leq |E|$. On the other hand, clearly $\Phi(i) \geq \sum_{v \notin \mathsf{Active}[i]} t(v)$ so together we have for the initial set $\sum_{v \notin S} t(v) \leq |E|$. Applying the assumption on $t(v)$, together with the last inequality we get:

$$|E| + |V| \leq \sum_{v \in V} (\deg(v)/2 + 1) \leq \sum_{v \in V} t(v) \leq \sum_{v \notin S} t(v) + \sum_{v \in S} t(v) \leq |E| + \sum_{v \in S} t(v).$$

Thus, $|V| \leq \sum_{v \in S} t(v)$. ∎

Note: A similar function to this one appears already in Berger's work [3].
The inequality $\sum_{v \notin S} t(v) \leq |E|$ can be extended for more general settings. For example, when $t(v) = \lceil \vartheta \cdot \deg(v) \rceil$, for a constant $\vartheta \in (1/2, 1]$, we can obtain a lower bound on $|S|$ of the form $\frac{|V|\delta(G)(2\vartheta-1)}{\Delta(G)+\delta(G)(2\vartheta-1)}$, where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees in G, respectively. This gives an approximation ratio of $\frac{\vartheta\left(\Delta(G)+\delta(G)(2\vartheta-1)\right)}{\delta(G)(2\vartheta-1)} = O(\frac{\Delta(G)}{\delta(G)})$.

When G is d-regular, the lower bound on S becomes $\frac{|V|(2\vartheta-1)}{2\vartheta}$ which gives an approximation ratio of $\frac{2\vartheta^2}{(2\vartheta-1)}$. Once the ratio between $\Delta(G)$ and $\delta(G)$ can be linear, we get the same inapproximability result as Chen's.

Theorem 3.4. *Assume that* Minimum Perfect Target Set *cannot be approximated within a factor of $f(n)$. Then for any constant $\vartheta \in [1/2, 1)$* Minimum Perfect Target Set *cannot be approximated within a factor of $f(\sqrt{n})$ when the threshold function is of the form $t(v) = \lceil \vartheta \cdot \deg(v) \rceil$, for every $v \in V$,*

Proof. Assume that there is such a lower bound $f(n)$. Let $G = (V, E)$ be a graph and let $t : V \to \mathbb{N}$ be an arbitrary threshold function. We construct a new graph G' with a new threshold function t' as follows. Consider a vertex $v \in V$. If $t(v) > \lceil \vartheta \deg(v) \rceil$ then add $t(v)/\vartheta - \deg(v)$ new dummy vertices all with threshold 1 and define $t'(v) = t(v)$. Otherwise, if $t(v) < \lceil \vartheta \deg(v) \rceil$ then add $\frac{\vartheta \deg(v) - t(v)}{1-\vartheta}$ new dummy vertices all with threshold $\frac{\vartheta}{1-\vartheta}$ and define $t'(v) = t(v) + \frac{\vartheta}{1-\vartheta}$. We also add $\frac{\vartheta}{1-\vartheta}$ new vertices, connect each of them to all the dummy vertices that where added at the last phase, and set their threshold to ϑ times their degree. Note that every vertex in G' has threshold $t'(v) = \left\lceil \vartheta \deg'(v) \right\rceil$, where $\deg'(v)$ is the degree of v in G'. The new graph G' has at most n^2 vertices, therefore, an $f(\sqrt{n})$-approximation for Minimum Perfect Target Set on G' with the threshold function t' would imply an $f(n)$-approximation on G. ∎

Remark. Note that the requirement that ϑ will be constant is just for simplicity of presentation. We can actually have the same results as long as $\frac{\vartheta}{1-\vartheta}$ does not exceed $d-1$.

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